POINTLIKE SETS: THE FINEST APERIODIC COVER OF A FINITE SEMIGROUP

Karsten HENCKELL
Division of Natural Sciences, New College of the University of South Florida, Sarasota, FL 34243, U.S.A.

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The research in this paper is motivated by the open question: "Is the complexity of a finite semigroup $S$ decidable?" Following the lead of the Presentation Lemma (Rhodes), we describe the finest cover on $S$ that can be computed using an aperiodic semigroup and give an explicit relation. The central idea of the proof is that an aperiodic computation can be described by a new 'blow-up operator' $H^\omega$. The proof also relies on the Rhodes expansion of $S$ and on Zeiger coding.

Guide to the paper

Chapter 1. Elementary definitions and notation should be omitted on first reading and used as a reference as needed.

Chapter 2. The Pl-functor defines pointlike sets in a general setting and shows by an abstract compactness argument that $\text{Pl}(S)$ can be computed by an aperiodic semigroup.

Chapter 3. Definition of $C^\omega(S)$ and $H^\omega$ defines $C^\omega(S)$, a collection of pointlike sets, in a constructive manner. $H^\omega$ is the 'blow-up-operator' that we will use in Chapter 5 to show $C^\omega(S) = \text{Pl}(S)$. It has some examples in the end.

Chapter 4. The Rhodes-expansion defines the tools needed in Chapter 5.

Chapter 5. $C^\omega(S) = \text{Pl}(S)$ shows the main result by actually constructing a relation $S \xrightarrow{R} \text{CP}(S)$ computing $C^\omega(S)$ with $\text{CP}(S)$ aperiodic. It uses $H^\omega$, generalized to $H^\omega$ on $C^\omega(S)$ 'to get rid of groups by blowing up'.

1. Elementary definitions and notation

In the following we list definitions and results used throughout Chapters 1–5. Most of this material is standard and is included for the convenience of the reader; for a more thorough treatment and additional background see [4], and especially
for relations, Tilson [8]. We assume the reader is familiar with basic semigroup theory.

All semigroups considered are finite, unless otherwise stated.

**Definition 1.1.** A semigroup is a finite set $S$ with an associative multiplication.

**Definition 1.2.** $S \times T$ denotes the Cartesian product of $S$ and $T$. $S \leq T$ means $S$ is a subsemigroup of $T$ (i.e., a subset closed under multiplication). $S \leq T_1 \times T_2$, or 'S is a subdirect product of $T_1$ and $T_2$', means $S \leq T_1 \times T_2$ and the projection maps

$$p_1: T_1 \times T_2 \rightarrow T_1 \quad \text{and} \quad p_2: T_1 \times T_2 \rightarrow T_2$$

project $S$ fully onto $T_1$ and $T_2$: $p_1(S) = T_1$, $p_2(S) = T_2$. If $S \subseteq T$, then $\langle S \rangle$ is the semigroup generated by $S$ in $T$.

**Definition 1.3.** $R$ is a relation between $S$ and $T$, or $R : S \rightarrow T$ or $S \xrightarrow{R} T$ if and only if $R$ is a relation.

(Note: We usually assume $R$ to be onto. In the literature this is usually not required. If $R$ has only $p_1(R) = S$, we write $S \xrightarrow{\overline{\text{onto}}} T$. We will not consider 'partial' relations, where $p_1(R) \subseteq S$.)

**Definition 1.4.** If $R$ is a function (i.e., $R(s) = \{t\}$ is one point), it is called a surmorphism (or epimorphism).

**Note.** We usually call $R$ also a homomorphism, although in the literature 'homomorphism' does not always mean $R$ is onto (i.e., $p_2(R) = T$).

**Note.** We usually denote homomorphisms with small Greek letters, like $\phi : S \rightarrow T$.

**Note.** Definitions 1.3 and 1.4 are 'elegant' definitions. In more concrete terms, $R$ is a relation if and only if $R(s_1) \cdot R(s_2) \subseteq R(s_1s_2)$ for all $s_1, s_2 \in S$. $\phi$ is a homomorphism if and only if $\phi(s_1) \cdot \phi(s_2) = \phi(s_1s_2)$ for all $s_1, s_2 \in S$. Yet another form is $R$ is a relation if and only if $s_1Rt_1$, $s_2Rt_2$ implies $s_1s_2Rt_1t_2$.

**Fact 1.5.** $R$ is a relation if and only if $R^{-1}$ is a relation, where $xR^{-1}y$ if and only if $yRx$. So $S \xrightarrow{R} T$ and $T \xrightarrow{R^{-1}} S$.

Often relations arise as a 'lift and push diagram' of homomorphisms:

$$\begin{array}{c}
R \xrightarrow{\theta} T \\
\downarrow \phi \\
S
\end{array}$$

(\star)
Given (\(\ast\)), one may define the relation \(\phi^{-1}\theta: S \to T\). Then

\[
\begin{array}{ccc}
R & \xrightarrow{\theta} & T \\
\downarrow{\phi} & \quad & \downarrow{\phi^{-1}\theta} \\
S & \xleftarrow{\psi} & T \\
\downarrow{p_1} & \quad & \downarrow{p_2} \\
& & \end{array}
\]

commutes, with \(\psi(r) = \{(s, t) \mid \phi(r) = s, \theta(r) = t\}\) and \(p_1^{-1}p_2 = \phi^{-1}\theta \leq S \times T\).

**Notation 1.6.** Let

\[
S^1 := \begin{cases}
S \cup \{1\} & \text{if } S \text{ is not a monoid (1 = identity of } S), \\
S & \text{if } S \text{ is a monoid;
}
\end{cases}
\]

\[
S^0 := \begin{cases}
S \cup \{0\} & \text{if } S \text{ does not have a zero } 0, \\
S & \text{if } S \text{ has a zero.
}
\end{cases}
\]

\(|A|\) denotes the cardinality of \(A\);

\(S\) denotes the free semigroup on letters \(s \in S\), i.e., all strings \((s_1, \ldots, s_n)\) with \(s_i \in S\) and concatenation as multiplication.

We write:

- \(s_1 \lesssim_s s_2\) if and only if \(S^1s_1 = S^1s_2\),
- \(s_1 \lesssim_R s_2\) if and only if \(s_1S^1 = s_2S^1\),
- \(s_1 \lesssim_\# s_2\) if and only if \(s_1S^1 \subseteq s_2S^1\),
- \(s_1 \lesssim_\triangledown s_2\) if and only if \(S^1s_1S^1 = S^1s_2S^1\),
- \(s_1 \lesssim_{\neq} s_2\) if and only if \(S^1s_1 \subseteq S^1s_2\),
- \(s_1 \lesssim_{\neq\#} s_2\) if and only if \(s_1S^1 \subseteq s_2S^1\),
- \(s_1 \lesssim_{\neq\triangledown} s_2\) if and only if \(S^1s_1 \subseteq S^1S^1S^1\),

and

If \(f: X \to Y\) is a function, we usually write \(f(x)\) for the value of \(f\) at \(x \in X\). \(F_R(X)\) denotes the semigroup of all functions \(f: X \to X\) with multiplication \([f \circ g](x) = g(f(x))\).

Because of the apparent reversal of symbols, often it is more natural to write the functions on the right, i.e. \(xf\) and \(x(f \circ g) = (xf)g\). We will also use this notation if the context makes it clear.

If \(s = (a, g, b) \in \mathcal{J}\) with respect to a fixed Rees-matrix coordinazation \(\mathcal{J}^0 \equiv\)
$\mathcal{M}(A, B, G, C)$ (see Definition 1.7), we write
g(s) = g(a, g, b) = g$ is the group-coordinate of $s$,
b(s) = b(a, g, b) = b$ is the $B$-coordinate of $s$.

We say $S$ is aperiodic if and only if maximal subgroups of $S$ are one point.

An important way to construct semigroups is via a ‘Rees-matrix construction’:

**Definition 1.7.** Let $S$ be a semigroup with zero 0, $A, B$ finite sets, $C$ a $B \times A$-matrix
with entries in $S$. Then

$$\mathcal{M}(A, B, S, C) := \{(a, s, b) | a \in A, \ b \in B, \ s \in S\}$$

with multiplication given by

$$(a, s, b) \cdot (a', s', b') := (a, sc(b, a')s', b'),$$

and all elements $(a, 0, b)$ identified as zero.

**Note 1.8.** Another way to construct semigroups is to ‘add a group of units $G’$, i.e.,
given $S$, form $S \cup G$, $G$ a group acting as permutations on the left and right of $S$,
and $\bot \in G$ the identity of $S$.

Rhodes [1] has shown that every semigroup can be constructed alternating the
‘Rees-matrix construction’ and ‘adding a group of units’ (up to a very nice division).
The $J$-class structure after two iterations (i.e., for 3-$J$-class semigroups) is worked
out in Austin and Rhodes (unpublished). We use this method to construct Example
3 in Chapter 2.

**Note 1.9.** It is well known that every $J$-class of $S$ can be described in this way, with
$S = G^0$, $G$ = Schützenberger-group. $C$ is called the structure-matrix, $\mathcal{J}$ – the set of
$R$-classes, $\mathcal{B}$ = the set of $L$-classes (see [4]).

**Definition 1.10.** A $J$-class of $S$ is called ‘$H$-trivial’ if and only if the Schützenberger-
group of $J$ is trivial, i.e., one point.

**Note 1.11 (Schützenberger-groups).** Let $H$ be an $H$-class of $S$. The Schützenberger-
group $G$ of $H$ is $\{s \in S | H \cdot s = H\}$, i.e., all permutations of $H$, made faithful on $H$.
(It is therefore not necessarily itself a subgroup of $S$.)

We have a homomorphism of the right idealizer of $H$ onto the Schützenberger-
group. Let $G'$ be any subgroup of the right idealizer mapping onto $G$. Since $G$ is
transitive, $h \cdot G' = h \cdot G = H$ for any $h \in H$. Now $G'$ is a subgroup of $S$, and in the
future we will call $G'$, abusing notation, also ‘the Schützenberger group’. Really all
we are interested in is

**Fact 1.12.** If $H$ is an $H$-class of $S$, then there exist groups $G', G''$ of $S$ such that
$H = h \cdot G = G'h$ for any $h \in H$. 
The powerset-functor and the union-map

**Definition 1.13.** Let \( P(S) := \{A \mid A \subseteq S\} \), the set of all subsets of \( S \). \( P(S) \) carries a natural multiplication, induced by multiplication in \( S \), via
\[
A \cdot B = \{a \cdot b \mid a \in A, b \in B\} \quad (A,B \in P(S)).
\]

**Definition 1.14.** Given a homomorphism \( \phi : S \to T \), define a map \( P(\phi) : P(S) \to P(T) \) by
\[
[P(\phi)](A) := \{\phi(a) \mid a \in A\}.
\]

\( P(\phi) \) is a homomorphism, so \( P \) is a functor \( \mathcal{F}_S \to \mathcal{F}_T \) (\( \mathcal{F}_S \) = category of finite semigroups).

**Fact 1.15.** \( S \trianglelefteq P(S) \) via the embedding \( S \trianglelefteq P(S), i(s) = \{s\} \) (singleton-set, containing \( s \) as only element). Abusing notation, we often write \( s \) for \( \{s\} \).

**Fact 1.16.** \( S \leq T \) implies \( P(S) \leq P(T) \).

**Definition 1.17.** Define \( P^0(S) := S \), \( P^{n+1}(S) := P(P^n(S)) \), \( n \geq 0 \).

**Fact 1.18.** \( P^n(S) \leq P^{n+1}(S) \), \( n \geq 0 \).

**Definition 1.19.** An important homomorphism is given by the familiar (set-theoretic) union-map \( \bigcup : P^2(S) \to P(S) \), namely if \( A \in P^2(S) \), \( A = \{A_1, \ldots, A_n\} \) with \( A_i \subseteq S \), then \( \bigcup A = A_1 \cup \cdots \cup A_n \in P(S) \). This map is easily seen to be a homomorphism.

**Note.** Thus \( \bigcup : P^{n+1}(S) \to P^n(S) \), \( n > 1 \) is defined, but not \( P(S) \to S \).

**Fact 1.20.** \( \bigcup \) is a retract, i.e., \( P(S) \leq P^2(S) \), \( P(S) \) (as considered \( \leq P^2(S) \)) maps onto \( P(S) \), and \( \bigcup \) is the inverse of the embedding \( A \trianglelefteq i(A) = \{A\} \), i.e., \( \bigcup \{A\} = A \).

**Definition 1.21.** \( P(S) \) carries a natural partial order, namely \( A \subseteq B \) (set-theoretic inclusion).

**Fact 1.22.** If \( A \subseteq B \) and \( C \in P(S) \), then \( CA \subseteq CB \) and \( AC \subseteq BC \).

**Fact 1.23.** If \( A \subseteq B, A,B \in P^2(S) \), then \( \bigcup A \subseteq \bigcup B \).

**Note 1.24.** If it is clear from the context, we write \( s \) for \( \{s\} \) and \( \{\{s\}\} \), etc. For example, if \( A = \{A_1, \ldots, A_n\} \in P^2(S) \), then \( s \cdot A = s \cdot \bigcup A \), which really means \( \bigcup \{\{s\} \cdot A\} = \{s\} \cdot \bigcup A \) (since \( \bigcup \{\{s\}\} = \{s\} \)).
Definition 1.25. Let $A \in P^2(S)$. Define $\text{sub}(A) = \{A_0 \in P(S) \mid A_0 \subseteq A_1 \text{ for some } A_1 \in A \text{ and } A_0 \notin \emptyset\}$, i.e. $\text{sub}(A) = A$ and all subsets of members of $A$ (excluding the empty set).

Relations and covers

Definition 1.26. A cover on a set $X$ is a set $D \in P^2(X)$ such that $\bigcup D = X$ and $\emptyset \notin D$ ($\emptyset$ = the empty set).

Note. Given a cover $D$, $\text{sub}(D)$ is also a cover. In applications one usually studies $\text{sub}(D)$.

Definition 1.27. A cover $D$ is called ‘full’ if and only if $\text{sub}(D) = D$.

Definition 1.28. We can put a partial ordering on covers by $C_1 \leq C_2$ if and only if for every $X \in C_1$ there exists a $Y \in C_2$ such that $X \subseteq Y$.

Note. $C_1, C_2 \in P^2(S) = P(P(S))$ are naturally partially ordered by inclusion: $C_1 \subseteq C_2$ if and only if $X \in C_1$ implies $X \in C_2$.

Fact 1.29. $C_1 \subseteq C_2$ implies $C_1 \leq C_2$ and if $C_2$ is full, $C_1 \leq C_2$ is equivalent to $C_1 \subseteq \text{sub}(C_2)$.

In fact, $C_1 \leq C_2$ could be defined as $C_1 \subseteq \text{sub}(C_2)$.

Fact 1.30. Both partial orders have a G.L.B., namely

$$C_1 \wedge C_2 := \{X \mid X = X_1 \cap X_2 \text{ with } X_1 \in C_1, \ X_2 \in C_2 \text{ and } X \neq \emptyset\}$$

for $\leq$ and

$$C_1 \cap C_2 = \{X \mid X \in C_1 \text{ and } X \in C_2 \text{ and } X \neq \emptyset\}$$

for $\subseteq$.

Fact 1.31. If $C_1, C_2$ are full, $C_1 \wedge C_2 = C_1 \cap C_2$.

Definition 1.32. A cover $D$ on a semigroup is called weakly preserved (preserved) if and only if for every $X \in D, s \in S, X \cdot s \in \text{sub}(D)$ (for every $X, Y \in D, X \cdot Y \in \text{sub}(D)$).

Fact 1.33. If $D$ is full, $D$ is preserved if and only if $D \leq P(S)$.

Definition 1.34. A cover-semigroup on $S$ is a cover $D$ on $S$ with

(i) $D$ is full;

(ii) $D$ is preserved, i.e., $D \leq P(S)$.

Definition 1.35. Given a cover $D$ on $S$, one may always construct the cover semigroup $\bar{D} := \text{sub}(D)$.
Fact 1.36. $\text{sub}(D) = \text{sub}(\text{sub}(D))$. 

Definition 1.37. Given a relation $S \xrightarrow{R} T$, define the cover $C(R) := \{ R^{-1}(t) \mid t \in T \}$. $C(R)$ is preserved, hence $\text{sub}(C(R))$ is a cover semigroup, denoted by $\mathcal{E}(R)$. In the sequel we will usually deal with $\mathcal{E}(R)$. 

Definition 1.38. We say ""$T$ computes the cover semigroup $D$ on $S$ (via $R$)"" if and only if $\mathcal{E}(R) = D$. 

Definition 1.39. We say ""$D$ is computable by an aperiodic semigroup"" if and only if there exists a relation $S \xrightarrow{R} C$ with $C$ aperiodic such that $C$ computes $D$ (via $R$). 

In the following, we wish to determine the finest cover-semigroup $D$ on a given set $S$, such that $D$ is computable aperiodically. We will give an effective procedure to determine $D (= C^\omega(S))$ and a specific relation $S \xrightarrow{R} C$ such that $\mathcal{E}(R) = D$ ($C = \text{CP}(S)$ is aperiodic). 

Definition 1.40. Given two relations $R_1 : S \rightarrow D_1$ and $R_2 : S \rightarrow D_2$ (i.e., $R_1 \subseteq S \times D_1$ and $R_2 \subseteq S \times D_2$), we can form $R_1 \times R_2 \subseteq (S \times D_1) \times (S \times D_2)$ and "take the diagonal", i.e., define $\Delta(R_1, R_2) := \{(s, x_1, x_2) \mid sR_1 x_1$ and $sR_2 x_2 \}$. Letting $\Delta(D_1, D_2) := \{(x_1, x_2) \mid x_1 \in D_1$ and $R_1^{-1}(x_1) \cap R_2^{-1}(x_2) \neq \emptyset \}$, we see $\Delta(R_1, R_2) \subseteq S \times \Delta(D_1, D_2)$, or $\Delta(R_1, R_2)$ is a relation $S \xrightarrow{\Delta} C(D_1, D_2)$. 

Definition 1.41. Extend Definition 1.40 by $\Delta(R_1, R_2, R_3) := \Delta(\Delta(R_1, R_2), R_3)$ and so on. 

Fact 1.42. The cover defined by $\Delta(R_1, R_2)$ is $C(R_1) \wedge C(R_2)$. 

Fact 1.43. $\mathcal{E}(\Delta(R_1, R_2)) = \mathcal{E}(R_1) \cap \mathcal{E}(R_2)$ and $\mathcal{E}(R_i) \subseteq \mathcal{E}(\Delta(R_1, R_2))$ follows from Facts 1.31 and 1.42. 

2. The Pl-functor 

Notation 2.1. Let $S$ be a semigroup, $K$ a collection of relations $S \xrightarrow{R} D(R)$, and let $D(K) := \{ D(R) \mid R \in K \}$ be the set of ranges of the relations $R \in K$. 

Definition 2.2. $X \subseteq S$ is pointlike with respect to $K$ if and only if for each $R \in K$ there exists a $c_0 \in D(R)$ such that $A \subseteq R^{-1}(c_0)$, or in short notation (see Chapter 1), $X \in \mathcal{E}(K)$. 

Notation 2.3. Let $\text{Pl}_K(S) := \{ X \subseteq S \mid X$ is pointlike with respect to $K \}$. 

Fact 2.4. $\Pi_K(S) = \bigcap_{R \in K} \mathcal{C}(R)$.

(Notice that this intersection is in fact finite, for there are only finitely many cover-semigroups on $S$.)

Proof. $X \in \Pi_K(S)$ if and only if $X \in \mathcal{C}(R)$ for every $R \in K$, if and only if $X \in \bigcap_{R \in K} \mathcal{C}(R)$. □

From now on, let $K$ be countable. (This is always guaranteed, if we restrict attention to finite semigroups.)

Fact 2.5. $\Pi_K(S)$ is computed by $\Delta(R_1, \ldots, R_N)$ for a finite number $N$ of relations $R_i \in K$.

Proof. $\Pi_K(S) = \bigcap_{R \in K} \mathcal{C}(R)$. Enumerate $R \in K$ as $K = \{R_i\}_{i \in \mathbb{N}}$. Then

\[(*) \quad \mathcal{C}(R_1) \supseteq \mathcal{C}(R_1) \cap \mathcal{C}(R_2) \supseteq \cdots \supseteq \bigcap_{1 \leq i \leq n} \mathcal{C}(R_i) \supseteq \cdots\]

and $\bigcap_{1 \leq i \leq n} \mathcal{C}(R_i)$ is computed by $\Delta(R_1, \ldots, R_n)$. Now some cover $C = \bigcap_{1 \leq i \leq N} \mathcal{C}(R_i)$ occurs infinitely often in $(*)$, so $\bigcap_{R \in K} \mathcal{C}(R) = \bigcap_{1 \leq i \leq N} \mathcal{C}(R_i)$ for some $N$. □

Fact 2.6. Assume that $K$ is closed under $\Delta$ (i.e., $R_1, R_2 \in K \Rightarrow \Delta(R_1, R_2) \in K$). Then $\Pi_K(S) = \mathcal{C}(R)$ for some $R \in K$.

Proof. Follows from Fact 2.5. □

In the following, we will place some more restrictions on $K$.

Definition 2.7. Let $\mathcal{D}$ be any collection of (finite) semigroups closed under $\preceq$ (i.e., if $D_1, D_2 \in \mathcal{D}$, $X \preceq D_1 \times D_2$, then $X \in \mathcal{D}$). Then let $K(\mathcal{D})$ be the set of all relations $S \rightarrow^R C$ with $C \in \mathcal{D}$. Obviously, $K(\mathcal{D})$ is closed under $\Delta$.

Definition 2.8. Let $\mathcal{C} :=$ the set of all aperiodic semigroups. We write $\Pi(S)$ for $\Pi_{K(\mathcal{C})}(S)$ in the following, and Fact 2.6 reads now (with $K = K(\mathcal{C})$):

Fact 2.9. $\Pi(S) = \mathcal{C}(R)$ for some $S \rightarrow^R C$ with $C$ an aperiodic semigroup, or $\Pi(S)$ is computable by an aperiodic semigroup.

Note. Fact 2.5 does not assure computability of $\Pi(S)$. Even though the intersection $\bigcap_{R \in K(\mathcal{C})} \mathcal{C}(R)$ is finite, we do not know when to stop forming $\mathcal{C}(R_1) \supseteq \cdots \supseteq \bigcap_{1 \leq i \leq n} \mathcal{C}(R_i) \supseteq \cdots$. On the other hand, Fact 2.5 gives a proof-scheme for proving that a certain cover-semigroup $C$ equals $\Pi(S)$:
Pointlike sets

Proof-scheme 2.10. Let D be a cover-semigroup on S. Then D = Pl(S), if D satisfies
(i) The elements of D are pointlike, i.e., D ≤ Pl(S);
(ii) D can be computed combinatorially, i.e., \( D = \mathcal{E}(R) \) with \( S \overset{R}{\longrightarrow} C \) and C combinatorial.

Proof. (i) assures \( D \leq \text{Pl}(S) \).
(ii) shows \( \text{Pl}(S) \leq D \) (since \( \text{Pl}(S) = \bigcap_{R \in \mathcal{K}(\mathcal{E})} \mathcal{E}(R) \), and \( D \) occurs in the intersection).
Together (i) and (ii) show \( D = \text{Pl}(S) \). \( \square \)

Note. We will define \( D := \mathcal{C}w(S) \) in Chapter 3, and Chapter 5 will construct \( S \overset{R}{\longrightarrow} \mathcal{C}P(S) \) with \( \mathcal{E}(R) = \mathcal{C}^{\omega}(S) \) and \( \mathcal{C}P(S) \) aperiodic.

We will postpone exploration of the functorial and other properties until later [2], so that we may then use \( \text{Pl}(S) = \mathcal{C}^{\omega}(S) \) to facilitate some proofs. However, most 'general' properties of \( \text{Pl}(S) \) are independent of that fact and could be proved earlier, just using the 'universal' definition of \( \text{Pl}(S) \) as \( \text{Pl}_{k(\mathcal{E})} \).

Excursion: transformation-semigroups

Definition 2.11. A finite transformation-semigroup (TSG) is a pair \((X, S)\) with \( X \) a finite, non-empty set and \( S \leq \mathcal{F}_R(X) \) a subsemigroup of the semigroup \( \mathcal{F}_R(X) \) of all functions on \( X \), written on the right.

Note. In the literature sometimes a more general approach is taken, allowing \( S \) to be not faithful on \( X \). However, this definition is general enough for our purpose.

All the notions defined in Chapter 1 can be generalized to TSG in the obvious way, for example

Definition 2.12. A relation of TSG's \((X, S) \overset{R}{\longrightarrow} (Z, T)\) is a pair of relations \( X \overset{R_1}{\longrightarrow} Z \), \( S \overset{R_2}{\longrightarrow} T \) such that
(i) \( S \overset{R_1}{\longrightarrow} T \) is an SG-relation;
(ii) \((xR_1) \cdot (s_1R_2) \subseteq (x \cdot s_1)R_1 \) for all \( x \in X \), \( s_1 \in S \) (compatibility).

Definition 2.13. The relation \( R = (R_1, R_2) \) defines a cover semigroup \( \mathcal{E}(R) := \mathcal{E}(R_2) \) on \( S \) as in Chapter 1. In addition, we get a 'cover on the letters', defined as \( C^X(R) := \text{sub}\{R_1^{-1}(z) \mid z \in \mathbb{Z}\} \).

Fact 2.14. Let \( S \overset{R}{\longrightarrow} T \) be a relation. Extend \( R \) to \( S^1, T^1 \) by \( 1R1 \) (this is again a relation). Then \((S^1, S) \overset{(R,R)}{\longrightarrow} (T^1, T)\) is a TSG-relation.
Definition 2.15. Let \( \bar{R} \) be a collection of TSG-relations \((X, S) \xrightarrow{R} (Y, T)\), \( R = (R_1, R_2) \). Then let
\[
\begin{align*}
P_{1,\bar{R}} &:= \{ S_0 \subseteq S \mid S_0 \in \mathcal{G}(R) \text{ for all } R \in \bar{R} \}, \\
P_{1,\bar{X}} &:= \{ X_0 \subseteq X \mid X_0 \in \mathcal{G}^X(R) \text{ for all } R \in \bar{R} \}.
\end{align*}
\]

With these definitions, all the facts of Chapter 2 extend to TSG. We list (without proof):

Fact 2.16.
\[
P_{1,\bar{R}}(X, S) = \bigcap_{R \in \bar{R}} \mathcal{G}(R), \quad P_{1,\bar{X}}(X, S) = \bigcap_{R \in \bar{R}} \mathcal{G}^X(R).
\]

Fact 2.17. If \( \bar{R} \) is countable, \( P_{1,\bar{R}}(X, S) = \mathcal{G}(\Delta(R_1, \ldots, R_N)) \), \( P_{1,\bar{X}}(X, S) = \mathcal{G}^X(\Delta(R_1, \ldots, R_N)) \) for a finite number \( N \) of relations \( R_i \in \bar{R} \).

Fact 2.18. If \( K \) is closed under \( \Delta \), then \( P_{1,\bar{R}}(X, S) = \mathcal{G}(R) \) and \( P_{1,\bar{X}}(X, S) = \mathcal{G}^X(R) \) for some \( R \in \bar{R} \).

Definition 2.19. Let \( \bar{D} \) be any collection of TSG's closed under \( \leq \). Let \( \bar{R}(\bar{D}) \) be the set of all relations \((X, S) \xrightarrow{R} (Y, C)\) with \((Y, C) \in \bar{D}\).

Proof-scheme 2.20. Let \((X, S)\) be a TSG. Let \( D \) be a cover on \( X \). Then \( D = P_{1,\bar{X}}(X, S) \), if \( D \) satisfies

(i) The elements of \( D \) are pointlike, i.e., \( D \subseteq P_{1,\bar{X}}(X, S) \);
(ii) \( D \) can be computed by an aperiodic semigroup, i.e., \( D = \mathcal{G}^X(R) \) with \((X, S) \xrightarrow{R} (Z, C)\) and \( C \) aperiodic.

Now the following connection exists between \( P_{1,\bar{R}} \) and \( P_{1,\bar{X}} \):

Definition 2.21. Let \( \bar{K} \) be a collection of TSG relations \((X, S) \xrightarrow{R_1, R_2} (Y, T)\), \( K \) be a collection of SG relations \( S \xrightarrow{R} S' \). \( \bar{K} \) and \( K \) are connected if and only if

(i) For every \( R = (R_1, R_2) \in \bar{K}, R_2 \in K \);
(ii) For every \( R \in K \), there exists \( \bar{R} \) such that \( \bar{R} = (R_1, R) \) and \( \bar{R} \in \bar{K} \) (in most applications \( \bar{R} = (R, R) : (S^1, S) \xrightarrow{T^1, T} (S^1, S) \) will do).

Fact 2.22. If \( K \) and \( \bar{K} \) are connected, then \( P_{1,K}(S) = P_{1,\bar{K}}(X, S) \).

Proof.
\[
P_{1,K}(S) = \bigcap_{R \in K} \mathcal{G}(R) \supseteq \bigcap_{R \in \bar{R}} \mathcal{G}(R) \quad \text{by (ii)}. \\
P_{1,\bar{R}}(X, S) = \bigcap_{R \in \bar{R}} \mathcal{G}(R) = \bigcap_{(R_1, R_2) \in \bar{K}} \mathcal{G}(R_1) \supseteq \bigcap_{R \in K} \mathcal{G}(R) \quad \text{by (i)}. \]
Fact 2.23. Let $S$ be a monoid, $K$ be a collection of relations on $S$ closed under $\leq \leq$, and $\overline{R} = \{ (X, S) \mid R = (R_1, R_2) (Y, T) \text{ and } R_2 \in K \}$ (i.e., all extensions of $R \in K$ to a TSG relation $\overline{R}$). Then $\Pi^X_R(X, S) = X \cdot \Pi^X_K(S)$ where $X \cdot \Pi^X_K(S)$ is short for \{ $x_0 \cdot A \mid x_0 \in X, A \in \Pi^X_K(S)$ \}.

Proof. Since $K$ and $\overline{R}$ are connected (Fact 2.14), $\Pi^X_K(X, S) = \Pi^X_K(S)$. So $X \cdot \Pi^X_K(S) = X \cdot \Pi^X_K(X, S) \subseteq \Pi^X_K(X, S)$.

For the converse let $R$ be the relation computing $\Pi^X_K(S)$ (which exists by Fact 2.18), $S \stackrel{R}{\longrightarrow} T$ and $\mathcal{E}(R) = \Pi^X_K(S)$. We will construct $(X, S) \overline{R} (Z, T)$, $\overline{R} = (\overline{R}_1, R)$ with $\mathcal{E}^X(\overline{R}) = X \cdot \Pi^X_K(S)$, proving $\Pi^X_K(X, S) \subseteq X \cdot \Pi^X_K(S)$.

So let $Z := X \times T$, and let $T$ act on $Z$ by $(x, t) \cdot t' = (x, t'')$. Define $X \overline{R}_1 X \times T$ by $\overline{R}_1(x_0) = \{(x, t) \mid x_0 \in X \cdot R^{-1}(t)\}$. (Here we use the assumption that $S$ is a monoid. It guarantees $\overline{R}_1(x_0) \neq \emptyset$, since $x_0 \in x_0 \cdot R^{-1}(R(1))$.)

Then $\overline{R}$ is a TSG-relation, and $\overline{R}_1^{-1}(x, t) = x \cdot R^{-1}(t)$, so $\mathcal{E}^X(\overline{R}) = X \cdot \Pi^X_K(S)$ by Proof-scheme 2.20.

Definition 2.24. Define $\Pi^X(X, S) = \Pi^X_K(X, S)$ and $\Pi(X, S) = \Pi^X_K(X, S)$ with $\overline{R} = \text{all TSG-relations } (X, S) \overline{R} (Y, T)$ with $T$ aperiodic.

Corollary 2.25. Let $S$ be a monoid. Then $\Pi^X(X, S) = X \cdot \Pi(S)$.

Proof. Definition 2.19 for $K = K(\mathcal{E})$ (all relations with aperiodic SG) and $\overline{R} = \text{all relations } (X, S) \overline{R} (Y, T)$ with $T$ aperiodic.

Note. Corollary 2.25 is the motivation for restricting the study to semigroups and the $\Pi$-functor.

The proof schemes 2.10 and 2.20 are due to Rhodes (unpublished).

3. Definition of $C^\alpha(S)$ and $H^\alpha$

In this chapter we define the crucial ‘blow-up-operator $H^\alpha$’ and $C^\alpha(S)$, the semigroup of ‘constructible’ pointlike sets of $S$. Three instructive examples are given at the end of the chapter.

Definition 3.1. Define a map $H : S \subseteq \text{into} P(S)$ by $H(s) := \{ t \mid t \in \mathcal{H}_s \}$ = the $\mathcal{H}$-class of $s \in S$. If clarity requires, we write $H^S(s)$ to indicate that the $\mathcal{H}$-classes are being taken with respect to $S$.

Note. $H$ is just a map, not a homomorphism.

Fact 3.2. If $S \subseteq T$, then $H^S(s) \subseteq H^T(s)$. 
Definition 3.3. Given \( T \subseteq P(S) \), define the important map \( \tilde{H}^T = H^T \circ \bigcup \). Since \( T \subseteq P(S) \), \( H^T(T) \subseteq P^2(S) \), so \( H^T \circ \bigcup(T) \subseteq P(S) \). \( \tilde{H}^T : T \to P(S) \) is called ‘H-amalgamation’. It takes \( A \in T \) (i.e., \( A \subseteq S \)), forms the \( \mathcal{H} \)-class of \( A \) (with respect to \( T \)), \( \mathcal{H}(A) = \{ A, A_1, \ldots, A_n \mid A_i, A \in \mathcal{H} \} \) and unions \( \tilde{H}^T(A) = A \cup A_1 \cup \cdots \cup A_n \). Hence

Definition 3.4. Let \( C^\omega(S) := \) unique smallest semigroup \( T \) such that
(i) \( S \neq T \neq P(S) \);
(ii) \( T \) is closed under subsets (i.e., \( \text{sub}(T) = T \));
(iii) \( T \) is closed under \( \mathcal{H} \)-amalgamation (with respect to \( T \)) (i.e., if \( P \in T \), then \( \tilde{H}^T(P) \in T \)).

Note. \( C^\omega(S) \) is well defined, since \( P(S) \) satisfies (i)-(iii), and if the \( S_i \) (\( i \in I \)) satisfy (i)-(ii), then \( \bigcap_{i \in I} S_i \) satisfies (i)-(iii).

Fact 3.5. (iii) in Definition 3.4 may be replaced by
(iii)’ For every \( P \in G \leq T \), \( G \) a subgroup of \( T \), \( \tilde{H}^T(P) \in T \).

Proof. (iii) implies (iii)’. Conversely, if \( T \) satisfies (i), (ii), (iii)’, and \( P \in T \), then \( H(P) = P \cdot G \) (\( G \) = Schützenberger-group), so \( \bigcup H(P) = \bigcup (P \cdot G) = P \cdot \bigcup G \in T \) by (iii)’. \( \square \)

An inductive definition of \( C^\omega(S) \) can be given as follows:

Definition 3.6. \( C^0(S) := S \), \( C^n(S) := \text{sub}(\langle \tilde{H}^n(X) \mid X \in C^{n-1} \rangle \cup C^{n-1}) \) (where \( \langle A \rangle \) denotes the semigroup generated by \( A \)), i.e., \( C^n(S) \) = the semigroup generated by \( C^{n-1} \) and all its \( \mathcal{H} \)-amalgams, closed under subsets. Then \( S \subseteq C^i(S) \subseteq C^{i+1} \subseteq P(S) \), so there exists an \( N \) such that \( C^N(S) = C^{N+1}(S) \). Then \( C^\omega(S) = C^N(S) \). This shows

Fact 3.7. \( C^\omega(S) \) can be effectively determined (is computable).

Proof. The iteration of Definition 3.6 has to stop in less than \( 2^{|S|} \) steps. (That the iteration may take more than 1 step is shown in Example 3.) \( \square \)

Fact 3.8. \( C^\omega(S) \subseteq \text{Pl}(S) \).

Proof. Look at the relation \( R : S \to \mathcal{C} \) with \( \mathcal{C}(R) = \text{Pl}(S) \), which exists by Fact 2.9. \( R \) extends in the obvious way to \( \bar{R} : \text{Pl}(S) \to \mathcal{C} \) (with \( \bar{R}(A) = \{ c_0 \in \mathcal{C} \mid A \subseteq R^{-1}(c_0) \} \)). Let \( A \in \text{Pl}(S) \), \( A \subseteq G \subseteq \text{Pl}(S) \), \( G \) a subgroup of \( \text{Pl}(S) \). So in

\[
\begin{array}{c}
R \\
\downarrow \quad \bar{R} \\
\text{Pl}(S)
\end{array}
\]

\[
\begin{array}{c}
\mathcal{P}_1 \\
\downarrow \quad \mathcal{P}_2 \\
C
\end{array}
\]

\[ R \xrightarrow{\mathcal{P}_2} C \]
we can lift \( G \leq \text{Pl}(S) \) to \( \tilde{G} \leq \tilde{R} \) (such that \( p_1(\tilde{G}) = G \)). \( p_2(\tilde{G}) \leq C \) is a subgroup, hence \( p_2(\tilde{G}) = c_0 \) for some \( c_0 \in C \), and \( G \subseteq \tilde{R}^{-1}(c_0) \). Since \( \tilde{R}^{-1}(c_0) = \{ X \in \text{Pl}(S) \mid X \subseteq \tilde{R}^{-1}(c_0) \} \), \( G \subseteq \bigcup \tilde{R}^{-1}(c_0) \subseteq \tilde{R}^{-1}(c_0) \), hence \( G \in \text{Pl}(S) \). This shows \( \text{Pl}(S) \) satisfies (iii)' . It also satisfies (i), (ii) in Definition 3.4, hence \( C^\omega(S) \leq \text{Pl}(S) \).

\[ \square \]

**Note.** The converse of Fact 3.8, namely \( \text{Pl}(S) \leq C^\omega(S) \), is the subject of Chapter 5.

*The \( \tilde{H} \)-operator (\( H \)-amalgamation)*

**Notation 3.9.** From now on, we will work in \( C^\omega(S) \leq P(S) \); all \( H \)-classes are taken with respect to \( C^\omega(S) \), so we write \( H, H \) sortly for \( H \in \text{C}^\omega(S) \), \( A, B, C, \ldots \) denote elements of \( C^\omega(S) \).

**Fact 3.10 (Properties of the \( \tilde{H} \)-operator).** (i) \( \tilde{H}(A) \supseteq A \); equality holds if and only if the \( G \)-class of \( A \) is \( H \)-trivial.

(ii) (\( \tilde{H} \) is local). There exist idempotents \( e(A), e'(A) \) such that \( \tilde{H}(A) = A \cdot e(A) = e'(A) \cdot A \). Moreover, \( e(A) = \tilde{H}(G) \) with \( G \) right Schützenberger group of \( A \), \( e'(A) = \tilde{H}(G') \) with \( G' \) left Schützenberger group of \( A \) and if \( 1 \in G, 1' \in G' \) are the identities of \( G, G' \), then \( A \cdot 1 = A, 1' \cdot A = A \).

(iii) \( \tilde{H}(A) \geq_X A \); \( \tilde{H}(A) = A \) if and only if the \( G \)-class of \( A \) is \( H \)-trivial. (So if the \( G \)-class of \( A \) is not \( H \)-trivial, \( \tilde{H}(A) \leq A \) in the \( \omega \)- and \( R \)-order. This fact, together with (i), is of crucial importance in this paper).

(iv) If \( A \not\leq B \) via \( X \cdot A = B, Y \cdot B = A \), then \( \tilde{H}(A) \not\leq \tilde{H}(B) \) via \( X \cdot \tilde{H}(A) = \tilde{H}(B), Y \cdot \tilde{H}(B) = \tilde{H}(A) \). Similar results are true for \( \tilde{R} \)- and \( G \)-equivalence.

(v) \( A \not\leq X \) implies \( \tilde{H}(A) = \tilde{H}(B) \).

(vi) \( A \not\leq X \) implies \( A \tilde{H}(A) = \tilde{H}(A) \).

(vii) \( A \not\leq X \) implies \( A \tilde{H}(A) = \tilde{H}(A) \).

**Negative results:**

(viii) (v) has no converse: \( \tilde{H}(A) = \tilde{H}(B) \), even though \( A \) is not \( H \)-equivalent to \( B \). Even adding \( A \tilde{H} B \) to the assumption does not help. See Example 1.

(ix) \( \tilde{H} : J_1 \rightarrow J_2 \) need not be onto \( J_2 \), see Example 2.

(x) \( \tilde{H}(A) \) is not necessarily \( H \)-trivial, and \( H^2 \neq H \). (This leads to the definition of \( H^\omega \).) See Example 1.

(xi) \( A \subseteq B \) does not imply \( \tilde{H}(A) \subseteq \tilde{H}(B) \), see Example 1.

(xii) \( A \leq_X B \) does not imply \( \tilde{H}(A) \leq_X \tilde{H}(B) \), see Example 1.

**Proof.** (i) \( \tilde{H}(A) \supseteq A \) follows directly from the definition. Let \( \tilde{H}(A) = A, A \tilde{H} A, A \neq A \); let \( G \) right Schützenberger group of \( H(A) \). With no loss of generality assume \( A_1 = A \cdot g, A = A \cdot e, g \in G, g \neq e, e = \text{identity of } G \). Then \( A_1 \cup A = A \), so \( A_1 \subseteq A \); but \( A \neq A \), so \( A_1 = A \cdot g \subseteq A \cdot e = A \). Multiplying by \( g^{-1} \in G \) (which is 1-1 on \( H(A) \)) \( A \subseteq A \cdot g \). Contradiction.
(ii) Since $H(A)$ can be written as $A \cdot G$ or $G' \cdot A$ with $G, G' \leq C^\omega(S)$ subgroups, 
$\tilde{H}(A) = \bigcup H(A) = \bigcup (A \cdot G) = A \cdot \bigcup G$ and $\tilde{H}(A) = \bigcup (G' \cdot A) = \bigcup G' \cdot A$. So let 
e(A) = \bigcup G, e'(A) = \bigcup G'. \bigcup G is an idempotent, since \( \bigcup G \cdot \bigcup G = \bigcup (G \cdot G) = \bigcup G \). Also $G \in C^\omega(S)$, since $G = \bigcup H(g_0)$, $g_0 \in G$.

(iii) The first part of (iii) follows from (ii). If $H(A) = \{A\}$, then $\tilde{H}(A) = \bigcup \{A\} = A$ trivially. Let $|H(A)| \geq 2$. By (ii), $\tilde{H}(A) \leq A$ in $\mathcal{L}$ and $\mathcal{R}$. The following argument is of critical importance in this paper:

Let $H(A) = A \cdot G$. If $\tilde{H}(A) = A \cdot \bigcup G$ was $\mathcal{R}$-equivalent to $A$, then the map 
given by multiplying with $\bigcup G$ on the right would be 1-1 on $H(A)$. But 
$A \cdot g \cdot \bigcup G = A \cdot \bigcup g \cdot G = A \cdot \bigcup G$ for all $g \in G$, so $\bigcup G$ is not 1-1 on $H(A)$ if $|G| \geq 2$.

The dual argument shows $\tilde{H}(A) < A$ in $\mathcal{R}$.

(iv) The maps $X \circ -$ and $Y \circ -$ are 1-1 on $H(A)$ and $H(B)$, so $X \cdot H(A) = H(B)$, $Y \cdot H(B) = H(A)$. Hence $X \cdot \tilde{H}(A) = X \cdot \bigcup H(A) = \bigcup H(B) = \tilde{H}(B)$ and so on.

(v) Clear.

(vi) $A \tilde{H}(X) = A \cdot X \cdot e(X) = A'Xe(X) = A'\tilde{H}(X)$.

(vii) $A \tilde{H}(X) = AXe(X)\mathcal{L}A'Xe(X) = A'\tilde{H}(X)$, dually for $\mathcal{R}$.

Fact 3.10(viii) suggests the following definition:

**Definition 3.11.** $H^\omega(A) := \tilde{H}^n(A)$ with $n$ large enough so that $\tilde{H}^{n+1}(A) = \tilde{H}^n(A)$. This defines a map $H^\omega: C^\omega(S) \to C^\omega(S)$.

**Fact 3.12.** $H^\omega$ has the following properties:

(i) $H^\omega(A) \supseteq A$, $H^\omega(A) = A$ if and only if the $\mathcal{J}$-class of $A$ is $H$-trivial.

(ii) ($H^\omega$ is local). $A \in C^\omega(S)$, then there exist idempotents $E_i$ ($i = 1, \ldots, n$ with $n \leq 2^{|S|}$) such that $H^\omega(A) = A \cdot E_1 \cdot \ldots \cdot E_n$. Similarly $H^\omega(A) = F_{m} \cdot \ldots \cdot F_1 A (F_1^2 = F_1)$. Moreover, $E_i = \bigcup G_i$ for some group $G_i \subseteq C^\omega(S)$, and with $1 \in G_i$ the identity $(AE_1 \cdot \ldots \cdot E_i-1, 1)G_i = AE_1 \cdot \ldots \cdot E_i-1$ and $(AE_1 \cdot \ldots \cdot E_i-1) \subseteq AE_1 \cdot \ldots \cdot E_i$.

(iii) $H^\omega(A) \approx_{\mathcal{F}} A$; $H^\omega(A) = A$ if and only if the $\mathcal{J}$-class of $A$ is $\mathcal{F}$-trivial. (So if the $\mathcal{J}$-class of $A$ is not $H$-trivial, $H^\omega(A) < A$ is strict.)

(iv) If $A \mathcal{J} B$ via $X \cdot A = B$, $Y \cdot B = A$, then $H^\omega(A) \mathcal{J} H^\omega(B)$ via $X \cdot H^\omega(A) = H^\omega(B)$, $Y \cdot H^\omega(B) = H^\omega(A)$. Similar results are true for $\mathcal{R}$ and $\mathcal{J}$-equivalences.

(v) $A \mathcal{J} B$ implies $H^\omega(A) = H^\omega(B)$.

(vi) $H^\omega(A)$ is $H$-trivial.

(vii) $(H^\omega)^2(A) = H^\omega(A)$.

**Negative results:**

Same as for $\tilde{H}$ (Fact 3.10), except now we have $(H^\omega)^2 = H^\omega$.

**Proof.** (i)–(v) follow by iteration from Fact 3.10. (vii) is a consequence of (vi).

(vi) Assume $H^\omega(A)$ not $H$-trivial; then by (iii) $H^\omega(H^\omega(A)) < H^\omega(A)$, a contradiction to the definition of $H^\omega$. □

The same examples as for $\tilde{H}$ apply to $H^\omega$ to show negative results.
Example 1. Let $G$ be a group. Then $\text{PI}(G) = P(G)$. We will investigate the structure of $P(G)$ somewhat, to get interesting examples for the behavior of $\mathcal{H}$ and $H^\omega$.

Definition 1. Let $|A|$ denote the cardinality of $A \leq G$.

Fact 2. $|A \cdot B| \geq \max(|A|, |B|)$, $A, B \in P(G)$.

Proof. Follows from the fact that every map $a \mapsto (\text{multiplication by } a \in G)$ is a 1-1 map.

Fact 3. $A \subseteq B$ implies $|A| = |B|$. Follows from (2).

Fact 4. $AX = B$ and $|A| = |B|$, then $Ax_0 = B$ for any $x_0 \in X$.

Proof. Since $-x_0$ is 1-1 and $|B| = |A|$, it is onto.

Fact 5. $A \subseteq B$, then there exist $g_0, g_1 \in G$ such that $g_0 Ag_1 = B$.

Proof. Follows from Facts 3 and 4.

So we see, that in order to show $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$-equivalence of elements in $P(G)$, one only has to look in $G$.

Notation 6. We write $\langle G_0 \triangleleft G \rangle$ for $G_0$ is a normal subgroup of $G$ (i.e., $gG_0g^{-1} = G_0$ for all $g \in G$).

Definition 7. Let $G_0 \leq G$ be a subgroup of $G$. Then let $N(G_0) := \text{maximal subgroup}$

Definition 8. $A \triangleleft B$, then there exist $g_0, g_1 \in G$ such that $g_0Ag_1 = B$.

Fact 8. $N(G_0) = \{g \in G \mid gG_0 = G_0g\}$.

Fact 9. $H(G_0) = \{G_0g \mid g \in N(G_0)\} = \text{all cosets of } G_0 \text{ in its normalizer}$.

Proof. If $X \not\subseteq G_0$, then by Fact 5, $X = \alpha G_0 = G_0\beta$, $\alpha, \beta \in G$. Then $\beta = an$ for some $n \in G_0$, so $\alpha G_0 = G_0\beta = G_0an$, hence $\alpha G_0 n^{-1} = \alpha G_0 = G_0\alpha = X$, $\alpha \in N(G_0)$ and $X = G_0\alpha$.

Conversely, if $g \in N(G_0)$, $G_0g = gG_0$ and $G_0gg^{-1} = g^{-1}gG_0 = G_0$, so $G_0g \not\subseteq G_0$.

Fact 10. $\tilde{H}(G_0) = N(G_0)$.

Proof. $\tilde{H}(G_0) = \bigcup \{G_0g \mid g \in N(G_0)\} = N(G_0)$.

Now assume that $N(G_0) \triangleleft G$, but $N(G_0) \neq G$, $N(G_0) \neq G_0$ (so we have $G_0 \not\triangleleft N(G_0) \triangleleft G$, but $G_0$ not normal in $G$). Then $\tilde{H}(G_0) = N(G_0)$, $\tilde{H}(N(G_0)) = G$, so $H \neq H^\omega$ and especially $H^\omega \neq H$.

Now assume also that $|G|/|N(G_0)| = 2$, so $G = N(G_0) \cup N(G_0)x$ for any $x \not\in N(G_0)$ and $N(G_0) \cdot x = x \cdot N(G_0)$. Then $G_0xG_0 \cdot x$ and $x \cdot G_0 \not\subseteq G_0$ and $\tilde{H}(G_0) \cdot x = [\tilde{H}(G_0)] \cdot x = N(G_0) \cdot x$, $\tilde{H}(x \cdot G_0) = x \cdot \tilde{H}(G_0) = x \cdot N(G_0)$.

Also notice that if $G_0 = N(G_0)$, $\tilde{H}(G_0) = G_0$. But let $1 \in G$ be the identity; then $\{1\} \subseteq G_0$, but $\tilde{H}\{1\} = G \not\subseteq G_0 = \tilde{H}(G_0)$. Also $G_0 \not\subseteq \{1\}$, but $G_0 \not\subseteq G = \tilde{H}(\{1\})$. Pointlike sets
Example 2. Let \( \mathbb{Z}_4 \) with generator \( z \) act on \( \mathcal{M} (A = \{a_1, a_2\}, B = \{b_1, b_2\}, \mathbb{Z}_2 = \{\pm 1\}, C = \{1, -1\}) \) by \( (g, b_1) \cdot z = (-g, b_2), (g, b_2) \cdot z = (g, b_1). \) (This action is linked with \( z \cdot (a_1, g) = (a_2, -g), z \cdot (a_2, g) = (a_1, g). \)) 

In \( C^w(S) \), we have the element \( \alpha = \{(a_1, 1, b_1), (a_1, 1, b_1)\} = (a_1, 1, b_1) = H(a_1, 1, b_1). \) \( (a_1, 1, b_1) \cdot \mathbb{Z}_4 = \{a_1, \pm 1, b_1, b_2\} = \beta \) since \( \alpha \cdot \mathbb{Z}_4 = \beta, \beta \cdot \alpha = \alpha, \alpha \) and \( \beta \) are \( \mathcal{R} \)-related.

However, \( \beta \) is not an \( H \)-amalgam of any \( x \mathcal{J}(a_1, 1, b_1), \beta \neq H(x) \) for all \( x \in C^w(S), x \mathcal{J}(a_1, 1, b_1). \) If it were, \( \beta = \beta_1 \cap \beta_2 \) with \( \{\beta_1, \beta_2\} \) forming an \( H \)-class.

Then (i), (ii) or (iii).

(i) \( \beta_1 = (a_1, 1, b_1) \) or \( b_2 \), \( \beta_2 = (a_1, 1, b_1) \) or \( b_1 \).

(ii) \( \beta_1 = \{(a_1, 1, b_1), (a_1, 1, b_2)\}, \beta_2 = \{(a_1, 1, b_1), (a_1, 1, b_2)\}. \)

(iii) \( \beta_1 = \{(a_1, 1, b_1), (a_1, 1, b_1)\}, \beta_2 = \{(a_1, 1, b_1), (a_1, 1, b_2)\}. \)

Case (iii) is ruled out, because \( \beta_1 \) and \( \beta_2 \) are not \( \mathcal{L} \)-equivalent. Case (i) and case (ii) are ruled out, since \( (b_1)z - 1, (b_2)z - 1. \)

Example 3. Let

\[
S^{(0)} := \mathcal{M} \left( A = \{a, b\}, B = \{a, b\}, G = \{1\}, C = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \right).
\]

Add an identity \( \bar{1} \) to \( S^{(0)} \) forming \( S^{(1)} := S^{(0)} + \bar{1}. \) Form a Rees-matrix semigroup with coefficients in \( S^{(1)} \) by

\[
S^{(2)} := \mathcal{M} \left( A' = \{1, 2\}, B' = \{1, 2\}, G = \{1\}, C' \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right).
\]

Now \( \mathbb{Z}_2 \) acts on \( C' \) via \( 1 \cdot z = 2, 2 \cdot z = 1 \) (\( z \) generator) and this action is linked. So add \( \mathbb{Z}_2 \) to \( S^{(2)} \) forming \( S^{(3)} := S^{(2)} + \mathbb{Z}_2. \) As can be easily seen, the semigroup-structure of \( S^{(3)} \) is now

\[
\begin{array}{c|cc}
\mathcal{J}_1 & \mathcal{J}_2 & \mathcal{J}_3 & \mathcal{J}_4 \\
\hline
1 & 2 & \mathbb{Z}_2; \\
1 & 0 & 0 & \\
2 & 0 & 0 & \\
\end{array}
\]

null \( \mathcal{J} \)-class elements are \( (x, \bar{1}, y), x, y \in \{1, 2\}; \)

\[
\begin{array}{c|cc}
1 & 2 \\
\hline
ab & ab \\
\end{array}
\]

regular, \( G = \{1\}; \)

\[
\begin{array}{c|cc}
1 & 2 \\
\hline
a & 01 & 00 \\
\hline
b & 00 & 10 \\
\end{array}
\]

zero.
Now consider
\[ x := \mathbb{Z}_2 \cdot (1, 1, 1) \cdot \mathbb{Z}_2, \text{ an element of } C^1(S); \]
\[ x^2 = \{(x, (a, 1, b), y)\} \cup \{x, (b, 1, a), y\} \quad (x, y \in \{1, 2\}); \]
\[ x^3 = \{(x, (a, 1, a), y)\} \cup \{x, (b, 1, b), y\}; \]
\[ x^4 = x^2. \]
So \( \{x^2, x^3\} \) is a group in \( C^1(S) \), hence \( \hat{H}(x^2) = x^2 \cup x^3 = \{(x; \alpha, 1, \beta, y) \mid x, y \in \{1, 2\}, \alpha, \beta \in \{a, b\}\} \) is an element of \( C^2(S) \).
However, \( \hat{H}(x^2) \) is not in \( C^1(S) \) already, as can be easily checked. This is an example for \( C^2(S) \neq C^1(S) \). It shows, moreover, that there are groups in \( C^\omega(S) \) that are not ‘derived from’ groups of \( S \), i.e., whose elements are not singletons (or subsets of groups). The group showing up here is a complicated subset of the bottom \( \mathcal{J} \)-class \( \mathcal{J}_3 \), created by the ‘falling’ of the middle-null-\( \mathcal{H} \) classes, after they were ‘spread out’ by the top group. One gets a good feel that \( C^\omega(S) \) is really a definition tied to the global structure of \( S \).

4. The Rhodes-expansion

The Rhodes-expansion, due to Rhodes, is an extremely useful tool in finite semigroup theory. In Subsection 4.1 we give a short definition and list some properties; for an extensive treatment of proofs, see [8]. In Subsection 4.2 we embed the Rhodes-expansion into a wreath-product, using a method originally due to Zeiger [9] and adapted to \( \hat{S} \) by Rhodes [6] (‘Zeiger-coding’). This embedding, extended to the infinite case, is also studied in [3].

4.1. Definition and elementary properties of the Rhodes-expansion

**Note 4.1.** Throughout this chapter \( \leq \) means \( \leq_{\varphi} \) as defined in Notation 1.6.

**Definition 4.2.** \( \hat{S} := \{(j_n, \ldots, j_1) \mid j_n < \cdots < j_1, j_i \in S^1, n \geq 1\}. \)

We will write suggestively \( j_n < \cdots < j_1 \) if \( (j_n, \ldots, j_1) \in \hat{S} \) (see Note 4.14 below concerning notation).

Define a canonical way to reduce \( \leq \)-chains to elements of \( \hat{S} \) in the following way:

**Definition 4.3.** Let \( x_n \leq \cdots \leq x_1 \) be a \( \leq \)-chain, \( x_i \in S^1, n \geq 1 \). Then the min-operator goes down the chain from right to left, striking all elements in a run of \( \mathcal{L} \)-equivalent elements but the last one.

To be more precise, let \( k = k(x_n \leq \cdots \leq x_1) \) be the number of \( \mathcal{L} \)-distinct elements in \( x_n \leq \cdots \leq x_1 \). Define
\[ \alpha_1 := \max \alpha \text{ such that } x_\alpha \mathcal{L} x_1 \text{ and } x_{\alpha + 1} \prec x_\alpha, \]
\[ \alpha_{n+1} := \max \alpha \text{ such that } x_\alpha \mathcal{L} x_{\alpha+1} \text{ and } x_{\alpha+1} \prec x_\alpha \]
\[(n \leq k - 1). \text{ Note that } \alpha_k = n. \]

Then

\[ \min(x_n \leq \cdots \leq x_1) := x_{\alpha_k} \prec x_{\alpha_{k-1}} \prec \cdots \prec x_{\alpha_1}. \]

In other words: if \( x_n \leq \cdots \leq x_1 \) is of the form

\[(x_n = x_{\alpha_k} \prec \cdots \prec x_{\alpha_{k-1}} \prec \cdots \prec x_{\alpha_2} \mathcal{L} \cdots \mathcal{L} x_{\alpha_1 + 1} \prec x_{\alpha_1} \mathcal{L} \cdots \mathcal{L} x_1), \]

then \( \min(x_n \leq \cdots \leq x_1) \) is as above.

**Definition 4.4.** Define a product on \( \hat{S} \) by

\[
(j_n \prec \cdots \prec j_1) \cdot (j'_n \prec \cdots \prec j'_1) = \min(j_n j'_{m} \leq \cdots \leq j_1 j'_m \prec \cdots \prec j'_1).
\]

It is easy to see that this product is associative. Some of the properties of \( \hat{S} \) are the following:

**Fact 4.5.** \( \hat{S} \) is generated by strings of length one.

**Definition 4.6.** Let \( \eta_S : \hat{S} \to S^1 \) be defined by

\[ \eta_S(j_n \prec \cdots \prec j_1) = j_n. \]

(\( \eta_S \) is called the 'canonical map'). Then

**Fact 4.7.** \( \eta_S \) is a \( \gamma \)-map.

**Fact 4.8.** Let \( \hat{s} = j_n \prec \cdots \prec j_1, \hat{t} = j'_m \prec \cdots \prec j'_1 \) be elements of \( \hat{S} \). Then

\[ \hat{s} \leq \hat{t} \text{ (in } \hat{S}) \iff m \leq n, j_i = j'_i \text{ (} i = 1, \ldots, m-1 \text{) and } j_m \mathcal{L} j'_m \text{ (in } S). \]

**Fact 4.9.**

\[ \hat{s} \mathcal{L} \hat{t} \text{ (in } \hat{S}) \iff m = n, j_i = j'_i \text{ (} i = 1, \ldots, m-1 \text{) and } j_n \mathcal{L} j'_n \text{ (in } S). \]

**Fact 4.10.** \( (\hat{S}, \hat{S}) \) is faithful.

**Proof.** 1 acts as left (but not right) identity:

\[ (1)(j_n \prec \cdots \prec j_1) = \min(1 \cdot j_n \leq j_n \prec \cdots \prec j_1) = (j_n \prec \cdots \prec j_1). \]
**Notation 4.11.** Every \( j_n < \cdots < j_1 \in S \) can be given a representation (which is *not* unique) as follows:

\[
j_n < \cdots < j_1 = (s_n \cdot \ldots \cdot s_1 < s_{n-1} \cdot \ldots \cdot s_1 < \cdots < s_1)
\]

with \( s_i \in S \) or inductively,

\[
j_1 = s_1, \quad j_{k+1} = s_{k+1} \cdot j_k \quad (k + 1 \leq n).
\]

**Fact 4.12.** Let \( s \in S \), \( k_0 \) be the first \( k \) such that \( j_k s \not\equiv \emptyset j_k \) (i.e., \( j_v s < \not\equiv j_v \) for \( e < k \)). Then \( j_{k_0 + \alpha} \in \mathcal{R} j_{k_0 + \alpha}, \quad \alpha = 1, \ldots, n - k_0 \).

**Proof.** By induction on \( x \);

\[
\hat{j}_{a+x} s = s_{a+x} j_{a+x-1} \cdot \mathcal{R} s_{a+x} j_{a+x-1} = j_{a+x+1}
\]
since \( j_{a+x-1} \cdot \mathcal{R} j_{a+x-1} \) by inductive assumption, and \( \mathcal{R} \) is a left congruence. \( \square \)

**Notation 4.13.** Let

\[
(s_n \ldots s_1 < \cdots < s_1) \cdot s = \min(s_n \ldots s_1 s \leq \cdots \leq s_1 s \leq s) = (s'_m \ldots s'_1 s \leq \cdots \leq s'_1 s)
\]

notationally. Then the \( s'_i \) can be chosen as follows: If \( s'_\beta \ldots s'_1 s = s_{k_0} \ldots s_1 s \), then

\[
s'_{\beta + \alpha} = s_{k_0 + \alpha}, \quad \alpha = 1, \ldots, n - k_0.
\]

(Follows from Fact 4.12).

**Note 4.14.** The notation \( j_n < \cdots < j_1 \) for an element in \( S \) is bad, but \( s_n \ldots s_1 < \cdots < s_2 s_1 \) is even worse. The reason is that an input sequence \( (s_1, \ldots, s_n) \), applied to the string (1), yields successively

\[
(1) \rightarrow s_1 \rightarrow \min(s_1 \leq 1) \rightarrow s_2 \rightarrow \min(s_1 s_2 \leq s_2 \leq 1) \rightarrow \cdots
\]

\[
\rightarrow s_n \rightarrow \min(s_1 \ldots s_{n-1} \leq s_2 \ldots s_n \leq \cdots \leq s_n \leq 1).
\]

(This is the famous 'reversal of time' connected with \( S \), see [3].) However, since it has been used in the literature, we are going to stick with it for the time being.

4.2. \((\hat{S}, \hat{S})\) is in a canonical wreath product

(See also [3] for an expanded version.)

**Note 4.15.** We now want to show that \((\hat{S}, \hat{S})\) can be embedded into a certain canonical wreath product. The embedding is done in three steps:

(i) We pass to an isomorphic semigroup \((\text{Code}(\hat{S}), \mathbb{Z}(S))\) using a 1-1 map 'Code' ('Zeiger-coding').

(ii) We define a coordinate map \( \text{Exp} : \text{Code}(\hat{S}) \xrightarrow{1-1} \Pi \) ('expansion into coord-
dinates, $\Pi = \mathcal{G}_N \times \cdots \times \mathcal{G}_1$ is roughly the Cartesian product of the $\mathcal{G}$-classes of $S$) and use it to construct an isomorphism

$$(\text{Code}(\mathcal{S}), Z(S)) \cong (\text{Exp}(\text{Code}(\mathcal{S})), \text{Exp}^{-1} Z(S) \text{Exp}).$$

(iii) We show that each element $a \in \text{Exp}^{-1} Z(S) \text{Exp}$ can be extended to a triangular map on all of $\Pi \supset \text{Exp}(\text{Code}(\mathcal{S}))$ with the correct component action, completing the embedding.

4.2.1. Zeiger-coding

The original idea of Zeiger-coding is due to Zeiger. Rhodes refined Zeiger's idea and adapted it to $\mathcal{S}$. We will present it here in an algebraic form, although it really is more a computer science idea on how to code nests $X_1 \supset X_2 \supset \cdots \supset X_n$.

**Definition 4.16.** Let $\tilde{S} := \{x_m \vartriangleleft \cdots \vartriangleleft x_1 \mid x_i \in S^1, m \geq 1\}$ be the set of all (strict) $\mathcal{G}$-chains on $S$.

**Definition 4.17.** Define a code-map $\text{Code} : \tilde{S} \xrightarrow{\text{1-1}} \tilde{S}$ as follows: For each $j = (a, g, b)$ let $t(j), \tilde{t}(j)$ be arbitrary fixed elements such that

$$(4.18) \quad j \cdot t(j) = (a, 1, 1) \quad \text{and} \quad (a, 1, 1) \tilde{t}(j) = j,$$

so

$$j \cdot t(j) \cdot \tilde{t}(j) = j.$$ 

Note that

$$(4.19) \quad j \not\triangleleft x \implies j t(j) = x \cdot t(x).$$

Let $\hat{j} = j_n \vartriangleleft \cdots \vartriangleleft j_1 \in \tilde{S}$. Define

$$(4.20) \quad x_1 := j_1, \quad x_n := j_n \cdot t(x_1) t(x_2) \cdots t(x_{n-1})$$

and

$$(4.21) \quad \text{Code}(\hat{j}) = (x_n \vartriangleleft \cdots \vartriangleleft x_1) \in \tilde{S}.$$ 

**Note 4.22.** To understand this definition better, take a representation of $\hat{j} = s_n \vartriangleleft \cdots \vartriangleleft s_1 \vartriangleleft s_2 s_1 \cdots s_1 (s_i \in S^1)$. Then

$$\text{Code}(\hat{j}) = \begin{cases} s_1 = x_1 \\ s_2 s_1 t(s_1) = x_2 \\ s_3 s_2 s_1 t(s_1) t(s_2 s_1 t(s_1)) = x_3 \\ \vdots \end{cases}.$$
or in general we have the recursion formulas

\[
\begin{align*}
J_1 & := s_1 \\
J_n & := s_n \cdot J_{n-1} \\
X_1 & := s_1 \\
X_n & := s_n \cdot X_{n-1} \cdot t(X_{n-1})
\end{align*}
\]

In words: \(J_n\) comes from \(J_{n-1}\) via hitting by \(s_n\) on the left, \(X_n\) comes from \(X_{n-1}\), first \textit{adjusted to standard element} \(X_{n-1} \cdot t(X_{n-1}) = (a, 1, 1)\), via hitting by \(s_n\) on the left. (This is the reason for Fact 4.29(ii) below.)

Clearly \(\text{Code}\) is 1-1 and maps into \(S\) (see Fact 4.26). In fact the inverse \(\text{Decode} : \text{Code}(S) \overset{1-1}{\text{onto}} S\) is given by

\begin{definition}
Let \((x_n, \ldots, x_1) \in \text{Code}(S)\). Define
\[
J_1 := x_1, \quad J_n := x_n \cdot \bar{t}(x_{n-1}) \cdots \bar{t}(x_1).
\]
Let \(\text{Decode}(x_n, \ldots, x_1) = (J_n < \cdots < J_1)\).
\end{definition}

\begin{fact}
\text{Decode} is the inverse of \text{Code}.
\end{fact}

\begin{proof}
By induction,
\[
x_k \cdot \bar{t}(x_{k-1}) \cdots \bar{t}(x_1) = s_k \cdot x_{k-1} \cdot t(x_{k-1}) \bar{t}(x_{k-1}) \cdots \bar{t}(x_1) \quad \text{(by (4.23))}
\]
\[
= s_k \cdot x_{k-1} \cdot \bar{t}(x_{k-2}) \cdots \bar{t}(x_1) \quad \text{(by (4.18))}
\]
\[
= s_k \cdot j_{k-1} \quad \text{(by induction hypothesis)}
\]
\[
= j_k.
\]
\end{proof}

\begin{fact}
With notation as above,
\begin{enumerate}
\item \(j_i \mathcal{R} x_i\) (1 \(\leq i \leq n\));
\item \(x_n \leq y \cdots \leq y x_1\) is strict.
\end{enumerate}
\end{fact}

\begin{proof}
(i) By induction
\[
x_k t(x_k) \mathcal{R} x_k \mathcal{R} j_k \quad \text{(by induction assumption)}
\]
so
\[
x_{k+1} = s_{k+1} x_k t(x_k) \mathcal{R} s_{k+1} j_k = j_{k+1}.
\]
(ii) \(x_n \leq y \cdots \leq y x_1\) by definition. (i) and \(j_n \leq \cdots \leq j_1\) show (ii).
\end{proof}

\begin{definition}
Using the 1-1 map \(\text{Code} : S \overset{1-1}{\text{onto}} \text{Code}(S) \leq S\), we get an isomorphism in the usual way, i.e., let
\[
Z(S) := \{\text{Decode} \circ \mathcal{S} \circ \text{Code} \mid \mathcal{S} \in S\}.
\]
\end{definition}
Then $Z(S)$ acts on $\text{Code}(\mathcal{S})$, and

$$(\mathcal{S}, \mathcal{S}) \equiv (\text{Code}(\mathcal{S}), Z(S)).$$

Notationally, the action of $Z(S)$ on $\text{Code}(\mathcal{S})$ can be described as follows (Notc: $Z(S)$ is generated by $\text{Decode} \circ (s) \circ \text{Code}, s \in S^1$):

**Notation 4.28.** Let $s \in S^1$ and let

$$(s_n \ldots s_1 < \cdots < s_1) \cdot s = (s'_m \ldots s'_1 s < \cdots < s'_1 s)$$

(see Notation 4.13). Let

$$x'_1 = s'_1 s, \quad x'_a = s'_a x'_{a-1} t(x'_{a-1}) \quad (a \leq m).$$

Then, notationally,

$$(x_n < \cdots < x_1) \cdot s = x'_m < \cdots < x'_1.$$

**Fact 4.29.** (i) If $j_n s <_S j_n$, then $x'_m < x_n$.

(ii) Assume $x_n \mathcal{R} x'_m$. Then

$$(x_{m+1} < \cdots < x_1) \cdot s = (x'_{m+1} < \cdots < x'_1)$$

with $x'_{m+1} = x_{n+1}$.

**Proof.** (i) $x'_m \mathcal{R} s_n \ldots s_1 s < s_n \ldots s_1 \mathcal{R} x_n$.

(ii) Let $\text{Decode}(x_{m+1} < \cdots < x_1) = (s_{m+1} s_n \ldots s_1 < \cdots < s_1)$. $x_n \mathcal{R} x'_m$ means $s_n \ldots s_1 \mathcal{R} s_n \ldots s_1$, and hence $(s_{m+1} s_n \ldots s_1 < \cdots < s_1) \cdot s = s_{m+1} \ldots s_1 s < s_{n+1} \ldots s_1 s < s'_m \ldots s'_1 s < \cdots < s'_1 s$ (with $s'_m \ldots s'_1 s = s_n \ldots s_1 s$, see Notation 4.13). Hence

$$x'_{m+1} = s_{n+1} x'_m t(x'_m)$$

(by (4.19))

$$= s_{n+1} x_n t(x_n)$$

(by (4.23)).

**Fact 4.30.** Let $(x_n < \cdots < x_1) \cdot s = (x'_m < \cdots < x'_1)$. Let $k_0$ be the first $k$ such that $x_k \mathcal{R} x'_0$ (where $(x_k < \cdots < x_1) \cdot s = (x'_m < \cdots < x'_1)$). Then $n - k = m' - \alpha$ and $x_{k+x} = x_{\alpha+x}$, $x = 1, \ldots, n-k$.

**Proof.** Follows from Fact 4.29(ii).

4.2.2. **Expansion into coordinates**

We now define the embedding of $(\mathcal{S}, \mathcal{S})$ into a wreath-product, using the following $\text{Exp}$ map (Expansion into coordinates).

**Definition 4.31.** In the following choose an arbitrary fixed ordering of the $\mathcal{J}$-classes
of \( S^1 \), compatible with the \( \mathcal{J} \)-order, i.e., enumerate the \( \mathcal{J} \)-classes of \( S^1 \) so that \( J_i \leq g J_k \) implies \( k < i \). Also for each \( B_i, A_i \) choose an ordering \( B_i = \{ b_1^{(i)}, \ldots, b_n^{(i)} \} \), \( A_i = \{ a_1^{(i)}, \ldots, a_m^{(i)} \} \). If the \( i \) is clear from the context, we denote \( b_i \) and \( a_i \) shortly by 1. So \( J_1 = G \), the group of units of \( S^1 \), and if \( N \) is the number of \( \mathcal{J} \)-classes of \( S^1 \), \( J_N = K(S^1) \) is the kernel of \( S \).

**Note 4.32.** It follows from the definition that if \( j \in J_k, s \in S^1, j \cdot s \notin J_k \), then \( j \cdot s \in J_i \) for some \( \mathcal{J} \)-class \( J_i \) with \( i > k \).

**Definition 4.33.** Give every \( \mathcal{J} \)-class \( J_i \) of \( S^1 \) an arbitrary fixed Rees-matrix representation, so that

\[
J_i^0 = \mathcal{M}^0(A_i, B_i, G_i, C_i) \quad (i = 1, \ldots, N)
\]

(with \( A_i \) the set of \( \mathcal{R} \)-classes of \( J_i \), \( B_i \) the set of \( \mathcal{P} \)-classes of \( J_i \), \( G_i \) the Schützenberger-group of \( J_i \), \( C_i \) the structure-matrix of \( J_i \)). Choose all \( B_i, A_i \) disjoint. Let \( b \) be a separate symbol, and define

**Definition 4.34.**

\[
B_i := B_i \cup \ldots \cup B_N, \quad G_i := (A_i \times G_i) \cup \{ b \},
\]

\[
\bar{B}_k := \bigcup_{N \leq i < k} B_i \cup \{ b \}, \quad \bar{G}_k := (A_k \times G_k) \cup \{ b \}.
\]

Call the \( \bar{B}_i \) "(generalized) B-coordinates" and \( \bar{G}_k \) "(generalized) G-coordinates".

\[
J_k := G_k \times B_k, \quad \Pi := J_N \times \cdots \times J_1.
\]

If necessary, we write \( \Pi(S) \) to indicate the semigroup whose \( \mathcal{J} \)-classes are used in \( J_i \).

**Notation 4.35.** We denote an element of \( \Pi \) by

\[
\vec{y} = (y_1, \ldots, y_N).
\]

Then \( y_i = y_i^{(a)} y_i^{(b)} \) with \( y_i^{(a)} \in \bar{G}_i, y_i^{(b)} \in \bar{B}_i \). (We will also denote initial substrings of an element of \( \Pi \) by \( \vec{y} \).)

Let \( x_i = (a_i, g_i, b_i) \in J_{n_i} \) \((i = 1, \ldots, k)\) in the above coordinatization and \( \mathcal{J} \)-class ordering. So \( n_1 < n_2 < \cdots < n_k \).

With every element \( \vec{x} = (x_n <_g \cdots <_g x_1) \in \vec{S} \) we want to associate an element of \( \Pi \), called the "coordinates of \( \vec{x} \)."

**Definition 4.36.** Let \( n_0 := 0, 1 \leq l \leq k \).

\[
C_\alpha(\vec{x}) = \begin{cases} 
(a_i, g_i, b_i) & \text{for } \alpha = n_i, \\
(b_i) & \text{for } n_{i-1} < \alpha < n_i, \\
(b, b) & \text{for } \alpha > n_k. 
\end{cases}
\]
Definition 4.37. Define $\text{Exp}(\tilde{x})$, the ‘expansion of $\tilde{x}$ (into coordinates)’ by
$$\text{Exp}(\tilde{x}) = C_N(\tilde{x}), \ldots, C_1(\tilde{x}).$$

Then $\text{Exp}$ is a map $\text{Exp} : \tilde{S} \rightarrow \Pi$, which is clearly 1-1. The ‘picture’ is:

\[ \begin{array}{ccc}
\mathcal{J} \times \mathcal{J}_1 & \text{put } b_1 \text{ in } \tilde{B}\text{-spot indicating that } b_1 \\
\text{\textit{J}-classes listed} & \text{is ‘coming up’; } b \text{ stands for blank.} \text{compatible with} \text{\textit{J}-order} & \\
\end{array} \]

\[ \begin{array}{c|c}
\mathcal{J}_1 & b_1 \\
\mathcal{J}_n & \vdots \\
\mathcal{J}_N & b \\
\end{array} \]

\[ \begin{array}{c}
b \hspace{1cm} \vdots \hspace{1cm} b_1 \hspace{1cm} \vdots \hspace{1cm} b_2 \hspace{1cm} \vdots \hspace{1cm} b_2 \\
\end{array} \]

\[ \begin{array}{c}
\mathcal{J}_n & a_1 \hspace{1cm} g_1 \\
\vdots & \vdots \\
\mathcal{J}_n & a_2 \hspace{1cm} g_2 \\
\mathcal{J}_n & \vdots \\
\mathcal{J}_n & b \\
\end{array} \]

Intuitively, $b$ stands for blank; $a_i, g_i, b_i \in J_{n_i}$ are put in the $J_{n_i}$-slot, and the ‘upcoming’ $b_i \in J_{n_i}$ are ‘slid-up’ or ‘advanced’ to the highest spot possible (i.e., not occupied by a ‘real’ $a_{i-1}g_{i-1}b_{i-1}$).

Remark 4.38.

\[ (\tilde{S}, \tilde{S}) \xrightarrow{\text{Code}} (\text{Code}(\tilde{S}), Z(\tilde{S})) \]

\[ (\Pi, \omega) \geq (\text{Exp}(\tilde{S}), \text{Exp}^{-1}\tilde{S}\text{Exp}) \xleftarrow{\text{Exp}^{-1}\text{Code}\text{Exp}} (\text{Exp Code}(\tilde{S}), \text{Exp}^{-1}Z(\tilde{S})\text{Exp}) \leq (\Pi, \omega) \]

$(\omega = \text{wreath product of } F_{\rho}(J_j)'s)$.

The map $\text{Code} = \text{Exp}^{-1}\text{Code Exp} : \text{Exp}(\tilde{S}) \rightarrow \text{Exp Code}(\tilde{S})$ is 1-1, onto (composition of 1-1 onto maps).

Moreover, it is easy to see from the definition of Code that Code is triangular (on Exp(\tilde{S})).
It is also easy to see that \( \text{Exp}^{-1} \hat{S} \text{Exp} \) is triangular, even without Advance \( b_i \); for meaning of Advance \( b_i \) see Remark 4.54(ii) below. Hence \( \text{Exp}^{-1} Z(S) \text{Exp} \) is triangular.

We will now prove that the component action of \( \alpha \in \text{Exp}^{-1} Z(S) \text{Exp} \) is remarkably simple, and that every element \( \alpha \in \text{Exp}^{-1} Z(S) \text{Exp} \) can be extended to all of \( \Pi \) in the obvious way, keeping the same simple component action.

### 4.2.3. Extending \( \text{Exp}^{-1} Z(S) \text{Exp} \) to all of \( \Pi \)

**Definition 4.39.** Putting Code and Exp together, we get a map

\[
\alpha = \text{Code} \circ \text{Exp} : \begin{array}{ccc}
\mathcal{L}\text{-chains} & \overset{1-1}{\text{into}} & \mathcal{I}\text{-chains} \\
\mathcal{S} & \overset{1-1}{\text{into}} & \mathcal{S} & \overset{1-1}{\text{into}} & \Pi
\end{array}
\]

Using \( \alpha, (\hat{S}, \hat{S}) \) can be transformed to a TSG on \( \alpha(\hat{S}) \subseteq \Pi \) by letting \( \hat{s} \in \hat{S} \) act as \( \alpha^{-1} \hat{s} \alpha \), i.e., let \( p \in \alpha(\hat{S}) \), \( p = \alpha(i) = \hat{i} \alpha \), define an action \( * \) of \( \hat{S} \) on \( \alpha(\hat{S}) \) by

\[
p \ast \hat{s} = p \alpha^{-1} \hat{s} \alpha = \hat{i} \alpha \alpha^{-1} \hat{s} \alpha = \hat{i} \hat{s} \alpha.
\]

So we get a TSG \( (\hat{S} \alpha, \hat{S}) \leq (\Pi, F_R(\Pi)) \) (via \( \alpha \)).

**Remark 4.40.** Of course, \( (\hat{S} \alpha, \hat{S}) \) may also be viewed as isomorphic to \((\text{Code}(\hat{S}), Z(S))\) via Exp. The crucial fact is then Fact 4.30, which translates into Remark 4.53 on \( \Pi \).

**Fact 4.41.** Let \( s \in S^1 \), \( \hat{s} = (s) \), the string of length one containing \( s \). (Remember Fact 4.5: \( \hat{s} \) is generated by \( s \).) Every \( \alpha^{-1} \hat{s} \alpha \) : \( \hat{S} \alpha \to \hat{S} \alpha \subseteq \Pi \) can be extended to a map \( \hat{s} : \Pi \overset{\text{into}}{\to} \Pi \) such that

- (i) \( \hat{s} \) restricted to \( \hat{S} \alpha = \alpha^{-1} \hat{s} \alpha \) (extension);
- (ii) \( \hat{s} \) is triangular on \( \Pi \);
- (iii) The component action of \( \hat{s} \) is as follows:
  - (a) on \( B \)-coordinates \( \hat{B}_i : (\hat{B}_i, 1^*) \),
  - (b) on \( G \)-coordinates \( \hat{G}_i : (\hat{G}_i, (1 \times G_i)^*) \) (where \( (1 \times G_i) \) is extended to \( b \) by \( b \cdot g = b \)).

Here the component action is permutation reset, with non-trivial groups only in \( G \)-coordinates.

**Note 4.42.** This gives another constructive proof of the prime decomposition theorem, by \( S \iff \hat{S} \leq (\Pi, (1 \times G_N)^* \circ 1^* \circ \ldots \circ (1 \times G_1)^* \circ 1^*) \).

**Proof of Fact 4.41.** We will construct \( \hat{s} \) by induction and simultaneously show (o)–(ii).
Notation 4.43. In the proof we will use as standard notation:

Representation

\[
\begin{array}{cccc}
\tilde{j} & s_1 & s_1' \\
< & < & < \\
\vdots & s_2 s_1 & \leftarrow \text{action } \circ s \\
< & < & < \\
\tilde{j}_n & s_n \cdots s_1 \\
< & < & < \\
\tilde{j}_{n+1} & s_{n+1} s_n \cdots s_1 & s_m' \cdots s_1' s &= s_{n+1} s_n \cdots s_1 s \\
\end{array}
\]

Note 4.44. Let \( \tilde{x} = \tilde{t}(x_1) \cdots \tilde{t}(x_1') \cdots \tilde{t}(x_m) \) for \( \tilde{x} = x_n < \cdots < x_1 \).

Definition 4.45. Let \( s(\tilde{x}) = \tilde{t}(x_n) \cdots \tilde{t}(x_1') \cdots \tilde{t}(x_m) \) for \( \tilde{x} = x_n < \cdots < x_1 \).

Note 4.46. Note 4.44 shows that

\[ b(x_m') = b(x_n) \cdot s(\tilde{x}), \quad g(x_m') = g(x_n) \cdot (b(x_n))[s(\tilde{x})]_G. \]

Definition 4.47. Call \( \bar{y} \in S\alpha \leq \Pi \) a 'legal' string. Also call any initial substring of a legal string legal. Call a string illegal if it is not legal.
Definition 4.48. If $\bar{y} = (y_k, \ldots, y_1)$ is legal, there exists a maximum $n$ such that $y_n^{(g)} \neq b$ ($n$ may be zero). Call this $n = n(\bar{y})$ the 'length of the entire part'. Let

$$(y_n, \ldots, y_1) := \text{the entire part of } \bar{y},$$

$$(y_k, \ldots, y_{n+1}) := \text{the rest (of } \bar{y}).$$

Fact 4.49. There exists a unique $j_n < \cdots < j_1 \in \S$ such that $\text{Exp Code}(j_n < \cdots < j_1) = (y_n, \ldots, y_1)$. 'The rest' is either $y_{n+k} = (b, b)$ for $x = 1, \ldots, k-m$ or $y_{n+k} = (b, b)$ with $b \in B_{k+a}$ for some $\alpha \geq 1$.

Proof. Follows from definition of Exp. □

We now proceed with the definition of $\S$ by induction:

$k = 1$. Remember $J_1^0 = (\M^0(1, 1, G, 1))$ with $G$ the group of units. Let $s = (a_0, g_0, b_0)$. Then $\S(y_1^{(b)}) = b_0$.

For $s \in G$, set

$$\S(y_1^{(g)}, y_1^{(b)}) = b.$$

For $s \in G$, set

$$\S(y_1^{(g)}, y_1^{(b)}) = \begin{cases} b & \text{if } y_1^{(b)} \neq 1 \in B_1, \\ y_1^{(g)}(y_1^{(b)})s & \text{if } y_1^{(b)} = 1. \end{cases}$$

This clearly satisfies (i) and (ii), and since $(s_n < \cdots < s_1)s = \min(s_n s \leq \cdots \leq s_1 s \leq s) = (s_n s < \cdots < s_1 s)$ with $s_k s \leq s$, $\S$ extends $\alpha^{-1}s\alpha$, i.e. (o).

Induction assumption 4.50. Assume $\S$ has been defined for $1 \leq i \leq k$, satisfying (o)–(ii) on $J_i \times \cdots \times J_1$.

Case 1. $\bar{y} = (y_k, \ldots, y_1)$ illegal. Then let $\S(y_{k+1}, \ldots, y_1) = b$. This clearly satisfies (i) and (ii), and (o) automatically.

Case 2. $\bar{y} = (y_k, \ldots, y_1)$ is legal. Assume $(y_n, \ldots, y_1)$, the entire part of $\bar{y}$, is $\text{Exp Code}(j_n < \cdots < j_1) = \text{Exp}(x_n < \cdots < x_1)$ (see Notation 4.43).

(i) Assume further $j \in R_j$ for some $1 \leq i \leq n$. Then Fact 4.30 shows that $\alpha^{-1}s\alpha$ fixes all coordinates after $n_i$ ($j_i \in J_{n_i}$), so we may extend it to $\S(y_{k+1}, \ldots, y_1) = y_{k+1}$ and satisfy (o)–(ii).

(ii) Assume now that $j_n s \not< R_j$.

Notation 4.51. Let

$$L(k) := \max \{ j_l : s \in J_k \},$$

$L(k) = 0$ if no such $l$ exist.
\[(L(k) = 0 \text{ means no } j_i \text{ above } J_k \text{ falls into } J_k \text{ under } s; \ L(k) = l_0 \text{ means some } j_i \text{ above } J_k \text{ falls into } J_k, \text{ and among all such } j_i, j_{l_0} \text{ is the last one to do so.)}\]

Let

\[M(k) := \min \text{ such that } L(m) \neq 0, \quad m \geq k\]

\[M(k) = 0 \quad \text{if no such } m \text{ exists.}\]

\[(M(k) = 0 \text{ means nothing falls into } J_k \text{ from above, and nothing falls by } J_k, \text{ landing somewhere below. } M(k) = k \text{ means something falls into } J_k. \ M(k) = m \text{ means something falls into a } J\text{-class below } k, \text{ i.e., ‘passes } k \text{ by’.)}\]

(a) \(M(k + 1) \neq 0\) (i.e., something falls into or past \(J_{k+1}\)). Put

\[s(y_{k+1}^{(k)}, \bar{y}) = b(j_{L(k+1)}, s \cdot t(x'_1) \ldots t(x'_{\alpha - 1}))\]

with \(\alpha\) such that \(x'_\alpha \in J_{k+1}\) and

(a) If \(y_{k+1}^{(b)} \in B_{k+1}\) or \(j_{n+1} \in J_{n+1}\), put

\[s(y_{k+1}^{(a)}, y_{k+1}^{(b)}, \bar{y}) = s(j_{L(k+1)} \cdot s \cdot t(x'_1) \ldots t(x'_{\alpha - 1}))\]

with \(\alpha\) such that \(x'_\alpha \in J_{k+1}\).

(b) If \(y_{k+1}^{(b)} \in B_{k+1}\), and \(J_{n+1} \cap J_{n+1}\), let \(y_{k+1}^{(b)} = (a_{k+1}, s_{k+1})\) and put

\[s(y_{k+1}^{(a)}, y_{k+1}^{(b)}, \bar{y}) = (a_{k+1}, s_{k+1} \cdot (y_{k+1}^{(b)})[s(x_i)])_G,\]

see Note 4.46.

This clearly satisfies (a), (i) and (ii).

(b) \(M(k + 1) = 0\) (i.e., nothing falls into or past \(J_{k+1}\)).

(a) If \(\text{Rest}(\bar{y})\) is all \((b, b)\)'s (i.e., \(y_{m+x} = (b, b)\)), let

\[s(y_{k+1}^{(b)}, \bar{y}) = b, \quad s(y_{k+1}^{(g)}, y_{k+1}^{(b)}, \bar{y}) = b.\]

(b) If \(\text{Rest}(\bar{y})\) is all \((b_{n+1}, b)\) (i.e., \(y_{m+x} = (b_{n+1}, b)\)), we have two cases:

(i) \(b_{n+1} \neq J_{k+1}\). Then

\[s(y_{k+1}^{(b)}, \bar{y}) = b_{n+1} \cdot s(\bar{x}) \quad \text{and} \quad s(y_{k+1}^{(g)}, y_{k+1}^{(b)}, \bar{y}) = b;\]

(ii) \(b_{n+1} \in J_{k+1}\). Then

\[s(y_{k+1}^{(b)}, \bar{y}) = b_{n+1} \cdot s(\bar{x}).\]

(Notice this is reset.)

Let \(y_{k+1}^{(g)} = (a_{k+1}, s_{k+1})\) and let \(y_{k+1}^{(b)}, \bar{y}\) be legal. Then put \(s(y_{k+1}^{(g)}, y_{k+1}^{(b)}, \bar{y}) = (a_{k+1}, s_{k+1} \cdot y_{k+1}^{(b)})[s(x_i)]_G\) (and \(s(b, y_{k+1}^{(b)}, \bar{y}) = b\)). Also \(s(y_{k+1}^{(g)}, y_{k+1}^{(b)}, x) = y_{k+1}^{(g)}\) for \(y_{k+1}^{(b)}, \bar{y}\) illegal. Again this satisfies (a)–(ii).

If \(\text{Rest}(\bar{y}) = 0\), then \(j_{n} \in J_{k}\), so since \(j_{n} \cdot s \in J_{l}\) by assumption, \(j_{l} \cdot s \in J_{l}\) with \(l \geq k + 1\). Hence \(M(k + 1) \neq 0\), which is case (a).
So for each generator $\mathcal{S} \in \mathcal{S}$ we have defined $\mathcal{S}: \Pi \rightarrow \Pi$ in triangular form with indicated component action, extending $\alpha^{-1}\mathcal{S}\alpha: \mathcal{S}\alpha \rightarrow \mathcal{S}\alpha$. Letting $Y := \mathcal{S}\alpha \subseteq \Pi$, $\theta: Y \rightarrow \mathcal{S}$, $\theta = \alpha^{-1}$, then $(\mathcal{S}\theta)(\mathcal{S}) = (\mathcal{S}\theta)(\mathcal{S})$ for $\mathcal{S} \in Y$, $\mathcal{S} \in \mathcal{S}$ generator and so by standard proof-scheme (see [7])

$$(\mathcal{S}, \mathcal{S}) \in (\Pi, (1 \times G_N)^* \circ \cdots \circ (1 \times G_1)^* \circ 1^*)$$

and, in fact,

$$(\mathcal{S}, \mathcal{S}) \leq (\Pi, (1 \times G_N)^* \circ 1^* \cdots \circ (1 \times G_1)^* \circ 1^*).$$

**Remark 4.53.** Fact 4.30 shows that $\alpha^{-1}\mathcal{S}\alpha: \mathcal{S}\alpha \rightarrow \mathcal{S}\alpha \subseteq \Pi$ has the 'Zeiger-property': "once $\alpha^{-1}\mathcal{S}\alpha$ permutes, it is identity further down".

We do not need this property in the proof of Chapter 5, but using additional care in the definition of the extension, $\mathcal{S}$ (as in Fact 4.41) can be made to have the additional property:

(iii) (Zeiger-property) Let $(y_n, \ldots, y_1) \in \Pi$, $\mathcal{S} \in \mathcal{S}$. Then there exists a $k_0$ such that

(i) $x \rightarrow s(x, y_{k_0}, \ldots, y_1)$ is a reset for $a < k_0$;

(ii) $x \rightarrow s(x, y_{k_0}, \ldots, y_{k_0-1}, \ldots, y_1)$ is a permutation;

(iii) $x \rightarrow s(x, y_{k_0-1}, \ldots, y_1)$ is identity for $1 \leq a \leq N - k_0$.

For details see [3].

**Remark 4.54 (Discussion of the proof of Fact 4.41).** The only interesting ideas in the proof can be shortly described as this:

(i) Fact 4.30 shows that the component action of $\alpha^{-1}\mathcal{S}\alpha$ will be reset for a while, then action of $\mathcal{S}$ on $J_k$ (which is in triangular form by the familiar $(a, g, b) \mathcal{S} = (a, g(b)\mathcal{S}, b \cdot s)$ with component action $(1 \times G_k)$ on the $A \times G$-coordinate), and identity thereafter. It is clear how to extend to $\mathcal{S}$ within given component action. This leaves $(B, RLM(S))$ as component action on $B_k$.

(ii) The 'Advance' $b_i$ are put in to reduce the component action on $B_k$. The idea is that if $j_k s < j_{k-1}$ and $j_{k-1} s < j_{k-1}$, then either

(a) $n_{k-1} = n_k - 1$, and so $j_{k-1} \cdot s$ falls into $J_n$, so that $b(j_k s) = b(j_{k-1} \cdot s)$ can be read off from above, or

(b) $n_k - n_{k-1} \geq 2$. Then in the $n_{k-1} + 1$-spot we will have $(b(j_k), b)$, so again we can read off $b(j_{k-1}) \cdot s$ from above (and in the $n_{k-1} + 1$-spot, $b(j_k) \cdot s$ gets reset to $b(j_{k-1}) \cdot s$, since $j_{k-1} s < j_{k-1}$).

5. $C^\omega(S) = P(S)$

We now have the tools to prove $C^\omega(S) = P(S)$. We show this by constructing a relation $S \rightarrow CP(S)$ such that $C(R) \leq C^\omega(S)$ (see Proof-scheme 2.10).
In Subsection 5.1 we define \( \hat{H}^\omega \), an adaptation of \( H^\omega \) to \( (C^\omega(S))^\circ \), and show that \( \alpha^{-1}\hat{H}^\omega\alpha \) (see Definition 4.39) can be extended to \( \mathcal{H} \), a triangular map on all of \( \Pi \) with very restrictive component action.

In Subsection 5.2 we combine the action of \( (C^\omega(S))^\circ \) on \( \Pi \) with \( \mathcal{H} \). The component action of \( \mathcal{H} \) is such that the combined component action of \( \mathcal{H} \)'s \( \mathcal{H} \) is aperiodic in every component (i.e., \( \mathcal{H} \) 'gets rid of the groups'). As a result we can construct the desired relation \( R \).

5.1. \( \hat{H}^\omega \) and \( \mathcal{H} \)

**Definition 5.1.** Let \( E(A) \) be such that \( H^\omega(A) = A \cdot E(A) \) (see Fact 3.12(ii)).

**Definition 5.2.** Use the map \( H^\omega : C^\omega(S) \to C^\omega(S) \) to define \( \hat{H}^\omega : (C^\omega(S))^\circ \to (C^\omega(S))^\circ \) as follows:

1. Extend \( H^\omega \) to \( C^\omega(S) \) by \( H^\omega(1) = 1 \) (if \( C^\omega(S) \) is not a monoid already. Note: If we use \( S \) throughout, then \( \{1\} \) is the identity on \( C^\omega(S) \)).
2. Let \( \hat{A} \in (C^\omega(S))^\circ \), \( \hat{A} = j_\alpha < \cdots < j_1 \) (\( j_i \in C^\omega(S) \)) and \( A_\alpha < \cdots < A_1 \) be some representation.

5.3. Define

\[
E_1 := E(j_1), \quad E_k := E(j_k \ldots E_{k-1}), \quad k \leq n.
\]

5.4. Define

\[
j'_1 = j_1E_1, \quad j'_k = j_kE_1 \ldots E_{k-1}E_k.
\]

Then

\[
j'_n \leq_\mathcal{H} \cdots \leq_\mathcal{H} j'_1
\]

since

\[
j'_k = j_k \ldots E_{k-1}E_k = A_k(j_{k-1} \ldots E_{k-1})E_k = A_kj'_{k-1}E_k
\]

and

\[
j'_{k-1} \geq_\mathcal{H} A_kj'_{k-1} \geq_\mathcal{H} H^\omega(A_kj'_{k-1}) = H^\omega(A_kj_{k-1} \ldots E_{k-1})
\]

and so let

\[
(5.5) \quad \hat{H}^\omega(j_n < \cdots < j_1) := \min(j'_n \leq_\mathcal{H} \cdots \leq_\mathcal{H} j'_1).
\]
The picture is this:

\[
\begin{array}{c}
\begin{array}{c}
A_1 \\
\uparrow \leq_{\mathcal{R}} \\
\downarrow <_{\mathcal{R}} \\
A_2
\end{array}
\quad \begin{array}{c}
A_1 E_1 \\
= H^\omega(A_1)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A_2 A_1 E_1 \\
\leq_{\mathcal{R}} \\
\geq_{\mathcal{R}} \\
A_3
\end{array}
\quad \begin{array}{c}
A_2 A_1 E_1 E_2 \\
= H^\omega(A_2 A_1 E_1) = H^\omega(A_2 H^\omega(A_1))
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A_3 A_2 A_1 E_1 E_2 \\
\leq_{\mathcal{R}} \\
\geq_{\mathcal{R}}
\end{array}
\quad \begin{array}{c}
A_3 A_2 A_1 E_1 E_2 E_3
\end{array}
\end{array}
\end{array}
\]

and take the min over the boxed expressions.

Note 5.7.

\[ j'_k = j_k E_1 \ldots E_{k-1} E_k = A_k(j_{k-1} E_1 \ldots E_{k-1}) E_k = A_k j'_{k-1} E(A_k j'_{k-1}) \]

would give another recursion formula.

Fact 5.8. (i) \( j_k \geq_{\mathcal{R}} j'_k \), \( k = 1, \ldots, n \).

(ii) \( j_k \subseteq j'_k \), \( k = 1, \ldots, n \).

(iii) If \( j_k \in J \) with \( J \) not \( H \)-trivial, then \( j'_k <_{\mathcal{R}} j_k \).

(iv) If \( j_k \mathcal{R} j'_k \) and \( j_{k+1} = A_{k+1} j_k \in J \) with \( J \) \( H \)-trivial, then \( j'_{k+1} = A_{k+1} j'_{k} \).

Proof. (i) trivial.

(ii) \( A_1 \subseteq A_1 E_1 = H^\omega(A_1) \) and if \( j_k \subseteq j'_k \), then \( j_{k+1} = A_{k+1} j_k \subseteq A_{k+1} j'_{k} \subseteq A_{k+1} j'_k \).

E(A_{k+1} j'_k) = j'_{k+1}.

(iii) \( j_k \geq_{\mathcal{R}} j'_k \) by (i), and if \( j_k \mathcal{R} j'_k \), then \( H^\omega(j'_k) <_{\mathcal{R}} j'_k \) (see Definition 5.2). Contradiction.

(iv) \( j'_{k+1} = A_{k+1} j'_k E(A_{k+1} j'_k) \). But \( j_k \mathcal{R} j_k \) implies \( A_{k+1} j'_k \mathcal{R} A_{k+1} j_k \in J \), so \( E(A_{k+1} j'_k) = 1 \) is trivial. \( \square \)
Note 5.9. $A_2 A_1 <_G A_1$ does not imply $H^\omega(A_2 H^\omega(A_1)) < H^\omega(A_1)$ (so taking a min is really necessary). As an example look at Example 1, Section 3. If $1 \leq G_0 \leq G$ is a nontrivial subgroup, $G_0 <_G \{1\}$. But $H^\omega(\{1\}) = G$, and $H^\omega(G_0 \cdot H^\omega(\{1\})) = H^\omega(G_0 \cdot G) = H^\omega(G) = G$ as well.

Key Fact 5.10. $\eta(\hat{A}) \subseteq \eta(\hat{H^\omega} (\hat{A}))$.

Proof. Fact 5.8(ii) for $k = n$. □

Fact 5.11. $(\hat{H^\omega})^2 = \hat{H^\omega}$.

Proof. Follows from $(H^\omega)^2 = H^\omega$ and $\hat{j}_k^i = H^\omega(A_k j_{k-1}^i)$. □

Definition 5.12. Let $H-T(\hat{S}) := \{(s_n < \cdots < s_1) \mid s_i \in J_k, \kappa\text{-trivial}\}$.

Fact 5.13. Then $\hat{H^\omega} : (C^\omega(S)) \rightarrow H-T((C^\omega(S))\hat{^\omega})$ and $\hat{H^\omega}$ is identity on $H-T((C^\omega(S))\hat{^\omega})$.

Proof. Trivial. □

Note 5.14. We now want to show that $\alpha^{-1}\hat{H^\omega} \alpha$ can be extended to $\kappa\eta$ on all of $\Pi$ with certain restrictive component action. Just as the same fact for $\alpha^{-1}s\alpha$ is a consequence of $(a,g,b)s = (a,g(b)s_g,b \cdot s)$ being in triangular form, Fact 5.15 gives rise to $\alpha^{-1}\hat{H^\omega} \alpha$ being triangular.

Fact 5.15. Let $J$ be a $\mathcal{J}$-class of $C^\omega(S)$. Then there exist functions $\phi, \psi$ such that $H^\omega(a,g,b) = (\phi(a),1,\psi(b))$ for all $(a,g,b) \in J$.

Proof. Follows from Fact 3.12. Also note that $\psi(b)$ determines the $\mathcal{J}$-class $J'$ of $H^\omega(a,g,b)$, and hence the group-coordinate $1 \in J'$. □

We can therefore write shortly

Definition 5.16.

$$H^\omega(b) := b(H^\omega(a,g,b))$$

which is independent of $a,g$ by Fact 5.15.
\[ H^\omega(a) := a(H^\omega(a, g, b)), \]
\[ H^\omega_{(b)}(g) := 1 \in J' \quad \text{where } J' \ni H^\omega(b). \]

**Definition 5.17.** Let \( \text{Down}(B_i) \leq F_R(B_i) \) be defined as follows:
\[
\text{Down}(\vec{B}_i) := \{ f : \vec{B}_i \to \vec{B}_i | f \text{ satisfies (i) and (ii) below} \}
\]
with
(i) If \( j \geq i, |G_j| \geq 2 \), then \( f : B_j \to B_k \) with \( k > j \) and \( B_k \) \( H \)-trivial;
(ii) If \( j \geq i, |G_j| = 1 \), then \( f : B_j \to \vec{B}_j \) and either
   (a) \( B_j \) can be written disjointly as \( B_j = B_j(0) \cup B_j(1) \) and \( f : B_j(0) \to B_k \) with \( k > j \)
   and \( B_k \) \( H \)-trivial. \( f \) is identity on \( B_j(1) \); or
   (b) \( f \) is a reset to \( \vec{b}_j \in \vec{B}_j \);
(iii) \( f(b) = b \).

In words: On the non-\( H \)-trivial \( B_j \)'s in \( \vec{B}_i \), \( f \) maps into \( B_k \) 'further down'; on the \( H \)-trivial \( B_j \)'s in \( \vec{B}_i \), \( f \) is either a reset or \( f \) is partial identity and maps 'further down' where it is not identity.

**Fact 5.18.** \( \text{Down}(\vec{B}_i) \) is aperiodic.

**Proof.** Obvious. \( \Box \)

**Fact 5.19.** Let \( H^\omega \) act on \( \vec{B}_i \) by Definition 5.16 (and \( H^\omega(b) = b \)). Then \( H^\omega \in \text{Down}(\vec{B}_i) \).

**Proof.** (i) follows from Fact 3.12.
(ii) If \( B_j \) \( H \)-trivial, \( H^\omega \) is identity on \( B_j \) by Fact 3.12. \( \Box \)

For definition of \( \alpha \) see Definition 4.39.

**Fact 5.20.** \( \alpha^{-1} H^\omega \alpha \) can be extended to a triangular map \( \mathcal{K} \) on \( \Pi J \), such that the component action of \( \mathcal{K} \) is as follows:
(i) On \( B \)-coordinates (\( \vec{B}_i, \text{Down}(\vec{B}_i)^* \));
(ii) On \( G \)-coordinates
   (a) If \( |G_i| \geq 2 \), \( (\vec{G}_i, \text{Reset}(b)) \),
   (b) If \( |G_i| = 1 \), \( (\vec{G}_i, 1^*) \).

We will use in the proof the following standard notation:

**Notation 5.21.**
Notice that

\[
x_{n+1} \xrightarrow{\text{Decode}} x_{n+1} \tilde{t}(x_n) \ldots \tilde{t}(x_1) = j_{n+1} \xrightarrow{\hat{H}^0} j'_m = j_{n+1} \cdot E_1 \ldots E_{n+1} \]

\[
= x_{n+1} \tilde{t}(x_n) \ldots \tilde{t}(x_1) E_1 \ldots E_{n+1} \]

\[
\xrightarrow{\text{Code}} x'_m = j'_m \cdot t(x'_1) \ldots t(x'_{m-1}),
\]

so all in all

\[
x_{n+1} \rightarrow x_{n+1} \tilde{t}(x_n) \ldots \tilde{t}(x_1) E_1 \ldots E_{n+1} t(x'_1) \ldots t(x'_{m-1})
\]

(compare with Note 4.44).

**Proof.** By induction.

(i) \(J_1\) = \(\mathcal{J}\)-class of the identity of \(C^\omega(S)^1\). Then let \(\mathcal{H}(y_1^{(b)}) = H^\omega(y_1^{(b)})\).

(ii) Let

\[
\mathcal{H}(y_1^{(b)}, y_1^{(b)}):= \begin{cases} 
  b & \text{if } y_1^{(b)} \in \overline{B}_2, \\
  H_2^\omega(y_1^{(b)}) \text{ with } \overline{b} = y_1^{(b)} & \text{if } y_1^{(b)} \in B_1.
\end{cases}
\]
This extends $\alpha^{-1}\widehat{H}\omega\alpha$ and satisfies (i) and (ii).

Now assume $\mathcal{H}$ has been defined on $J_k \times \cdots \times J_1$, extending $\alpha^{-1}\widehat{H}\omega\alpha$, with component action as given.

Case 1. $(y_k, \ldots, y_1) \in J_k \times \cdots \times J_1$ is illegal (see Definition 4.47). Let $\mathcal{H}(y_{k+1}, \ldots, y_1) = \text{Reset}(b, b)$. This satisfies (i) and (ii).

Case 2. $\vec{y} = (y_k, \ldots, y_1)$ is legal.

As before, define

$$L(k) := \begin{cases} \max\{j_i \in J_k\}, & \text{if no such } l \text{ exists.} \\ 0 & \text{if no such } l \text{ exists.} \end{cases}$$

(5.22)

$$M(k) := \begin{cases} \min\{L(k) \neq 0\}, & \text{if no such } m \text{ exists.} \\ 0 & \text{if no such } m \text{ exists.} \end{cases}$$

(i) Assume $M(k+1) \neq 0$ (i.e., under $\widehat{H}\omega$ something falls by $J_{k+1}$ from above).

$$\mathcal{H}(y_{k+1}, \vec{y}) = b(j_{k+1}(t(x_1') \ldots t(x_{m-1})))$$

with $\alpha$ such that $x_{m}' \in J_{k+1}$

$$\mathcal{H}(y_{k+1}, y_{k+1}', \vec{y}) = \begin{cases} (1) \text{ If } x_{n+1} \mathcal{R} x_{n+1} \bar{t}(x_{n}) \ldots \bar{t}(x_{n})E_1 \ldots E_{n+1}t(x_1') \ldots t(x_{m-1}) & = y_{k+1}' \text{ (note that in this case } J_{k+1} \text{ is aperiodic and } A \text{-coordinate does not change);} \\ (2) \text{ If } x_{n+1} >_{\mathcal{R}} x_{n+1}, & = \begin{cases} (a) \text{ if } M(k+1) = k+1, \\ (b) \text{ if } M(k+1) > k+1. \end{cases} \end{cases}$$

So component action is either identity or reset. This extends $\alpha^{-1}\widehat{H}\omega\alpha$, satisfying (i) and (ii).

(ii) Now assume $M(k+1) = 0$, i.e., nothing falls from above by or into $J_{k+1}$.

Case $\alpha$. $\text{Rest}(\vec{y})$ has $y_{m+x} = (b, b)$. Then let $\mathcal{H}(y_{k+1}, \vec{y}) = (b, b)$.

Case $\beta$. $\text{Rest}(\vec{y})$ has $y_{m+x} = (b, b)$. Then $b = b(x_{n+1})$ in Notation 4.43. So let

$$\mathcal{H}(y_{k+1}'\beta, \vec{y}) = H(b \cdot \bar{t}(x_{n}) \ldots \bar{t}(x_{n})E_1 \ldots E_\bar{t}(x_1') \ldots t(x_{m-1})).$$

(This is a reset.)

$$\mathcal{H}(y_{k+1}', y_{k+1}', \vec{y}) = \begin{cases} b & \text{if } b \notin J_{k+1} \text{ or } y_{k+1}', \vec{y} \text{ illegal or } |G_{k+1}| \geq 2, \\ y_{k+1}' & \text{if } b = y_{k+1}' \in J_{k+1} \text{ and } |G_{k+1}| = 1. \end{cases}$$

Again this extends $\alpha^{-1}\widehat{H}\omega\alpha$, satisfying (i) and (ii).
Case $y$ (critical case). Assume $\text{Rest} = 0$, and also $j_n R j_{m-1} \in J_k$ (otherwise we are in (i)). Then $J_k$ is $H$-trivial.

For $y_{k+1}^{(b)} \in B_j$ not $H$-trivial, put

$$\mathcal{H}(y_{k+1}^{(b)}, \vec{y}) = H(y_{k+1}^{(b)} i(x_n) \ldots i(x_1) E_1 \ldots E_n t(x'_1) \ldots t(x'_m))$$

which is in $B_j \subset B_j$.

If $y_{k+1}^{(b)} \in B_j$, $j \geq k + 1$, $B_j$ $H$-trivial, then by 4.23

$$j'_m = A_{n+1} j_{m-1}$$

so

$$x'_m = A_{n+1} x_{m-1} t(x'_{m-1}) = A_{n+1} x_n t(x_n) = x_{n+1}.$$ Hence we can put

$$\mathcal{H}(y_{k+1}^{(b)}, \vec{y}) = y_{k+1}^{(b)}$$

for $y_{k+1}^{(b)} \in B_j$ $H$-trivial $\cup \{b\}$ and

$$\mathcal{H}(y_{k+1}^{(g)}, y_{k+1}^{(b)}, \vec{y}) = \begin{cases} y_{k+1}^{(g)} & \text{if } y_{k+1}^{(b)} \in J_{k+1} \text{ $H$-trivial,} \\ b & \text{otherwise.} \end{cases}$$

This extends $\alpha^{-1} \overrightarrow{H^\omega}$, and component action on $\overline{B}_{k+1}$ is partial identity, rest down and on $G_{k+1}$ identity or reset.

So we have defined $\mathcal{H}$ with indicated component action in triangular form on $\Pi$, extending $\alpha^{-1} \overrightarrow{H^\omega}$.

5.2. CP(S)

Define a TSG $\text{CP}(S) = (X, C)$ (′Computes Pointlikes′).

**Definition 5.23.**

$$X := (C^\omega(S))^\times,$$

$$C := \{s \overrightarrow{H^\omega} \ldots \overrightarrow{H^\omega} s_n \overrightarrow{H^\omega} | s \in S, n \geq 1\}$$

(identifying $s \in S$ with $\langle \{s\} \rangle \in (C^\omega(S))^\times$).

**Note.** We will use $\text{CP}(S)$ also for the SG $C$ if it is clear from the context.

**Fact 5.24.** $\text{CP}(S)$ is aperiodic.

**Proof.** Using the map $\alpha : (C^\omega(S))^\times \xrightarrow{\text{Code}_{\text{into}}} C^\omega(S) \xrightarrow{\text{Exp}_{\text{into}}} \Pi$ we get an isomorphism

$$(X, C) \cong (X \alpha, \{\alpha^{-1} c \alpha | c \in C\}).$$
Now
\[ \alpha^{-1}s_1 ... \hat{H}^\omega \alpha = (\alpha^{-1}s_1 \alpha) \alpha^{-1} ... (\alpha^{-1}\hat{H}^\omega \alpha) \]
and using the extensions \( \mathcal{H} \) for \( \alpha^{-1}\hat{H}^\omega \alpha \) and \( \delta \) for \( \alpha^{-1}s \alpha \) we get
\[
(X, C) \equiv (\Pi, \{\delta_1 ... \mathcal{H}\delta_n \mathcal{H} | \delta_i \in S\})
\]
\[
\leq (\Pi, F_\mathcal{R} (\mathcal{G}_k) \circ ... \circ F_\mathcal{R} (\mathcal{G}_1) \circ F_\mathcal{R} (B_1)).
\]

Let \( \text{CA}(Y_k) \) denote the component action of \( \mathcal{H} \) at \( Y_k \), \( \text{CA}(Y_k) \) denote the component action of \( \delta \) at \( Y_k \). Then the component action of \( C \) at \( Y_k \) is given by
\[
C(Y_k) = (\text{CA}(Y_k)) \text{CA}(Y_k)).
\]

We claim this to be aperiodic for any \( Y_k \).

Case \( a \). For general group coordinates \( Y^{(g)} \) we have
(i) If \( |G_k| \geq 2 \), then \( \text{CA}(Y_k) = \{\text{Reset}(b)\} \), so \( C(Y_k) = \{\text{Reset}(b)\} \).
(ii) If \( |G_k| = 1 \), \( \text{CA}(Y_k) = \text{CA}(Y_k) = 1^* \).

Case \( b \). For general \( \mathcal{B} \)-coordinates \( Y^{(b)} \) we have
\[
\text{CA}(Y_k) = \text{Down}(B_k), \quad \text{CA}(Y_k) = 1^*
\]
so
\[
C(Y_k) \leq \text{Down}(B_k) \cup \{1^* \}.
\]

In any case \( C(Y_k) \) is aperiodic, hence \( (\Pi, \{\delta_1 ... \mathcal{H}\delta_n \mathcal{H} | \delta_i \in S\}) \equiv (X, C) \) is aperiodic.

**Definition 5.25.** Define a TSG-relation \((S^{(i)}, S) \stackrel{R=\theta, \phi}{\rightarrow} (X, C)\) by
\[
\phi(s) := \{s_1 ... \hat{H}^\omega s_n \hat{H}^\omega | s_1 ... s_n = s\},
\]
\[
\theta(s) := \{A_n < ... < A_1 | s \in A_n, A_i \in C^\omega(S)\} \quad \text{for } s \in S,
\]
\[
\theta(1) := (1) \quad \text{(string of length one, 1 = identity of } C^\omega(S) \text{ if } S \text{ does not have an identity, and hence } C^\omega(S) \text{ does not have one either)}
\]

**Fact 5.26.** (i) \( \phi \) is onto.
(ii) \( \theta \) is onto, \( \theta(s) \neq \emptyset \).
(iii) \( \phi \) is an SG-relation.
(iv) \( (\theta, \phi) \) is a TSG-relation.

**Proof.** (i) \( s_1 ... \hat{H}^\omega s_n \hat{H}^\omega \in \phi(s_1 ... s_n) \).
(ii) Obvious.
(iii)
\[
\phi(s) \cdot \phi(t) = \{s_1 ... \hat{H}^\omega s_n \hat{H}^\omega t_1 ... \hat{H}^\omega t_m \hat{H}^\omega | s_1 ... s_n = s, t_1 ... t_m = t\}
\]
\[
\subseteq \phi(st) \quad \text{since } s_1 ... s_n t_1 ... t_m = st.
(iv) \[ \theta(s) \cdot \phi(t) = \{ (A_n \cdots < A_1) t_1 \cdots \hat{H}^\omega t_m \hat{H}^\omega \mid s \in A_n, \ t_1 \cdots t_n = t \}. \]

But \( s \cdot t_1 \cdots t_n = st \in \eta(\alpha) \) for all \( \alpha \in \theta(s) \cdot \phi(t) \), so \( \theta(s) \cdot \phi(t) \subseteq \theta(st) \). For \( 1, \theta(1) \cdot \phi(t) = \{(1) t_1 \cdots \hat{H}^\omega t_m \hat{H}^\omega \} \) and \( t \in \eta(\alpha) \) for all \( \alpha \in \theta(1) \cdot \phi(t) \). Hence \( \theta(1) \cdot t = \theta(t) \geq \theta(1) \cdot \phi(t) \). \( \square \)

**Definition 5.27.** Let \( \sigma = s_1 \cdots \hat{H}^\omega s_n \hat{H}^\omega \in C \). Define

\[ \text{Pl}(\sigma) := \eta((1) \cdot \sigma). \]

**Fact 5.28.** \( s_1 \cdots s_n \in \text{Pl}(\sigma) \).

**Proof.** By induction

\[ (1) \cdot \{ s_1 \} \hat{H}^\omega = \hat{H}^\omega \{ s_1 \} < 1 = H^\omega \{ s_1 \} < 1 \]

and \( s_1 \in H^\omega \{ s_1 \} \). Let \( \sigma_i := s_1 \hat{H}^\omega \cdots \hat{H}^\omega s_i \hat{H}^\omega \). Assume \( s_1 \cdots s_i \in \text{Pl}(\sigma_i) \). Then

\[ \eta((1) \cdot \sigma_i s_i+1 \hat{H}^\omega) \]

since

\[ \eta((1) \sigma_i) \ni s_1 \cdots s_i, \quad \eta((1) \sigma_i s_i+1) \ni s_1 \cdots s_is_i+1, \]

and

\[ \eta((1) \sigma_i s_i+1 H^\omega) \ni \eta((1) \sigma_i s_i+1) \ni s_1 \cdots s_is_i+1. \]

**Fact 5.29.** \( s \phi \sigma = s \in \text{Pl}(\sigma) \) follows from Fact 5.28.

**Fact 5.30.** \( \{ \phi^{-1}(\sigma) \mid \sigma \in C \} \subseteq C^\omega(S) \).

**Proof.** \( \phi^{-1}(\sigma) \subseteq \text{Pl}(\sigma) \) and \( \text{Pl}(\sigma) = \eta((1) \cdot \sigma) \in C^\omega(S) \). Hence \( C(\phi) \subseteq C^\omega(S) \subseteq \text{Pl}(S) \).

Since \( C \) is aperiodic, \( C(\phi) \supset \text{Pl}(S) \) (see Fact 2.4), so we have

**Fact 5.31.** \( \mathcal{C}(\phi) = C^\omega(S) = \text{Pl}(S) \).

**Note.** The computation of \( \text{Pl}(S) \) via \( \text{CP}(S) \) is special in the following sense: If \( A \theta (A < A_2 < \cdots < A_1) \), \( s \phi s \hat{H}^\omega \), we have: There exists a sequence of pointlikes \( E_1 \cdots E_n \) such that \( \eta((A < \cdots < A_1) \cdot s \cdot \hat{H}^\omega) = A \cdot s \cdot E_1 \cdots E_n \). Furthermore, the \( E_i = \bigcup G_i \) with \( l_i \) fixing \( A \cdot s \cdot E_1 \cdots E_i \).

It was at first thought that this condition ("local computation") would greatly help in the decidability of \( \#_G(S) \).

**Note 5.32.** A rough estimate (with \( |S| \leq 2^{|S|}, |F_R(X)| \leq |X|^{|X|}, |C^\omega(S)| \leq |P(S)| = 2^{|S|}) shows \( |\text{CP}(S)| \leq \alpha^2 \) with \( \alpha = |C^\omega(S)| \leq 2^{|S|} \), which is quite large.
In terms of 'multiplicities', \( A \in \mathcal{C}^a(S) \) is represented by
\[
\bar{A} := \{ A < A_1 < \cdots A_i | A_i \in \mathcal{C}^a(S) \},
\]
and roughly \(|\bar{A}| \leq |(\mathcal{C}^a(S))^a| \leq 2^{O(S)}\), so we have large multiplicities.

The construction in Fact 2.23 shows that \( \text{Pl}^X(X, S) \) can be computed with \((X \times \text{CP}(S), \text{CP}(S))\). If we assume that there exists an \( x_0 \in X \) such that for every \( x \in X \) there exists an \( A \in \text{Pl}(S) \) with \( x = x_0 \cdot A \), this can be slightly reduced:

\((\mathcal{C}^a(S), \text{CP}(S))\) then computes \( \text{Pl}^X(X, S) \) as well by letting \( \theta : X \to \mathcal{C}^a(S) \) be defined by
\[
\theta(x) := \{ A_n < \cdots A_1 | x = x_0 \cdot A_n \},
\]
\( \phi : S \to \text{CP}(S) \) as before.

Then again \( Y \in \text{Pl}^X(X, S) \) is represented by \( \bar{Y} := \{ A_n < \cdots A_1 | x_0 \cdot A_n = Y \} \) and roughly the multiplicities are \(|\bar{Y}| \leq 2^{O(S)}\).

One may ask if these multiplicities are really necessary to compute \( \text{Pl}^X(X, S) \). We believe that the above bound may still be improved considerably.

The following example shows that some multiplicities may be necessary, though, to compute \( \text{Pl}^X(X, S) \) by an aperiodic semigroup.

**Example 4.** Let \( X := \{0, 1, 2, 3, 4, 1', 1", 4', 4"\} \) and \( S \leq F_R(X) \) be generated by
\[
1 = \text{identity on } X,
\]
\[
\alpha : \begin{cases}
1 \rightarrow 2 \\
2 \rightarrow 3 \\
3 \rightarrow 4 \\
4' \rightarrow 4''
\end{cases},
\quad
\beta : \begin{cases}
2 \rightarrow 3 \\
3 \rightarrow 4 \\
4 \rightarrow 1 \\
1' \rightarrow 1''
\end{cases}
\quad
\gamma : \begin{cases}
1 \rightarrow 1' \\
1'' \rightarrow 2 \\
4 \rightarrow 4'
\end{cases}
\quad
\delta : \begin{cases}
4 \rightarrow 4'
\end{cases}
\]
and \( \alpha, \beta, \gamma, \delta : x \to 0 \) unless otherwise indicated.

**Fact 1.** The maximal pointlikes in \( \text{Pl}^X(X, S) \) are \( \{1, 3, 0\}, \{2, 4, 0\} \) and \( \{x, 0\} \) with \( x \in \{1', 1", 4', 4"\} \).

**Proof.** (a) The above sets are indeed pointlike, since \( 1 \overset{\alpha \beta}{\rightarrow} 3 \overset{\alpha \beta}{\rightarrow} 1 \), so \( \{1, 3\} \) is pointlike; \( \{1, 3\} \overset{\beta}{\rightarrow} \{4, 0\} \overset{\beta}{\rightarrow} \{1, 0\} \) shows \( \{4, 0\} \) and \( \{1, 0\} \) are pointlike; \( \{1, 0\} \overset{\alpha \beta}{\rightarrow} \{3, 0\} \overset{\alpha \beta}{\rightarrow} \{1, 3, 0\} \) shows \( \{1, 3, 0\} \) is pointlike. Similarly \( \{2, 4, 0\} \) is pointlike. \( \{1, 3, 0\} \overset{\alpha}{\rightarrow} \{2, 4, 0\} \) proves \( \{2, 4, 0\} \) to be pointlike, \( \{1, 0\} \overset{\gamma}{\rightarrow} \{1, 0\} \) etc. show \( \{x, 0\} \), \( x \in \{1', 1", 4', 4"\} \) to be pointlike.

(b) To show that they are maximal, we could compute \( \mathcal{C}^a(S) \) and \( X \cdot \mathcal{C}^a(S) = \text{Pl}^X(X, S) \) by Corollary 2.25. It is easier to use Proof-scheme 2.20, though, and construct a relation that actually computes them using an aperiodic semigroup:

Let \( \tilde{X} := X \cup \{1, 3\}, \{2, 4\}, (\delta, 1), (y, 2) \}. \) Let \( S \leq F_R(\tilde{X}) \) be generated by
\[ \tilde{I} = \text{identity on } \tilde{X} \]

\[ \bar{\alpha} : \begin{cases} 
1 \to \{2, 4\} \\
(\delta, 1) \to 2 \\
2 \to 3 \\
(\gamma, 2) \to 3 \\
3 \to 4 \\
4' \to 4'' \\
\{1, 3\} \to \{2, 4\} \\
\{2, 4\} \to 3 
\end{cases} \]

\[ \beta : \begin{cases} 
2 \to \{1, 3\} \\
(\gamma, 2) \to 3 \\
3 \to 4 \\
4 \to 1 \\
1' \to 1'' \\
\{2, 4\} \to \{1, 3\} \\
\{1, 3\} \to 4 
\end{cases} \]

\[ \bar{\gamma} : \begin{cases} 
1 \to 1' \\
(\delta, 1) \to 1' \\
\{1, 3\} \to 1' \\
1'' \to (\gamma, 2) 
\end{cases} \]

\[ \delta : \begin{cases} 
4 \to 4' \\
\{2, 4\} \to 4' \\
4'' \to (\delta, 1) 
\end{cases} \]

(and \( x \to 0 \) where not specified).

Notice the 'multiplicities': \{1\} is represented as \{1\} and as \((\delta, 1)\), \{2\} as \{2\} and \((\gamma, 2)\), i.e., we 'remember' if \( \delta \) or \( \gamma \) 'was used last'.

Also note the choice of 'blow-ups': for \( \bar{\alpha} \) it occurs at \( 1 \to \{2, 4\} \), for \( \beta \) at \( 2 \to \{1, 3\} \).

Define \((X, S) \xrightarrow{R=(\theta, \phi)} (\tilde{X}, \tilde{S})\) by

\[ \phi(\sigma) = \{ \bar{\sigma}_1 \ldots \bar{\sigma}_k \mid \sigma_1 \ldots \sigma_k = \sigma \} \quad (\sigma \in S), \]

\[ \theta(x) = \{ y \in \tilde{X} \mid x \in y \} \quad (x \neq 0), \]

\[ \theta(0) = \tilde{X} \]

(where \( x \in y \) if \( x = y \) or \( x \in y \) and also \( 1 \in (\delta, 1) \), \( 2 \in (\gamma, 2) \)). It is easy to see that this is indeed a TSG-relation and that it computes the above sets.

To see that \( \tilde{S} \) is aperiodic observe that

(i) All generators (\( \neq \tilde{1} \)) have \( \bar{\sigma}^n = \text{constant} \) for \( n \) large enough.

(ii) \( \bar{\alpha} \bar{\beta} \bar{\alpha} = \{2, 4\} \) or \( 0 \) for all \( x \in \tilde{X} \).

(iii) By (i) and (ii) it follows that if \( \bar{\sigma} = \bar{\sigma}_1 \ldots \bar{\sigma}_n \) is a non-trivial group-element of \( \tilde{S} \) (i.e., \( \bar{\sigma}^2 \neq \bar{\sigma} \)), it must contain \( \bar{\delta} \) or \( \bar{\gamma} \) in its decomposition. Assume \( \bar{\sigma}_1 = \bar{\gamma} \) and \( i \) is the first index such that \( s \cdot \bar{\delta}_i : \bar{\gamma} \), (the argument for \( \bar{\delta} \) is similar), and that \( \tilde{x} = \bar{\sigma}_1 \ldots \bar{\sigma}_{i-1} \to \tilde{x} \) is a non-trivial cycle. Then \( \tilde{x} = \bar{\sigma}_1 \ldots \bar{\sigma}_{i-1} \to 1 \), and we can assume without loss of generality that \( \tilde{x} = 1 \) and \( \bar{\sigma}_i = \bar{\gamma} \). \( 1 \bar{\gamma} = 1' \) implies \( \bar{\sigma}_2 = \bar{\beta} \) and \( \bar{\sigma}_1 = \bar{\gamma} \), so \( \bar{\sigma} = \bar{\gamma} \bar{\beta} ? \bar{T} \) with \( \bar{T} : 2 \to 1'' \). (Since \( \tilde{x} = \bar{\sigma}_1 \ldots \bar{\sigma}_{i-1} \to \tilde{x} \) is non-trivial.) But then \( 1 \bar{\sigma} \) is \( 1'' \), \( \bar{\gamma} \bar{\beta} \bar{\sigma} 0 \), a contradiction. Therefore a non-trivial group-element \( \bar{\sigma} \) cannot exist.

**Fact 2.** Let \( \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta} : PL^X(X, S) \to PL^X(X, S) \) be defined so that \( Y \bar{\sigma} \geq Y \cdot \sigma \) for all \( Y \in PL^X(X, S) \), \( \sigma \in \{ \alpha, \beta, \gamma, \delta \} \). Then \( \tilde{S} = \langle \tilde{I}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta} \rangle \) is not aperiodic. (In other words, it is impossible to act with \( \sigma \) on \( PL^X(X, S) \), computing \( PL^X(X, S) \) with an aperiodic semigroup. Multiplicities have to be introduced to do so - e.g. \( \{1\} \) is represented by \( 1 \) and \( (\delta, 1) \) in our computation.)
Proof. We omit the 0 in the following argument, e.g. writing \{1,3\} for both \{1,3,0\} and \{1,3\} and 2 for \{2,0\} and \{2\}.

(i) \frown 1 is forced. Otherwise \frown \{1,3\} \mapsto \{2,4\} \mapsto \{4\} \mapsto \{2,4\} pointlike. Similarly \frown \{2,4\} \mapsto \{3,1\} \mapsto \{1\} shows (ii) \frown 2 is forced.

(iii) \frown 2 is forced. Otherwise \frown \{2,4\} \mapsto \{4\} \mapsto \{1\} \mapsto \{4\} pointlike. Similarly \frown \{1,3\} \mapsto \{1\} shows (iv) \frown 3 is forced.

(v) \frown \{2,4\} \mapsto \{3,1\} \mapsto \{2,4\} makes \{1,2,3,4\} pointlike. Similarly \frown \{1,3\} \mapsto \{2,4\} \mapsto \{1,3\} shows (vi) \frown 4 is forced.

(vii) \frown \{4,2\} makes \{1,2,3,4\} pointlike. Similarly, using \frown \{1,3\} \mapsto \{4,2\} \mapsto \{1\}

(viii) \frown \{1\} \mapsto \{4\} \mapsto \{1\} a non-trivial cycle.

Afterword

Assume #G(S) = 1 and D is a way of 'putting S up into a wreath product at complexity', with

\[
\begin{align*}
C \circ G \circ C \longrightarrow G \circ C \longrightarrow C \\
\leq & \leq & \leq \\
D: & T_1 \xrightarrow{\gamma_1} T_2 \xrightarrow{\gamma_2} T_3 \\
\theta: & T_1 \xrightarrow{\gamma_1} T_2 \xrightarrow{\gamma_2} T_3 \\
S & \xrightarrow{(R = \theta^{-1} \gamma_1 \gamma_2)}
\end{align*}
\]

Even though we have determined Pl(S) and exhibited a particular way \(S \xrightarrow{R} CP(S)\) of computing it, and \(S \xrightarrow{R} T_1\) will have \(Pl(S) \leq C(R)\) as covers, there still might not be any way of using CP(S) alone to put S up at complexity, i.e. \(S \leftarrow T \leq C \circ G \circ CP(S)\) might not exist...!

\(T_3\) might use multiplicities 'designed to make adding on a group easy'.

We will explore these questions in a forthcoming paper [2] (where we give an example that CP(S) is indeed 'not necessarily good enough to put S up at complexity'.
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References

[2] K. Henckell, Necessary and sufficient conditions for $c(S)=1$, to be published.