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Korovkin type approximation theorems obtained through generalized statistical convergence

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ABSTRACT

The concept of λ-statistical convergence was introduced in [M. Mursaleen, λ-statistical convergence, Math Slovaca, 50 (2000) 111–115] by using the generalized de la Vallée Poussin mean. In this work we apply this method to prove some Korovkin type approximation theorems.

1. Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [1] and Steinhaus [2] independently in the same year 1951, and since then several generalizations and applications of this notion have been investigated by various authors.

Let K be a subset of $\mathbb{N}$, the set of natural numbers. Then the *asymptotic density* of K, denoted by $\delta(K)$, is defined as

$$\delta(K) = \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

The number sequence $x = (x_j)$ is said to be *statistically convergent* to the number ℓ if for each $\epsilon > 0$,

$$\lim_n \frac{1}{n} |\{j \leq n : |x_j - \ell| \geq \epsilon\}| = 0.$$

In this case, we write $st - \lim x_k = L$.

Remark 1.1. It is well known that every statistically convergent sequence is convergent, but the converse is not true. For example, suppose that the sequence $x = (x_n)$ is defined as

$$x = (x_n) = \begin{cases} \sqrt{n}, & \text{if } n \text{ is square} \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that the sequence $x = (x_n)$ is statistically convergent to 0, but it is not convergent.

The idea of λ-statistical convergence was introduced in [3] as follows:

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Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n + 1, \quad \lambda_1 = 0.$$

The generalized de la Vallée Poussin mean is defined by

$$t_n(x) =: \frac{1}{\lambda_n} \sum_{j \in I_n} x_j$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_j)$ is said to be (V, λ)-*summable* to a number ℓ (see [4]) if

$$t_n(x) \to \ell \quad \text{as } n \to \infty.$$

A sequence $x = (x_j)$ is said to be *strongly* (V, λ)-*summable* to a number ℓ if

$$\frac{1}{\lambda_n} \sum_{j \in I_n} |x_j - \ell| \to 0 \quad \text{as } n \to \infty.$$

We denote it by $x_j \to \ell[V, \lambda]$ as $j \to \infty$.

Let $K \subseteq \mathbb{N}$ be a set of positive integers; then

$$\delta_\lambda(K) = \lim_n \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq j \leq n : j \in K\}|$$

is said to be the λ-*density* of K.

If $\lambda_n = n$, the λ-density reduces to the natural density. Also, since $(\lambda_n/n) \leq 1$, $\delta(K) \leq \delta_\lambda(K)$ for every $K \subseteq \mathbb{N}$.

The number sequence $x = (x_j)$ is said to be λ-*statistically convergent* to the number ℓ if for each $\epsilon > 0$, $\delta_\lambda(K_\epsilon) = 0$, where $K_\epsilon = \{j \in I_n : |x_j - \ell| > \epsilon\}$, i.e.

$$\lim_n \frac{1}{\lambda_n} |\{j \in I_n : |x_j - \ell| > \epsilon\}| = 0.$$

In this case we write $st_\lambda\text{-}\lim_j x_j = \ell$ and we denote the set of all λ-statistically convergent sequences by S_λ.

Remark 1.2. As in Remark 1.1, it is observed that if a sequence is (V, λ)-summable to a number ℓ, then it is also λ-statistically convergent to the same number ℓ but the converse need not be true. For example, let the sequence $z = (z_k)$ be defined by

$$z_k = \begin{cases} k, & \text{if } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Then x is λ-statistically convergent to 0 but not (V, λ)-summable.

Recently some Korovkin type approximation theorems were proved in [5–7] by using statistical convergence, lacunary statistical convergence and statistical summability $(C, 1)$, respectively. In this work, we prove some analogues of the classical Korovkin theorem via λ-statistical convergence. The classical Korovkin approximation theorem can be stated as follows (see [8–10]):

Suppose that (T_n) is a sequence of positive linear operators from $C[a, b]$ into $C[a, b]$. Then

(i) $\lim_n \|T_n(f, x) - f(x)\|_{C[a,b]} = 0$, for all $f \in C[a, b]$, if and only if
(ii) $\lim_n \|T_n(f_i, x) - f_i(x)\|_{C[a,b]} = 0$, for $i = 0, 1, 2$, where $f_0(x) = 1, f_1(x) = x$ and $f_2(x) = x^2$.

2. Main results

Let $C[a, b]$ be the space of all functions f continuous on $[a, b]$. We know that $C[a, b]$ is a Banach space with norm $\|f\|_\infty := \sup_{a \leq x \leq b} |f(x)|, f \in C[a, b]$. Suppose that $T_n : C[a, b] \to C[a, b]$. We write $T_n(f, x)$ for $T_n(f(t), x)$ and we say that T is a positive operator if $T(f, x) \geq 0$ for all $f(x) \geq 0$.

Theorem 2.1. *Suppose that $T_n : C[a, b] \to C[a, b]$ is a sequence of positive linear operators satisfying the following conditions:*

$$st_\lambda - \lim \|T_n(1, x) - 1\|_\infty = 0, \tag{2.1}$$

$$st_\lambda - \lim \|T_n(t, x) - x\|_\infty = 0, \tag{2.2}$$

$$st_\lambda - \lim \|T_n(t^2, x) - x^2\|_\infty = 0. \tag{2.3}$$

Then for any function $f \in C[a, b]$ bounded on the whole real line, we have

$$st_\lambda - \lim \|T_n(f, x) - f(x)\|_\infty = 0.$$

Proof. Since $f \in C[a, b]$ and f is bounded on the whole real line, we have

$$|f(x)| \leq M, \quad -\infty < x < \infty.$$

Therefore,

$$|f(t)-f(x)| \le 2M, \quad -\infty < t, x < \infty. \tag{2.4}$$

Also, since $f \in C[a,b]$ we have that f is continuous on $[a,b]$, i.e.

$$|f(t)-f(x)| < \epsilon, \quad \forall |t-x| < \delta. \tag{2.5}$$

Using (2.4), (2.5) and putting $\psi(t) = (t-x)^2$, we get

$$|f(t)-f(x)| < \epsilon + \frac{2M}{\delta^2}\psi, \quad \forall |t-x| < \delta.$$

This means that

$$-\epsilon - \frac{2M}{\delta^2}\psi < f(t)-f(x) < \epsilon + \frac{2M}{\delta^2}\psi.$$

Now we can apply $T_n(1,x)$ to this inequality since $T_n(f,x)$ is monotone and linear. Hence,

$$T_n(1,x)\left(-\epsilon - \frac{2M}{\delta^2}\psi\right) < T_n(1,x)\,(f(t)-f(x)) < T_n(1,x)\left(\epsilon + \frac{2M}{\delta^2}\psi\right).$$

Note that x is fixed and so $f(x)$ is a constant number. Therefore,

$$-\epsilon T_n(1,x) - \frac{2M}{\delta^2}T_n(\psi,x) < T_n(f,x) - f(x)T_n(1,x) < \epsilon T_n(1,x) + \frac{2M}{\delta^2}T_n(\psi,x). \tag{2.6}$$

But,

$$\begin{aligned} T_n(f,x) - f(x) &= T_n(f,x) - f(x)T_n(1,x) + f(x)T_n(1,x) - f(x) \\ &= [T_n(f,x) - f(x)T_n(1,x)] + f(x)[T_n(1,x) - 1]. \end{aligned} \tag{2.7}$$

Using (2.6) and (2.7), we have

$$T_n(f,x) - f(x) < \epsilon T_n(1,x) + \frac{2M}{\delta^2}T_n(\psi,x) + f(x)(T_n(1,x)-1). \tag{2.8}$$

Now, let us estimate $T_n(\psi,x)$:

$$\begin{aligned} T_n(\psi,x) &= T_n((t-x)^2,x) = T_n(t^2-2tx+x^2,x) \\ &= T_n(t^2,x) - 2xT_n(t,x) + x^2T_n(1,x) \\ &= [T_n(t^2,x)-x^2] - 2x[T_n(t,x)-x] + x^2[T_n(1,x)-1]. \end{aligned}$$

Using (2.8), we get

$$\begin{aligned} T_n(f,x) - f(x) &< \epsilon T_n(1,x) + \frac{2M}{\delta^2}\{[T_n(t^2,x)-x^2] - 2x[T_n(t,x)-x] + x^2[T_n(1,x)-1]\} + f(x)(T_n(1,x)-1) \\ &= \epsilon[T_n(1,x)-1] + \epsilon + \frac{2M}{\delta^2}\{[T_n(t^2,x)-x^2] - 2x[T_n(t,x)-x] + x^2[T_n(1,x)-1]\} \\ &\quad + f(x)(T_n(1,x)-1). \end{aligned}$$

Since ϵ is arbitrary we can write

$$\begin{aligned} \|T_n(f,x)-f(x)\|_\infty &\le \left(\epsilon + \frac{2Mb^2}{\delta^2} + M\right)\|T_n(1,x)-1\|_\infty + \frac{4Mb}{\delta^2}\|T_n(t,x)-x\|_\infty + \frac{2M}{\delta^2}\|T_n(t^2,x)-x^2\|_\infty \\ &\le K\left(\|T_n(1,x)-1\|_\infty + \|T_n(t,x)-x\|_\infty + \|T_n(t^2,x)-x^2\|_\infty\right), \end{aligned} \tag{2.9}$$

where $K = \max\left(\epsilon + \frac{2Mb^2}{\delta^2} + M, \frac{4Mb}{\delta^2}\right)$. For $\epsilon' > 0$, write

$$D = \left\{n \in I_m : \|T_n(1,x)-1\|_\infty + \|T_n(t,x)-x\|_\infty + \|T_n(t^2,x)-x^2\|_\infty \ge \frac{\epsilon'}{K}\right\},$$

$$D_1 = \left\{n \in I_m : \|T_n(1,x)-1\|_\infty \ge \frac{\epsilon'}{3K}\right\},$$

$$D_2 = \left\{n \in I_m : \|T_n(t,x)-x\|_\infty \ge \frac{\epsilon'}{3K}\right\},$$

$$D_3 = \left\{n \in I_m : \|T_n(t^2,x)-x^2\|_\infty \ge \frac{\epsilon'}{3K}\right\}.$$

Then $D \subset D_1 \cup D_2 \cup D_3$, and so $\delta_\lambda(D) \leq \delta_\lambda(D_1) + \delta_\lambda(D_2) + \delta_\lambda(D_3)$.
Therefore, using conditions (2.1)–(2.3), we get

$$st_\lambda - \lim \|T_n(f, x) - f(x)\|_\infty = 0.$$

This completes the proof of the theorem. □

Remark 2.1. (i) We get the classical Korovkin theorem by letting $n \to \infty$ in (2.9). (ii) By taking $\lambda_n = n$ in our theorem, we get Theorem 1 of [5].

In the following we give an example of a sequence of positive linear operators satisfying the conditions of Theorem 2.1 but not satisfying the conditions of the Korovkin theorem.

Example 2.1. Consider the sequence of classical Bernstein polynomials

$$B_n(f, x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}; \quad 0 \leq x \leq 1.$$

Let the sequence (P_n) be defined by $P_n : C[0, 1] \to C[0, 1]$ with $P_n(f, x) = (1+z_n)B_n(f, x)$, where z_n is defined as above. Then

$$B_n(1, x) = 1, \qquad B_n(t, x) = x, \qquad B_n(t^2, x) = x^2 + \frac{x - x^2}{n},$$

and the sequence (P_n) satisfies the conditions (2.1)–(2.3). Hence we have

$$st_\lambda - \lim \|P_n(f, x) - f(x)\|_\infty = 0.$$

On the other hand, we get $P_n(f, 0) = (1 + z_n)f(0)$, since $B_n(f, 0) = f(0)$, and hence

$$\|P_n(f, x) - f(x)\|_\infty \geq |P_n(f, 0) - f(0)| = z_n|f(0)|.$$

We see that (P_n) does not satisfy the classical Korovkin theorem, since $\limsup_{n\to\infty} z_n$ does not exist.

Next we study a Korovkin type theorem for a sequence of positive linear operators on $L_p[a, b]$ via λ-statistical convergence.

Theorem 2.2. *Let (A_n) be the sequence of positive linear operators $A_n : L_p[a, b] \to L_p[a, b]$ and let the sequence $\{\|A_n\|\}$ be uniformly bounded. Suppose that*

$$st_\lambda - \lim \|A_n(1, x) - 1\|_{L_p} = 0,$$
$$st_\lambda - \lim \|A_n(t, x) - x\|_{L_p} = 0,$$

and

$$st_\lambda - \lim \|A_n(t^2, x) - x^2\|_{L_p} = 0.$$

Then for any function $f \in L_p[a, b]$, we have

$$st_\lambda - \lim \|A_n(f, x) - f(x)\|_{L_p} = 0.$$

Remark 2.2. We can reformulate the above theorem under the same hypothesis as follows; that is, if

$$st - \lim \|B_n(1, x) - 1\|_{L_p} = 0,$$
$$st - \lim \|B_n(t, x) - x\|_{L_p} = 0,$$

and

$$st - \lim \|B_n(t^2, x) - x^2\|_{L_p} = 0,$$

hold. Then for any function $f \in L_p[a, b]$, we have

$$st - \lim \|B_n(f, x) - f(x)\|_{L_p} = 0,$$

where $B_n = \frac{1}{\lambda_n}\sum_{k\in I_n} A_k$.

Remark 2.3. By Theorem 2.1 of [4], we have (i) $x_k \to L[V, \lambda] \Rightarrow x_k \to L(S_\lambda)$ but not the converse, (ii) if $x = (x_k)$ is bounded and $x_k \to L(S_\lambda)$, then $x_k \to L[V, \lambda]$ and hence $x_k \to L(C, 1)$ provided x is not eventually constant. We use this observation to prove the following result.

Theorem 2.3. *Let $T_n : C[a,b] \to C[a,b]$ be a sequence of positive linear operators satisfying the conditions (2.2) and (2.3) of Theorem 2.1 and*

$$\lim_n \|T_n(1,x) - 1\|_\infty = 0. \tag{2.1$'$}$$

Then for any $f \in C[a,b]$, we have

$$\lim_m \frac{1}{\lambda_m} \sum_{n \in I_m} \|T_n(f,x) - f(x)\|_\infty = 0.$$

Proof. From the condition (2.1′), it follows that $\|T_n(1,x)\|_\infty \le M'$, for some constant $M > 0$ and for all $n = 1, 2, 3, \ldots$. Hence, for $f \in C[a,b]$, we have

$$\|T_n(f,x) - f(x)\|_\infty \le \|f\|_\infty \|T_n(1,x)\|_\infty + \|f\|_\infty \le M(M' + 1). \tag{2.10}$$

Since (2.1′) implies (2.1), by Theorem 2.1 we get

$$st_\lambda - \lim \|T_n(f,x) - f(x)\|_\infty = 0. \tag{2.11}$$

By Remark 2.3, (2.10) and (2.11) together give the desired result.
This completes the proof of the theorem. □

3. λ-statistical order

In this section we deal with the order of λ-statistical convergence of a sequence of positive linear operators.

Definition 3.1. The number sequence $x = (x_k)$ is λ*-statistically convergent to the number L with degree* $0 < \beta < 1$ if for each $\epsilon > 0$,

$$\lim_n \frac{1}{(\lambda_n)^{1-\beta}} |\{j \in I_n : |x_j - \ell| > \epsilon\}| = 0.$$

In this case, we write

$$x_k - L = (st_\lambda)\text{-}o(k^{-\beta}), \quad \text{as } k \to \infty.$$

Theorem 3.1. *Suppose that $T_n : C[a,b] \to C[a,b]$ is a sequence of positive linear operators satisfying the following conditions:*

$$\|T_n(1,x) - 1\|_\infty = st_\lambda\text{-}o(n^{-\beta_1}),$$
$$\|T_n(t,x) - x\|_\infty = st_\lambda\text{-}o(n^{-\beta_2}),$$
$$\|T_n(t^2,x) - x^2\|_\infty = st_\lambda\text{-}o(n^{-\beta_3}).$$

Then for any function $f \in C[a,b]$, we have

$$\|T_n(f,x) - f(x)\|_\infty = st_\lambda\text{-}o(n^{-\beta}), \quad \text{as } n \to \infty,$$

where $\beta = \min\{\beta_1, \beta_2, \beta_3\}$.

Proof. We can rewrite the inequality (2.9) as follows:

$$\frac{\|T_n(f,x) - f(x)\|_\infty}{(\lambda_k)^{1-\beta}} \le \left(\epsilon + \frac{2Mb^2}{\delta^2} + M\right) \frac{\|T_n(1,x) - 1\|_\infty}{(\lambda_k)^{1-\beta_1}} \left(\frac{(\lambda_k)^{1-\beta_1}}{(\lambda_k)^{1-\beta}}\right) + \frac{4Mb}{\delta^2} \frac{\|T_n(t,x) - x\|_\infty}{(\lambda_k)^{1-\beta_2}} \left(\frac{(\lambda_k)^{1-\beta_2}}{(\lambda_k)^{1-\beta}}\right) + \frac{2M}{\delta^2} \frac{\|T_n(t^2,x) - x^2\|_\infty}{(\lambda_k)^{1-\beta_3}} \left(\frac{(\lambda_k)^{1-\beta_3}}{(\lambda_k)^{1-\beta}}\right).$$

Hence,

$$\|T_n(f,x) - f(x)\|_\infty = st_\lambda\text{-}o(n^{-\beta}), \quad \text{as } n \to \infty,$$

where $\beta = \min\{\beta_1, \beta_2, \beta_3\}$.
This completes the proof of the theorem. □

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