On Existence of Positive Solutions and Bounded Oscillations for Neutral Difference Equations

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A criterion for the existence of a positive solution for the first order difference equation

\[ A(x_n - cx_{n-m}) + p_n x_{n-k} = 0, \quad p_n \geq 0 \]

is established. Results are also obtained for the oscillation and nonoscillation of solutions of a second order difference equation

\[ A^2(x_n - cx_{n-m}) = p_n x_{n-k}, \]

where the \( p_n \) are either of constant sign or are oscillatory. Some of the results are sharp. © 1992 Academic Press, Inc.

1. Introduction

For the last few years the oscillation and nonoscillation of solutions of delay difference equations are being extensively investigated [2–10]. In particular, the oscillations of the solutions of the neutral difference equation

\[ A(x_n - cx_{n-m}) + p_n x_{n-k} = 0, \quad c, p_n \geq 0 \] (1.1)

have been investigated in [3, 9], where \( A \) denotes the forward difference operator \( Ax_n = x_{n+1} - x_n \). Equation (1.1) was first considered by Brayton and Willoughby [1] from the numerical analysis point of view.

We let

\[ M = \max\{m, k\}, \quad \text{where } m \text{ and } k \text{ are nonnegative integers}. \]

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Then by a solution of Eq. (1.1) we mean a sequence \( \{x_n\} \) which is defined for \( n \geq -M \) and which satisfies Eq. (1.1) for \( n = 0, 1, \ldots \). Clearly, if
\[
x_n = A_n \quad \text{for} \quad n = -M, \ldots, -1, 0
\] (1.2)
are given, then Eq. (1.1) has a unique solution satisfying the initial conditions (1.2). We assume throughout that \( p_n \) cannot be eventually identically zero. A nontrivial solution \( \{x_n\} \) of Eq. (1.1) is said to be oscillatory if for every \( N > 0 \) there exists an \( n \geq N \) such that \( x_n x_{n+1} \leq 0 \). Otherwise it is nonoscillatory. The oscillatory behavior of an ordinary differential equation and its discrete analogue can be quite different. For example, it is well known that every solution of the Logistic equation
\[
x'(t) = r x(t) \left( 1 - \frac{x(t)}{K} \right)
\] (1.3)
is monotonic. But its discrete analogue
\[
x_{n+1} = m x_n (1 - x_n)
\]
has a chaotic solution for some \( m \). In this paper first we consider the existence of a positive solution of Eq. (1.1) with \( p_n \geq 0 \) or \( p_n < 0 \). Then we discuss the second order neutral difference equation
\[
A^2 (x_n - c x_{n-m}) = p_n x_{n-k},
\] (1.4)
where the \( p_n \) are either of constant sign or are oscillatory. We obtain results both for oscillation and for nonoscillation. Some of the results are sharp.

2. First Order Equation (1.1)

For Eq. (1.1) with \( 0 < c < 1 \) we obtained in [9] a sufficient condition for the existence of a positive solution. In this paper we are interested in establishing a condition for the existence of a decaying positive solution for Eq. (1.1). To that end our result is the following

**Theorem 2.1.** Assume that

(i) \( 0 \leq c < 1 \),
(ii) \( c \frac{n}{n-m} + n \sum_{i=k}^{\infty} \frac{p_i}{i-k} \leq 1 \), for all sufficiently large \( n \).

Then Eq. (1.1) has a positive solution \( \{x_n\} \) which satisfies \( \lim_{n \to \infty} x_n = 0 \).
Proof. Consider the Banach Space $l^N_\infty$ of all real sequences $y = \{y_n\}$ where $n \geq N$ with sup norm $\|y\| = \sup_{n \geq N} |y_n|$. We define a subset $S$ in $l^N_\infty$ as

$$S = \{ y \in l^N_\infty : 0 \leq y_n \leq 1, n \geq N \}.$$ 

We define a partial order on $l^N_\infty$ in the usual way. That is, $x, y \in l^N_\infty$, $x \leq y$ means that $x_n \leq y_n$ for $n \geq N$.

We will avoid introducing equivalence classes in $l^N_\infty$. Thus, if for any $x, y \in l^N_\infty$, $x_n = y_n$ for all $n \geq 1$ we will consider such sequences to be the same. Then for every subset $A$ of $S$ both $\inf A$ and $\sup A$ exist in $S$. Now we define an operator $T : S \to l^N_\infty$. For $y \in S$

$$T y_n = \begin{cases} 
  c y_{n-m} \left( \frac{n}{n-m} \right) + n \sum_{i=n}^\infty \frac{p_i y_{i-k}}{i-k}, & \text{if } n \geq N_1, \\
  \frac{n}{N_1} T y_{N_1} + \left( 1 - \frac{n}{N_1} \right), & \text{if } N \leq n < N_1. 
\end{cases}$$ (2.1)

By (ii), it is obvious that $T S \subseteq S$, and that $T$ is an increasing mapping. By Knaster's fixed point theorem [11] there is $y \in S$ such that

$$y_n = c y_{n-m} \left( \frac{n}{n-m} \right) + n \sum_{i=n}^\infty \frac{p_i y_{i-k}}{i-k}, \quad n \geq N_1,$$ (2.2)

and $y_n > 0$ for $n \geq N$. If we set

$$x_n = \frac{y_n}{n}$$ (2.3)

then from (2.2) we have

$$x_n = c x_{n-m} + \sum_{i=n}^\infty p_i x_{i-k}$$

and so

$$A(x_n - c x_{n-m}) + p_n x_{n-k}, \quad n \geq N_1,$$

which shows that $\{x_n\}$ is a positive solution of Eq. (1.1) with

$$\lim_{n \to \infty} x_n = 0.$$

This completes the proof.
EXAMPLE 2.1. Consider

\[ A \left( x_n - \frac{1}{2} x_{n-1} \right) + \frac{(n - 1)^2}{2(n + 1)(n + 2)(n + 3)} x_{n-1} = 0. \]  

(2.4)

It is easy to see that conditions (i) and (ii) of Theorem 2.1 are satisfied. Therefore Eq. (2.4) has a decaying positive solution. In fact,

\[ x_n = \frac{1}{n + 2}, \quad n \geq 2 \]  

(2.5)

is such a solution.

Remark 2.1. Even for the special case when \( c = 0 \) and Eq. (1.1) reduces to a delay difference equation it is difficult to find conditions for the existence of a positive solution. In a recent paper [10] one finds a condition for the existence of a positive solution, which is not expressed in terms of coefficients and delay arguments and hence cannot be easily verified, our condition (ii) is easy to verify since it depends explicitly on \( p_n, m, \) and \( k. \)

Now we consider an equation of unstable type namely,

\[ A(x_n - cx_{n-m}) = p_n x_{n-k}, \quad p_n \geq 0. \]  

(2.6)

The Eq. (2.6) can have an oscillatory solution. For example,

\[ A(x_n - x_{n-1}) = \frac{4n^2 - 2}{n(n + 1)} x_{n-1} \]

has a solution \( x_n = (-1)^n (1/n) \) which is oscillatory. We shall prove that Eq. (2.6) always has a nonoscillatory solution.

**THEOREM 2.2.** Equation (2.6) has always a positive solution.

*Proof.* For \( p_n \geq 0 \) one can find a sequence \( \{H_n\} \) such that \( \sum_{i=n}^{\infty} p_i H_i = \infty \) and

\[ \frac{p_n}{\sum_{i=n}^{\infty} p_i H_i} \to 0 \quad \text{as} \quad n \to \infty. \]  

(2.7)

Now we define a sequence as

\[ z_n = \prod_{l=n_0}^{n} \sum_{k=n_0}^{l} \prod_{j=n_0}^{k} \sum_{i=n_0}^{j} p_i H_i. \]  

(2.8)
Then
\[ \frac{z_{n-m}}{z_n} = \prod_{i=n-m+1}^{n} \prod_{j=n_0}^{k} \prod_{i=n_0}^{j} p_i H_i \to 0 \quad \text{as} \quad n \to \infty. \] (2.9)

By L'Hopital's rule \[12\] we have that
\[ \sum_{n, i=n_0}^{n} \frac{p_i z_{i-k}}{z_n} \to 0 \quad \text{as} \quad n \to \infty. \] (2.10)

With \( l_N \) and \( S \) as defined in Theorem 2.1 we define an operator \( T \),
\[ T y_n = \begin{cases} \frac{1}{2z_n} + c \frac{z_{n-m}}{z_n} y_{n-m} + \frac{1}{z_n} \sum_{i=N_1}^{n-1} (p_i z_{i-k} y_{i-k}) & \text{if} \quad n \geq N_1 + 1 \\ 1 & \text{if} \quad N \leq n \leq N_1, \end{cases} \] (2.11)
where \( N = N_1 - \max\{k, m\} \) and \( N_1 \) is chosen so large that
\[ \frac{z_n}{z_n} \geq 1 \]
\[ \frac{z_{n-m}}{z_n} \leq \frac{1}{2} \]
\[ \frac{z_{n-m}}{z_n} + \frac{1}{z_n} \sum_{i=N_1}^{n} p_i z_{i-k} \leq \frac{1}{2} \] (2.12)
for \( n \geq N_1 \). We note that, in view of (2.9) and (2.10), such an integer \( N_1 \) does exist. By (2.11) and (2.12) we have \( 0 \leq T y_n \leq 1 \), for \( n \geq N \), which implies that \( TS \subseteq S \). For any \( y, y^* \in S \) we obtain
\[ |T y_n - T y_n^*| \leq c \frac{z_{n-m}}{z_n} |y_{n-m} - y_{n-m}^*| + \frac{1}{z_n} \sum_{i=N_1}^{n} p_i z_{i-k} |y_{i-k} - y_{i-k}^*|, \]
from which it follows that
\[ \|T y - T y^*\| = \sup_{n \geq N} |T y_n - T y_n^*| \leq \frac{1}{2} \|y - y^*\|. \]

Thus \( T \) is a contraction on \( S \). Consequently there is an element \( y \in S \) such that \( T y = y \). Thus
\[ y_n = \begin{cases} \frac{1}{2z_n} + c \frac{z_{n-m}}{z_n} y_{n-m} + \frac{1}{z_n} \sum_{i=N_1}^{n-1} p_i z_{i-k} y_{i-k} & \text{if} \quad n \geq N_1 + 1 \\ 1 & \text{if} \quad N \leq n \leq N_1. \end{cases} \] (2.13)
It is obvious that \( y_n > 0 \) for \( n \geq N \). Now we set

\[ x_n = y_n z_n > 0. \]

From (2.13) we have

\[ x_n = \frac{1}{2} + cx_{n-m} + \sum_{i=N_1}^{n-1} p_i x_{i-k}, \quad n \geq N_1 + 1. \quad (2.14) \]

Hence

\[ A(x_n - cx_{n-m}) = p_n x_{n-m}, \]

which shows that \( \{x_n\} \) is a positive solution of (2.6). Which completes the proof.

**Remark 2.2.** The conclusion of Theorem 2.2 was unknown even for the constant coefficient case [3].

From the expression (2.14) describing the solution of (2.6) we can obtain more information about the solutions of (2.6).

**Corollary 2.1.**

(i) If \( c > 0 \), Eq. (2.6) has a positive solution \( \{x_n\} \) with

\[ \lim_{n \to \infty} x_n = l \]

where \( l \) is either a positive number or \( +\infty \).

(ii) If \( c = 1 \), Eq. (2.6) has a solution \( \{x_n\} \) with \( \lim_{n \to \infty} x_n = +\infty \).

(iii) If \( c > 1 \), Eq. (2.6) has a solution \( \{x_n\} \) with

\[ x_n \geq c^{(n-n_0)/m} x_{n_0}, \]

implying that the solution tends to infinity exponentially.

(iv) If \( 0 \leq c \leq l \) and \( \sum_{i=n_0}^{\infty} p_i = \infty \), Eq. (2.6) has a solution \( \{x_n\} \) with

\[ \lim_{n \to \infty} x_n = \infty. \]

(v) If \( 0 < c < 1 \) and \( \sum_{i=n_0}^{\infty} p_i = \infty \) then every bounded solution of (2.6) is either oscillatory or it tends to zero as \( n \to \infty \).

**Proof.** In fact from (2.14) we have

\[ x_n \geq \frac{1}{2} + cx_{n-m} \geq \frac{1}{2} + c(\frac{1}{2} + cx_{n-2m}) \geq \cdots, \]

from which (i) and (ii) follow, and (iii) follows from

\[ x_n \geq cx_{n-m} \geq \cdots \geq c^l x_{n-1m}. \]

The assertion (iv) follows from (2.14) directly. In order to prove (v) we let
\{x_n\} be a bounded positive solution of (2.6), then \(Az_n \geq 0\), \(\lim_{n \to \infty} z_n = l\) exists where \(l\) is finite. If \(l > 0\) then

\[
l - z_{n_0} = \sum_{i = n_0}^{\infty} p_i x_{i - k}.
\]  

(2.15)

In this case \(x_n \geq (1/2) l\) and (2.15) leads to the fact that \(\sum_{i = n_0}^{\infty} p_i < \infty\) which is a contradiction. If \(l < 0\)

\[
x_n \leq c x_n m \leq \cdots \leq c^l x_{n - lm},
\]

which implies that \(\lim_{n \to \infty} x_n = 0\). This completes the proof.

**Example 2.2.** Consider

\[
A(x_n - x_{n - 1}) = p_n x_{n - 1}, \quad n \geq 2,
\]

where

\[
p_n = (n^2 - 1)(n - 1).
\]

By Corollary 2.1, (2.16) has a solution with \(\lim_{n \to \infty} x_n = \infty\). In fact, \(x_n = n!\) is such a solution of (2.16). For (2.6) with \(c > 1\) there can exist a non-oscillatory solution tending to nonzero constant as \(n \to \infty\). For example, consider

\[
A(x_n - 2x_{n - 1}) = \frac{3}{n^2} x_{n - 1}
\]

(2.17)

which has a bounded solution \(x_n = 1 + 1/n\). Thus we have the following:

**Theorem 2.3.** Assume that \(c > 1\) and that

\[
\sum_{i = n_0}^{\infty} p_i = \infty.
\]

(2.18)

Then every bounded solution of Eq. (2.6) satisfies

\[
\lim \inf_{n \to \infty} |x_n| = 0.
\]

(2.19)

**Proof.** Let \(\{x_n\}\) be a bounded positive solution of (2.6). Then \(Az_n \geq 0\) where \(z_n = x_n - cx_{n - m}\). If \(z_n > 0\), then the boundedness implies that \(\lim_{n \to \infty} z_n = l > 0\). In which case \(x_n \geq z_n \geq l/2 > 0\) for all sufficiently large \(n\). From (2.6) we have

\[
\frac{l}{2} \sum_{i = n_0}^{\infty} p_i \leq \sum_{i = n_0}^{\infty} p_i x_{i - k} < \infty,
\]
which is a contradiction. If $z_n < 0$ then $\lim_{n \to \infty} z_n = \beta \leq 0$, we also have that $\sum_{i=n_0}^{\infty} p_i x_{i-k} < \infty$, which implies that $\lim \inf_{n \to \infty} x_n = 0$. We shall prove that $\lim \sup_{n \to \infty} x_n = 0$ also. If there exists two sequences $\{x'_n\}$ and $\{x''_n\}$ such that

$$\lim \sup_{n \to \infty} x_n = \lim_{n' \to \infty} x'_n = \alpha > 0$$

and

$$\lim \inf_{n \to \infty} x_n = \lim_{n'' \to \infty} x''_n = 0$$

then

$$\lim_{n' \to \infty} z_{n'} + m = \lim_{n'' \to \infty} x_{n''} + m \geq 0,$$

which implies that $\beta = 0$. Then

$$\lim_{n' \to \infty} z_{n'} + m = 0 = \lim_{n'' \to \infty} x_{n''} + m - c \lim_{n'' \to \infty} x''_n,$$

that is,

$$\lim_{n'' \to \infty} x_{n''} + m = c\alpha > \alpha,$$

which contradicts the definition of $\alpha$. Therefore

$$\lim_{n \to \infty} x_n = 0.$$

**Example 2.3.** Consider

$$\Delta(x_n - 2x_{n-1}) = \frac{n-1}{n(n+1)} x_{n-1}$$

(2.20)

which satisfies the assumptions of Theorem 2.3. In fact, (2.20) has a solution $\{x_n\} = 1/n$.

We consider next the forced equation

$$\Delta(x_n + cx_{n-m}) - p_n x_{n-k} = F_n$$

(2.21)

and prove the following:

**Theorem 2.4.** Assume that

(i) $c > 0$, $p_n \geq 0$.

(ii) $\Delta f_n = F_n$ and $\lim_{n \to \infty} f_n = +\infty$, $\lim \inf_{n \to \infty} f_n = -\infty$. Then every bounded solution of (2.21) oscillates.
Proof. In fact, let \( \{x_n\} \) be a bounded positive solution of (2.21). Then, with \( z_n = x_n + cx_{n-m} \) we have

\[ A(z_n - f_n) \geq 0. \]

Then \( z_n - f_n \) is nondecreasing. Since \( z_n > 0 \) and \( f_n \) is oscillating we must have \( z_n \geq f_n \), which violates condition (ii).

**Example 2.4.** Consider

\[ (2.22) \]

where

\[ f_n = (-1)^n (2n+1), \quad f_n = (-1)^n n. \]

Equation (2.22) satisfies the condition (ii). Therefore every bounded solution of (2.22) oscillates. One such solution is \( x_n = (-1)^n (1/n) \).

**Remark 2.3.** For \( p_n \leq 0 \), condition (ii) is a sufficient condition for all solutions of (2.22) to be oscillatory [9]. But for \( p_n \geq 0 \), condition (ii) cannot guarantee the oscillation of unbounded solutions. For example, consider

\[ (2.23) \]

where

\[ p_n = \frac{4n^3 - 2n - 2}{n(n^2 - 1)} > 0 \]

\[ F_n = (-1)^{n+1} (2n+1), \quad f_n = (-1)^n n. \]

Equation (2.23) satisfies all the assumptions of Theorem 2.3. But (2.23) has a solution \( x_n = n^2 \) which is nonoscillatory.

### 3. Second Order Difference Equations

Consider the second order neutral difference equation

\[ A^2(x_n - cx_{n-m}) = p_n x_{n-k}, \quad n \geq n_0, \]

where \( p_n \geq 0, p_n \neq 0, c \geq 0 \).
**Theorem 3.1.** Equation (3.1) always has a positive solution which tends to infinity as \( t \to \infty \).

*Proof.* For a given \( p_n \geq 0 \), there exists a sequence \( \{ H_n \} \) such that

\[
\sum_{i=n_0}^{\infty} p_i H_i = \infty, \quad \frac{p_i}{\sum_{i=n_0}^{n} p_i H_i} \to 0 \quad \text{as} \quad n \to \infty.
\]

Define a sequence \( \{ z_n \} \) as

\[
z_n = \prod_{l=n_0}^{n} \sum_{k=n_0}^{l} \sum_{i=n_0}^{j} p_i H_i.
\]

As in Section 1 there exists a large number \( N_1 \) such that

\[
z_n \geq 1, \quad c \frac{z_{n-m}}{z_n} < \frac{1}{2}
\]

\[
c \frac{z_{n-m}}{z_n} + \frac{1}{z_{n-i=n_0}} \sum_{i=n_0}^{n-1} (n - 1 - i) p_i z_{i-k} \leq \frac{1}{2} \quad \text{for} \quad n \geq N_1.
\]

Define a subset \( S \) in the Banach Space \( l_{x_c}^N \) as

\[
S = \{ \{ x_n \} \in l_{x_c}^N : 0 \leq x_n \leq 1, n \geq n_0 \}.
\]

Define an operator \( T \) on \( S \),

\[
T y_n = \begin{cases} 
\frac{c}{z_n} \frac{z_{n-m}}{z_n} y_{n-m} + \frac{1}{z_{n-i=n_0}} \sum_{i=n_0}^{n-2} (n - 1 - i) p_i y_{i-k} z_{i-k} + \frac{1}{2 z_n} & \text{if} \quad n \geq N \\
1 & \text{if} \quad n_0 \leq n < N_1.
\end{cases}
\]

By (3.2), \( TS \subset S \). Let \( y, y^* \in S \), then

\[
|Ty_n - Ty_n^*| \leq \frac{c}{z_n} \frac{z_{n-m}}{z_n} |y_{n-m} - y_{n-m}^*| + \frac{1}{z_{n-i=n_0}} \sum_{i=n_0}^{n-2} (n - 1 - i) p_i z_{i-k} |y_{i-k} - y_{i-k}^*| \\
\leq \frac{1}{2} \| y - y^* \|, \quad n \geq N_1.
\]
Hence

\[ \| Ty - Ty^* \| = \sup_{n \geq n_0} |Ty_n - Ty_n^*| \]

\[ = \sup_{n \geq N_1} |Ty_n - Ty_n^*| \]

\[ \leq \frac{1}{2} \| y - y^* \|, \]

which shows that \( T \) is a contraction on \( S \). Therefore there is an element \( y \in S \) such that

\[ y_n = c \frac{z_{n-m}}{z_n} y_{n-m} + \frac{1}{n-1} \sum_{i=m}^{n-2} (n-1-i) p_i y_{n-i} z_{n-i} + \frac{1}{22z_n}, \quad n \geq N_1 \]

\[ y_n = 1, \quad n_0 \leq n < N_1. \]

Set

\[ x_n = y_n z_n. \]

Then

\[ x_n = c x_{n-m} + \frac{1}{n-2} \sum_{i=n_0}^{n-2} (n-1-i) p_i x_{n-i}, \quad n \geq N_1 \]

\[ x_n = z_n, \quad n_0 \leq n < N_1. \]

Thus

\[ \Delta^2(x_n - c x_{n-m}) = p_n x_{n-k}, \quad n \geq N_1, \]

that is, \( \{x_n\} \) is a positive solution of (3.1). Now we shall show that

\[ \lim_{n \to \infty} x_n = \infty. \]

In fact, set

\[ u_n = x_n - c x_{n-m}. \]

Then

\[ u_n = \sum_{i=n_0}^{n-2} (n-1-i) p_i x_{n-i} + \frac{1}{2} > 0 \]

\[ \Delta u_n = \sum_{i=n_0}^{n-1} p_i x_{n-i} > 0 \]

\[ \Delta^2 u_n = p_n x_{n-k} > 0. \]

Hence \( \lim_{n \to \infty} u_n = \infty \). But \( x_n > u_n \) so \( \lim_{n \to \infty} x_n = \infty \). The proof is complete.
Now we consider the second order equation with oscillatory coefficients,
\[ \Delta^2(x_{n} - cx_{n-m}) + p_n x_{n-k} = 0, \quad (3.3) \]
where \( p_n \) is allowed to be oscillatory.

We have the following:

**Theorem 3.2.** Assume that

(i) \( c > 1, m, k \geq 1 \),

(ii) \( \sum_{i=N}^{\infty} i |p_i| < \infty \).

Then (3.3) has a bounded positive solution.

**Proof.** Let \( N \) be sufficiently large so that
\[ \sum_{i=N}^{\infty} i |p_i| < \frac{c - 1}{4} \quad \text{for} \quad n \geq N. \]

Define a set \( \Gamma \) in \( l_{\infty}^{N} \) as
\[ \Gamma = \left\{ y_n \in l_{\infty}^{N} : \frac{c}{2} \leq y_n \leq 2c, n \geq N \right\}. \]

Define a map \( T \) on \( \Gamma \) as
\[ Tx_n = \frac{3}{2} (c - 1) + \frac{1}{c} x_{n+m} + \frac{1}{c} \sum_{i=n+m}^{\infty} [i - (n + m - 1)] p_i x_{i-k}, \quad n \geq N, \]
\[ = 2c, \quad n_0 \leq n \leq N. \]

It is obvious that \( TT \subset \Gamma \).

Let \( x, y \in \Gamma \) then we have
\[ |Tx_n - y_n| \leq \frac{1}{c} |x_{n+m} - y_{n+m}| \]
\[ + \frac{1}{c} \sum_{i=n+m}^{\infty} [i - (n + m - 1)] |p_i| |x_{i-k} - y_{i-k}| \]
\[ \leq \frac{1}{c} \left( 1 - \frac{c - 1}{4} \right) \|x - y\|. \]

Hence \( (1/c)(1 + (c - 1)/4) < 1 \), and hence \( T \) is a contraction on \( \Gamma \). We must have an element \( \{x_n\} \in \Gamma \) such that
\[ x_n = \frac{3}{2} (c - 1) + \frac{1}{c} x_{n+m} + \frac{1}{c} \sum_{i=n+m}^{\infty} [i - (n + m - 1)] p_i x_{i-k}. \]
Set $y_n = x_n/c$, then

$$y_{n+m} - cy_n = -\frac{3}{2} (c - 1) - \sum_{i=n+m}^{\infty} [i - (n + m - 1)] p_i y_{i-k}.$$

Hence

$$A^2 (y_{n+m} - cy_n) = p_{n+m} y_{n+m-k},$$

which shows that $\{y_n\}$ is a solution of (3.3) with $1/2 \leq y_n \leq 2$. This completes the proof.

4. BOUNDED OSCILLATIONS

By Theorem 3.1 it is clear that it is not possible to find criteria for all the solutions of Eq. (3.1) to be oscillatory. However, we can prove the following regarding the bounded oscillations of (3.1).

**Theorem 4.1.** Assume that

(i) $0 < c < 1$, $k \geq 1$,
(ii) $p_i > 0$ and

$$\limsup_{n \to \infty} \sum_{i=n-k}^{n-1} [i - (n - k - 1)] p_i > 1.$$ 

Then every bounded solution of (3.1) is oscillatory.

**Proof:** If not, there is a bounded nonoscillatory solution $\{x_n\}$ of (3.1) which, without loss of generality, we can assume to be positive. Set $z_n = x_n - cx_{n-m}$. Then $A^2 z_n > 0$. In view of the boundedness of $z_n$ we must have $A z_n \leq 0$ eventually.

**Case 1.** Let $z_n > 0$, for all $n \geq N$. Summing (3.1) we have

$$\sum_{N=n-k}^{n-1} (Az_n - Az_N) = \sum_{N=n+k}^{n-1} \sum_{i=N}^{n-1} p_i x_{i-k}$$

or

$$(k-1) Az_n - (z_n - z_{n-k}) \geq \sum_{N=n-k}^{n-1} \sum_{i=N}^{n-1} p_i z_{i-k} \geq z_{n-1-k} \sum_{i=n-k}^{n-1} [i - (n - k - 1)] p_i.$$
Thus
\[ z_n - z_{n-k} + z_{n-1} - k \left( \sum_{i=n-k}^{n-1} [i-(n-k-1)] p_i \right) \leq 0. \]

After rewriting the above inequality we have
\[ z_n - Az_{n-k-1} + z_{n-1} - k \left( \sum_{i=n-k}^{n-1} [i-(n-k-1)] p_i - 1 \right) \leq 0, \]
which contradicts the condition (ii).

**Case 2.** Let \( z_n < 0 \) for all sufficiently large \( n \), then \( Az_n \leq 0 \) and \( z_n \leq -1 < 0 \) for all sufficiently large \( n \). On the other hand, \( z_n < 0 \) implies that
\[ x_n \leq cx_{n-m} \leq \cdots \leq c^j x_{n-jm}, \]
which leads to the conclusion that \( \lim_{n \to \infty} x_n = 0 \). Consequently \( \lim_{n \to \infty} z_n = 0 \), a contradiction. Thus the proof is complete.

**Theorem 4.2.** Assume that

(i) \( c > 1 \)

(ii) \( p_n \geq 0 \) and \( \sum_{i=m_0}^{\infty} ip_i = \infty \). Then every bounded solution of (3.1) oscillates.

**Proof.** To the contrary let \( \{x_n\} \) be a bounded positive solution of (3.1). Then \( A^2 z_n > 0, Az_n < 0 \) eventually. If \( z_n < 0 \) eventually, then
\[ \lim_{n \to \infty} z_n = -\alpha, \]
where \( \alpha \) is some positive constant. For sufficiently large \( n \) we have
\[ -\alpha < x_n - cx_{n-m} < -\frac{\alpha}{2c}. \]

Hence
\[ x_{n-m} > \frac{\alpha}{2c}. \tag{4.1} \]

Substituting (4.1) into (3.1) we have
\[ A^2 z_n \geq \frac{\alpha}{2c} p_n. \tag{4.2} \]
Summing it we have

$$\Delta z_n - \Delta z_N \geq \frac{\alpha}{2c} \sum_{i=N}^{n-1} p_i. \quad (4.3)$$

If $\sum_{i=n}^{\infty} p_i = \infty$, then we get a contradiction. So we assume that $\sum_{i=N}^{\infty} p_i < \infty$. Then from (4.3)

$$\Delta z_i \big|_{i=\infty} - \Delta z_n \geq \frac{\alpha}{2c} \sum_{i=n}^{\infty} p_i.$$

Hence

$$-\Delta z_n \geq \frac{\alpha}{2c} \sum_{i=n}^{\infty} p_i.$$

Summing it from $n$ to $\bar{n} - 1 > n$, we have

$$z_n \geq z_\bar{n} + \frac{\alpha}{2c} \sum_{i=n}^{\infty} \sum_{i=i}^{x} p_i \geq z_\bar{n} + \frac{\alpha}{2c} \sum_{i=n}^{\infty} (\bar{n} - n) p_i. \quad (4.4)$$

We allow $\bar{n} \to \infty$ then the right hand side of (4.4) tends to $+\infty$, a contradiction. Thus $z_n > 0$ eventually, and hence

$$x_n \geq cx_{n-m}$$

which implies that $x_n \geq d > 0$, $d$ is a constant. Thus from (3.1) we have

$$\Delta^2 z_n \geq dp_n. \quad (4.5)$$

As before (4.5) leads to a contradiction. Thus the proof is complete.

Combining Theorems 3.2 and 4.2 we have the following:

**Corollary 4.1.** If $c > 1$, $p_i \geq 0$ then every bounded solution of (3.1) is oscillatory if and only if

$$\sum_{i=N}^{\infty} ip_i = \infty. \quad (4.6)$$

**Example 4.1.** Consider

$$A(x_n - 2x_{n-1}) = \frac{2(n-1)(6n^3 + 10n^2 - 5n - 5)}{(n-1)n(n+1)(n+2)} x_{n-2} \quad (4.7)$$
which satisfies the assumptions of Theorem 4.2. Therefore every bounded solution of (4.7) is oscillatory. In fact, \( x_n = (-1)^n (1/n) \) is such a solution.

**References**


