Computers and Mathematics with Applications 62 (2011) 2591–2600

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Shortest path problem with uncertain arc lengths

Yuan Gao *

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

ARTICLE INFO

Article history:
Received 16 November 2010
Received in revised form 23 July 2011
Accepted 25 July 2011

Keywords:
Shortest path problem
Uncertainty theory
Uncertain programming
Dijkstra algorithm

ABSTRACT

Uncertainty theory provides a new tool to deal with the shortest path problem with nondeterministic arc lengths. With help from the operational law of uncertainty theory, this paper gives the uncertainty distribution of the shortest path length. Also, it investigates solutions to the α-shortest path and the most shortest path in an uncertain network. It points out that there exists an equivalence relation between the α-shortest path in an uncertain network and the shortest path in a corresponding deterministic network, which leads to an effective algorithm to find the α-shortest path and the most shortest path. Roughly speaking, this algorithm can be broken down into two parts: constructing a deterministic network and then invoking the Dijkstra algorithm.

1. Introduction

The shortest path problem concentrates on finding a path with minimum distance, time, or cost from the source node to the destination node. Many programming problems, including transportation, routing, communications, and supply chain management, can be regarded as special cases of the shortest path problem. In the 1950s and 1960s, some successful algorithms were proposed or developed by Bellman [1], Dijkstra [2], Dreyfus [3], and Floyd [4], which made the shortest path problem occupy a central position in a network. In this paper, these algorithms will be referred to as classical algorithms. In classical algorithms, the network is required to have deterministic arc lengths. However, because of failure, maintenance, or other reasons, the arc lengths are nondeterministic in many situations. As a result, it is improper to employ classical algorithms in these situations. Some researchers believed that these nondeterministic phenomena conform to randomness, and they introduced probability theory into the shortest path problem; see Frank [5], Hall [6], Loui [7], Mirchandani [8], for example. Since Dubois and Prade [9] proposed the fuzzy shortest path problem (FSPP) in 1980, fuzzy theory has begun to attract network researchers. Some researchers, such as Klein [10], Ji and Iwamura [11], Lin and Chen [12], and Okada [13], have done a lot of work in this field.

In 2007, Liu [14] proposed uncertainty theory to describe nondeterministic phenomena, especially expert data and subjective estimation. From then on, uncertainty theory has provided a new approach to deal with nondeterministic factors in programming problems. Obviously, the length of each path in an uncertain network is uncertain, and we cannot get a shortest path in the normal sense. What we investigate is the distribution of the shortest path length. In 2010, Liu [15] proposed the concepts of α-shortest path and the most shortest path in an uncertain network. These two concepts are duals of each other, and they are the optimal paths under some confidence level constrains. In practice, the α-shortest path and the most shortest path are of important significance. However, the method to find α-shortest path or the most shortest path is not given.

This paper is concerned with two things: (i) the uncertainty distribution of the shortest path length, and (ii) an effective method to find the α-shortest path and the most shortest path in an uncertain network. In the past literature, the

* Tel.: +86 1051533473.
E-mail address: g-y08@mails.tsinghua.edu.cn.

0898-1221/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2011.07.058

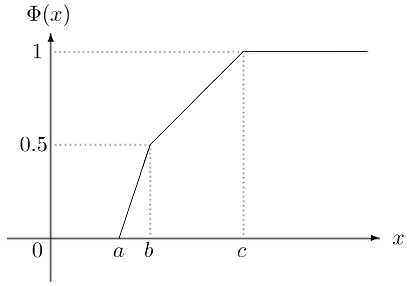

Fig. 1. Zigzag uncertainty distribution.

optimal solution to the shortest path problem in a nondeterministic network was usually obtained by methods of heuristic algorithms, Monte Carlo simulation, or assuming a ranking index for comparing arc lengths. These methods lack rigor or portability. What is worse, some of them are of high computational complexity. Fortunately, under the framework of uncertainty theory, the uncertainty distribution of the shortest path length is obtained quite easily. Also, it is found that the α-shortest path of an uncertain network is just the shortest path in the corresponding deterministic network. This conclusion provides an effective method to find the α-shortest path in an uncertain network; that is, the α-shortest path can be obtained by simply invoking any classical algorithm in a deterministic network. In this paper, the classical algorithm we employ is the Dijkstra algorithm. After obtaining the α-shortest path, we can get the most shortest path easily, which is the dual problem of obtaining the α-shortest path.

The remainder of this paper is organized as follows. In Section 2, some basic concepts and properties of uncertainty theory used throughout this paper are introduced. In Section 3, the operational law and related theorem are discussed in detail. In Section 4, the uncertain shortest path problem is described. Section 5 gives the uncertainty distribution of the shortest path length. Section 6 derives the method to find the α-shortest path and the most shortest path in an uncertain network. In Section 7, an example is given to illustrate the conclusions presented in Section 6. Section 8 gives a brief summary to this paper.

2. Preliminaries

Founded in 2007, uncertainty theory is a new branch of mathematics. Up to now, theory and practice have shown that uncertainty theory is an efficient tool to deal with nondeterministic information, especially expert data and subjective estimation. From a theoretical aspect, an uncertain process [16], and uncertain differential equation [17], and uncertain set theory [18] have been established. From a practical aspect, uncertain programming [19], uncertain inference [20], uncertain statistics [21], uncertain optimal control [22], etc., have also developed quickly. In short, uncertainty theory is increasingly being researched and used.

In this section, we introduce some foundational concepts and properties of uncertainty theory, which will be used throughout this paper.

Let Γ be a nonempty set, and let $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is assigned a number $\mathcal{M}\{\Lambda\} \in [0, 1]$. In order to ensure that the number $\mathcal{M}\{\Lambda\}$ has certain mathematical properties, Liu [14,21] presented four axioms: (1) normality, (2) self-duality, (3) countable subadditivity, and (4) product measure axioms. If the first three axioms are satisfied, the set function $\mathcal{M}\{\Lambda\}$ is called an uncertain measure.

Definition 1 (*Liu [14]*)**.** Let Γ be a nonempty set, $\mathcal{L}$ a σ-algebra over Γ, and $\mathcal{M}$ an uncertain measure. Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Definition 2 (*Liu [14]*)**.** An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

The uncertainty distribution of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \le x\}$ for any real number x. For example, the zigzag uncertain variable $\xi \sim \mathcal{Z}(a, b, c)$ has an uncertainty distribution (Fig. 1)

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a \\ (x-a)/2(b-a), & \text{if } a \le x \le b \\ (x+c-2b)/2(c-b), & \text{if } b \le x \le c \\ 1, & \text{if } x \ge c. \end{cases}$$

Definition 3 (*Liu [23]*)**.** The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}\{\xi_i \in B_i\}\right\} = \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Definition 4 (*Liu [21]*)**.** An uncertainty distribution Φ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$.

Obviously, zigzag uncertain variable has a regular uncertainty distribution. If Φ is regular, uncertainty distribution Φ is continuous and strictly increasing at each point x with $0 < \Phi(x) < 1$. Also, Φ^{-1} is continuous and strictly increasing in (0, 1). What is more, we have $\mathcal{M}\{\xi \le t\} = \mathcal{M}\{\xi < t\}$, provided that ξ has a regular uncertainty distribution.

We usually assume that all uncertainty distributions in practical applications are regular. Otherwise, a small perturbation can be imposed to obtain a regular one. In the following sections, we will see that the inverse uncertainty distribution $\Phi^{-1}(\alpha)$ has some good operational properties, which makes the solution to uncertain programming problems easy to obtain.

3. The operational law

In uncertainty theory, the operation of independent uncertain variables follows the operational law.

Theorem 1 (*Liu [23], operational law*)**.** *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables, and let $f : \Re^n \to \Re$ be a measurable function. Then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable such that*

$$\mathcal{M}\{\xi \in B\} = \begin{cases} \sup\limits_{f(B_1,B_2,\ldots,B_n)\subset B} \min\limits_{1\le i\le n} \mathcal{M}_i\{\xi_i \in B_i\}, & \text{if } \sup\limits_{f(B_1,B_2,\ldots,B_n)\subset B} \min\limits_{1\le i\le n} \mathcal{M}_i\{\xi_i \in B_i\} > 0.5 \\ 1 - \sup\limits_{f(B_1,B_2,\ldots,B_n)\subset B^c} \min\limits_{1\le i\le n} \mathcal{M}_i\{\xi_i \in B_i\}, & \text{if } \sup\limits_{f(B_1,B_2,\ldots,B_n)\subset B^c} \min\limits_{1\le i\le n} \mathcal{M}_i\{\xi_i \in B_i\} > 0.5 \\ 0.5, & \text{otherwise}, \end{cases}$$

where $B, B_1, B_2, \ldots, B_n$ are Borel sets of real numbers.

Example 1. If $h > 0$, $k > 0$, we have

$$h\mathcal{Z}(a_1, b_1, c_1) + k\mathcal{Z}(a_2, b_2, c_2) = \mathcal{Z}(ha_1 + ka_2, hb_1 + kb_2, hc_1 + kc_2).$$

If the function f in the operational law has some additional feature, such as monotonicity, a very useful conclusion can be obtained.

A real function $f(x_1, x_2, \ldots, x_n)$ is said to be strictly increasing if f satisfies the following conditions:

(1) $f(x_1, x_2, \ldots, x_n) \ge f(y_1, y_2, \ldots, y_n)$ when $x_i \ge y_i$ for $i = 1, 2, \ldots, n$;
(2) $f(x_1, x_2, \ldots, x_n) > f(y_1, y_2, \ldots, y_n)$ when $x_i > y_i$ for $i = 1, 2, \ldots, n$.

Theorem 2. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively, and let $f : \Re^n \to \Re$ be a continuous and strictly increasing function. Then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an inverse uncertainty distribution:*

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)).$$

Proof. Since f is an increasing function, we have

$$\{f(\xi_1, \ldots, \xi_n) \le f(\Phi_1^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))\} \supset \left\{(\xi_1 \le \Phi_1^{-1}(\alpha)) \cap \cdots \cap (\xi_n \le \Phi_n^{-1}(\alpha))\right\}.$$

Because $\xi_1, \xi_2, \ldots, \xi_n$ are independent and regular, for each $\alpha \in (0, 1)$, it follows that

$$\begin{aligned} &\mathcal{M}\{\xi \le f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))\} \\ &\quad = \mathcal{M}\{f(\xi_1, \xi_2, \ldots, \xi_n) \le f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))\} \\ &\quad \ge \mathcal{M}\left\{(\xi_1 \le \Phi_1^{-1}(\alpha)) \cap (\xi_2 \le \Phi_2^{-1}(\alpha)) \cap \cdots \cap (\xi_n \le \Phi_n^{-1}(\alpha))\right\} \\ &\quad = \mathcal{M}\{\xi_1 \le \Phi_1^{-1}(\alpha)\} \wedge \mathcal{M}\{\xi_2 \le \Phi_2^{-1}(\alpha)\} \wedge \cdots \wedge \mathcal{M}\{\xi_n \le \Phi_n^{-1}(\alpha)\} \\ &\quad = \Phi_1(\Phi_1^{-1}(\alpha)) \wedge \Phi_2(\Phi_2^{-1}(\alpha)) \wedge \cdots \wedge \Phi_n(\Phi_n^{-1}(\alpha)) \\ &\quad = \alpha. \end{aligned}$$

It is assumed that f is strictly increasing; then, we have

$$\{f(\xi_1, \ldots, \xi_n) > f(\Phi_1^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))\} \supset \left\{\{\xi_1 > \Phi_i^{-1}(\alpha)\} \cap \cdots \cap \{\xi_n > \Phi_n^{-1}(\alpha)\}\right\}.$$

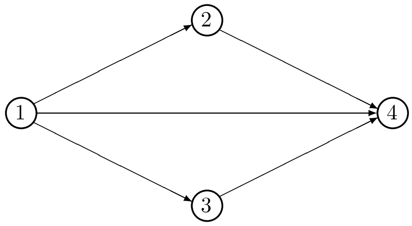

Fig. 2. Network N for Example 2.

Because $\xi_1, \xi_2, \ldots, \xi_n$ are independent and regular, for each $\alpha \in (0, 1)$, it follows that

$$\begin{aligned}
&\mathcal{M}\{\xi > f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))\} \\
&\quad = \mathcal{M}\{f(\xi_1, \xi_2, \ldots, \xi_n) > f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))\} \\
&\quad \geq \mathcal{M}\left\{(\xi_1 > \Phi_1^{-1}(\alpha)) \cap (\xi_2 > \Phi_2^{-1}(\alpha)) \cap \cdots \cap (\xi_n > \Phi_n^{-1}(\alpha))\right\} \\
&\quad = \mathcal{M}\{(\xi_1 > \Phi_1^{-1}(\alpha))\} \wedge \mathcal{M}\{(\xi_2 > \Phi_2^{-1}(\alpha))\} \wedge \cdots \wedge \mathcal{M}\{(\xi_n > \Phi_n^{-1}(\alpha))\} \\
&\quad = (1 - \mathcal{M}\{\xi_1 \leq \Phi_1^{-1}(\alpha)\}) \wedge (1 - \mathcal{M}\{\xi_2 \leq \Phi_2^{-1}(\alpha)\}) \wedge \cdots \wedge (1 - \mathcal{M}\{\xi_n \leq \Phi_n^{-1}(\alpha)\}) \\
&\quad = (1 - \Phi_1(\Phi_1^{-1}(\alpha))) \wedge (1 - \Phi_2(\Phi_2^{-1}(\alpha))) \wedge \cdots \wedge (1 - \Phi_n(\Phi_n^{-1}(\alpha))) \\
&\quad = 1 - \alpha;
\end{aligned}$$

thus,

$$\begin{aligned}
\mathcal{M}\{\xi \leq f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))\} &= 1 - \mathcal{M}\{\xi > f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))\} \\
&\leq 1 - (1 - \alpha) = \alpha.
\end{aligned}$$

It follows that $\mathcal{M}\{\xi \leq f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))\} = \alpha$, for each $\alpha \in (0, 1)$.

Obviously, $f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))$ is a strictly increasing and continuous function of α. The definition of a regular uncertainty distribution tells us

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)).$$

The theorem is proved. □

Theorem 2 gives a simple method to get the uncertainty distribution of $f(\xi_1, \xi_2, \ldots, \xi_n)$. Although in most uncertain programming problems we can only get the distribution in a numerical sense, it is never too much to stress the important role of Theorem 2. In this paper, the solution to the uncertain shortest path problem is just based on this theorem.

4. Problem description

In general, a deterministic network is denoted as $N = (V, A)$, where $V = \{1, 2, \ldots, n\}$ is a finite set of nodes, and $A = \{(i, j) \mid i, j \in V\}$ is the set of arcs.

Denote $w = \{w_{ij} \mid (i, j) \in A\}$ as the set of arc lengths. Then in network $N = (V, A)$, from the source node to the destination node, the shortest path length is a function of w, which is denoted as f_{SP} in this paper. Given w, $f_{SP}(w)$ can be obtained by using the Dijkstra algorithm. The following example illustrates function f_{SP}.

Example 2. The network $N = (V, A, \xi)$ is shown in Fig. 2.

When $w = (w_{12}, w_{13}, w_{14}, w_{24}, w_{34}) = (1, 3, 4, 1, 3)$, the shortest path is $1 \to 2 \to 4$; that is, $f_{SP}(w) = w_{12} + w_{24} = 2$. When $w = (w_{12}, w_{13}, w_{14}, w_{24}, w_{34}) = (3, 3, 4, 3, 3)$, the shortest path is $1 \to 4$; that is, $f_{SP}(w) = w_{14} = 4$.

In this paper, the nondeterministic factor in network $N = (V, A)$ is only the length of each arc. We employ uncertainty theory to deal with this nondeterministic factor. Some assumptions are listed as follows.

(1) There is only one source node and only one destination node.
(2) There is no cycle in each network.
(3) The length of each arc(i, j) is a positive uncertain variable ξ_{ij}.
(4) All the uncertain variables ξ_{ij} are independent.

Define $\xi = \{\xi_{ij} \mid (i, j) \in A\}$. We can denote the network with uncertain arc length as $N = (V, A, \xi)$; its shortest path length is $f_{SP}(\xi)$. Obviously, $f_{SP}(\xi)$ is an uncertain variable.

For the network $N = (V, A, \xi)$, assume that the uncertainty distribution of $f_{SP}(\xi)$ is $\Psi(x)$; i.e., $\Psi(x) = \mathcal{M}\{f_{SP}(\xi) \leq x\}$. From now on, we use following representation to denote path P:

$$P = \{x_{ij} \mid (i, j) \in A\},$$

where $x_{ij} \in \{0, 1\}$ is a decision variable; that is, when $x_{ij} = 0$, arc(i, j) is not in path P; when $x_{ij} = 1$, arc (i, j) is in path P. Then the length of P is

$$l(P) = \sum_{(i,j)\in A} x_{ij}\xi_{ij}.$$

It is clear that $l(P)$ is also an uncertain variable. Assume that node 1 is the source and node n is the destination. Then P is a path from source node 1 to destination node n if and only if

$$\begin{cases} \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1, & i = 1, \\ 0, & 2 \le i \le n-1, \\ -1, & i = n, \end{cases} \\ x_{ij} = \{0, 1\}, \quad \forall (i, j) \in A. \end{cases}$$

In 2010, Liu [15] gave the concepts of the α-shortest path and the most shortest path in an uncertain network $N = (V, A, \xi)$.

Definition 5 (*Liu [15]*). In network $N = (V, A, \xi)$, P_0 is a path from the source node to the destination node. Then P_0 is called the α-shortest path if

$$T_\alpha = \min\{T \mid \mathcal{M}\{l(P_0) \le T\} \ge \alpha\} \le \min\{T \mid \mathcal{M}\{l(P) \le T\} \ge \alpha\}$$

for all paths P from the source node to the destination node, where α is a predetermined confidence level.

The practical meaning of the α-shortest path is natural. Given an $\alpha \in (0, 1)$, we hope to get a smallest length T (denoted as T_α) and a path P_0, where the uncertain variable $l(P_0)$ is less than T_α with confidence level α. Here, path P_0 is just the α-shortest path.

Definition 6 (*Liu [15]*). In network $N = (V, A, \xi)$, P_0 is a path from the source node to the destination node. Then P_0 is called the most shortest path if

$$\mathcal{M}\{l(P_0) \le T\} \ge \mathcal{M}\{l(P) \le T\}$$

for all paths P from the source node to the destination node, where T is a predetermined length.

Given a predetermined confidence length T, the most shortest path P_0 is the optimal path which is less than T with the largest confidence level. It is clear that the α-shortest path and the most shortest path are duals of each other.

In contrast to the situation in a deterministic network, the shortest path length $f_{SP}(\xi)$ is an uncertain variable in an uncertain network. The α-shortest path and the most shortest path are the optimal paths under some confidence level. With a change of confidence level, the optimal paths will also change. Then, for uncertain network $N = (V, A, \xi)$, what is the uncertainty distribution of f_{SP}? And how can we get the α-shortest path and the most shortest path? In Sections 5 and 6, we will give the answers.

5. The uncertainty distribution of $f_{SP}(\xi)$

For a network, the shortest path length $f_{SP}(w)$ is a continuous and increasing function with respect to each component of w. It is also clear that reducing the length of each arc leads to a smaller shortest path length; that is,

$$f_{SP}(x) < f_{SP}(y),$$

where $x = \{x_{ij} \mid (i, j) \in A\}$, $y = \{y_{ij} \mid (i, j) \in A\}$, and $x_{ij} < y_{ij}$. Thus, f_{SP} is a strictly increasing function.

According to Theorem 2, we can easily obtain the inverse uncertainty distribution of $f_{SP}(\xi)$.

Theorem 3. *In network $N = (V, A, \xi)$, ξ_{ij} has a regular uncertainty distribution Φ_{ij}. Then, the inverse uncertainty distribution of $f_{SP}(\xi)$ is determined by*

$$\Psi^{-1}(\alpha) = f_{SP}(\Phi_{ij}^{-1}(\alpha) \mid (i, j) \in A).$$

Through Theorem 3, we can get the uncertainty distribution of $f_{SP}(\xi)$ in a numerical sense easily. The following example illustrates this.

Example 3. Still use the network $N = (V, A, \xi)$ presented in Example 2. Here, $\xi_{12} = 1$, $\xi_{14} = 4$, $\xi_{34} = 1$, $\xi_{13} \sim \mathcal{Z}(2, 3, 3.5)$, and $\xi_{24} \sim \mathcal{Z}(2.5, 3, 4)$. For simplicity, if $\xi_{ij} = c$ is a constant, we set $\Phi_{ij}^{-1}(\alpha) = c$, for any $\alpha \in (0, 1)$.

When $\alpha = 0.9$, it is easy to get $\Phi_{12}^{-1}(\alpha) = 1$, $\Phi_{13}^{-1}(\alpha) = 3.4$, $\Phi_{14}^{-1}(\alpha) = 4$, $\Phi_{24}^{-1}(\alpha) = 3.8$, and $\Phi_{34}^{-1}(\alpha) = 1$, as shown in Fig. 3.

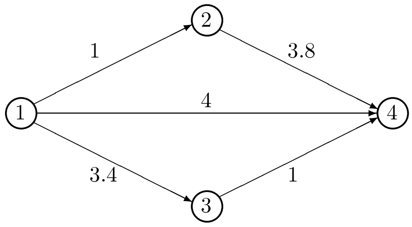

Fig. 3. Network N when $\alpha = 0.9$.

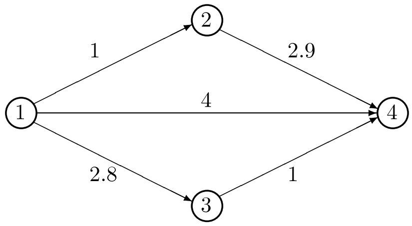

Fig. 4. Network N when $\alpha = 0.4$.

Then

$$\begin{aligned}\Psi^{-1}(0.9) &= f_{SP}(\Phi_{12}^{-1}(0.9), \Phi_{13}^{-1}(0.9), \Phi_{14}^{-1}(0.9), \Phi_{24}^{-1}(0.9), \Phi_{34}^{-1}(0.9)) \\ &= f_{SP}(1, 3.4, 4, 3.8, 1) \\ &= 4;\end{aligned}$$

that is, $\mathcal{M}\{f_{SP}(\xi) \le 4\} = \Psi(4) = 0.9$.

When $\alpha = 0.4$, it is easy to get $\Phi_{12}^{-1}(\alpha) = 1$, $\Phi_{13}^{-1}(\alpha) = 2.8$, $\Phi_{14}^{-1}(\alpha) = 4$, $\Phi_{24}^{-1}(\alpha) = 2.9$, and $\Phi_{34}^{-1}(\alpha) = 1$, as shown in Fig. 4.

Then

$$\begin{aligned}\Psi^{-1}(0.4) &= f_{SP}(\Phi_{12}^{-1}(0.4), \Phi_{13}^{-1}(0.4), \Phi_{14}^{-1}(0.4), \Phi_{24}^{-1}(0.4), \Phi_{34}^{-1}(0.4)) \\ &= f_{SP}(1, 2.8, 4, 2.9, 1) \\ &= 2.8 + 1 = 3.8;\end{aligned}$$

that is, $\mathcal{M}\{f_{SP}(\xi) \le 3.8\} = \Psi(3.8) = 0.4$.

For any $\alpha \in (0, 1)$, we can obtain $\Psi^{-1}(\alpha)$ by repeating this process, and then get the uncertainty distribution Ψ satisfying any predetermined accuracy.

6. The α-shortest path and the most shortest path

For confidence level α, the following theorem gives a method to get the α-shortest path.

Theorem 4. *In network $N = (V, A, \xi)$, ξ_{ij} has a regular uncertainty distribution Φ_{ij}, $(i, j) \in A$. Then, the α-shortest path of $N = (V, A, \xi)$ is just the shortest path of network $\bar{N} = (\bar{V}, \bar{A})$; $\bar{V} = V$, $\bar{A} = A$, and the length of* $\mathrm{arc}(i, j) \in \bar{A}$ *is $\Phi_{ij}^{-1}(\alpha)$.*

Proof. Assume that node 1 is the source and node n is the destination. According to Definition 5, the α-shortest path P_0 is the optimal solution to the following uncertain programming model:

$$\begin{cases} \min & T_0 \\ \text{s.t.} & \mathcal{M}\{l(P) \le T_0\} = \mathcal{M}\left\{\displaystyle\sum_{(i,j)\in A} x_{ij}\xi_{ij} \le T_0\right\} \ge \alpha, \\ & \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1, & i = 1, \\ 0, & 2 \le i \le n-1, \\ -1, & i = n, \end{cases} \\ & x_{ij} = \{0, 1\}, \quad \forall (i, j) \in A. \end{cases} \tag{1}$$

Since each ξ_{ij} has a regular uncertainty distribution, $\mathcal{M}\left\{\sum_{(i,j)\in A} x_{ij}\xi_{ij} \le T_0\right\} \ge \alpha$ can be replaced by $\mathcal{M}\left\{\sum_{(i,j)\in A} x_{ij}\xi_{ij} \le T_0\right\} = \alpha$ in the model. Then, using the inverse uncertainty distribution, model (1) can be equivalently transformed to the following deterministic model:

$$\begin{cases} \min & T_0 \\ \text{s.t.} & \displaystyle\sum_{(i,j)\in A} x_{ij}\Phi_{ij}^{-1}(\alpha) \le T_0, \\ & \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1, & i = 1, \\ 0, & 2 \le i \le n-1, \\ -1, & i = n, \end{cases} \\ & x_{ij} = \{0, 1\}, \quad \forall (i,j) \in A. \end{cases} \tag{2}$$

Model (2) is equivalent to

$$\begin{cases} \min & \displaystyle\sum_{(i,j)\in A} x_{ij}\Phi_{ij}^{-1}(\alpha) \\ \text{s.t.} & \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1, & i = 1, \\ 0, & 2 \le i \le n-1, \\ -1, & i = n, \end{cases} \\ & x_{ij} = \{0, 1\}, \quad \forall (i,j) \in A. \end{cases} \tag{3}$$

In fact, the solution to model (3) is just the shortest path of the deterministic network $\bar{N} = (\bar{V}, \bar{A})$; $\bar{V} = V, \bar{A} = A$, and the length of arc$(i, j) \in \bar{A}$ is $\Phi_{ij}^{-1}(\alpha)$. We can find the shortest path of network $\bar{N} = (\bar{V}, \bar{A})$ by using the Dijkstra algorithm. The theorem is proved. □

In the past, the methods of heuristic algorithms, Monte Carlo simulation, or assuming a ranking index for comparing arc lengths were usually employed to find the optimal solution to nondeterministic programming problems. Although the stability of these methods is always tested and verified by numerical experiments in such papers, the rigor and portability of these methods are not good. In other words, these methods are only second best. Theorem 4 provides a better method to find the α-shortest path in an uncertain network; that is, we only need to employ the Dijkstra algorithm to find the shortest path of a corresponding deterministic network. Obviously, this method also has a low computational complexity, which is the same as that of the Dijkstra algorithm, namely $O(n^2)$.

After obtaining the α-shortest path, we can obtain the most shortest path by using the following theorem.

Theorem 5. *In network $N = (V, A, \xi)$, ξ_{ij} has a regular uncertainty distribution Φ_{ij}, $(i, j) \in A$, and the shortest path length $f_{SP}(\xi)$ has an uncertainty distribution Ψ. Given a predetermined length T_0, the most shortest path P_0 is just the α-shortest path of $N = (V, A, \xi)$, where $\alpha = \Psi(T_0)$.*

Proof. Assume that node 1 is the source and node n is the destination. According to Definition 6, the most shortest path P_0 is the optimal solution to the following uncertain programming model:

$$\begin{cases} \max & \alpha \\ \text{s.t.} & \mathcal{M}\{l(P) \le T_0\} = \mathcal{M}\left\{\displaystyle\sum_{(i,j)\in A} x_{ij}\xi_{ij} \le T_0\right\} \ge \alpha, \\ & \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1, & i = 1, \\ 0, & 2 \le i \le n-1, \\ -1, & i = n, \end{cases} \\ & x_{ij} = \{0, 1\}, \quad \forall (i,j) \in A. \end{cases}$$

Since each ξ_{ij}, $(i, j) \in A$ has a regular uncertainty distribution Φ_{ij}, respectively, the above model is equivalent to the following programming model:

$$\begin{cases} \max & \alpha \\ \text{s.t.} & \displaystyle\sum_{(i,j)\in A} x_{ij}\Phi_{ij}^{-1}(\alpha) \le T_0, \\ & \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1, & i = 1, \\ 0, & 2 \le i \le n-1, \\ -1, & i = n, \end{cases} \\ & x_{ij} = \{0, 1\}, \quad \forall (i,j) \in A. \end{cases} \tag{4}$$

Let $\alpha' = \Psi(T_0)$, and let P' be the α'-shortest path of $N = (V, A, \xi)$. Theorems 3 and 4 give

$$\Psi^{-1}(\alpha') = f_{SP}(\Phi_{ij}^{-1}(\alpha') \mid (i,j) \in A) = T_0,$$
$$\sum_{(i,j)\in P'} \Phi_{ij}^{-1}(\alpha') = T_0,$$

which means that P' is a feasible solution to model (4), and α' is the objective. In fact, α' is also the optimal objective. We will prove this below.

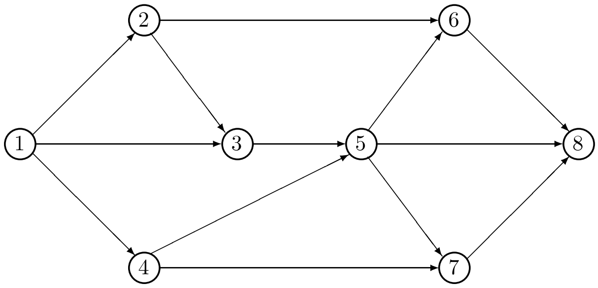

Fig. 5. Network N for Example 4.

Table 1
List of arc lengths.

arc(i, j)	ξ_{ij}	arc(i, j)	ξ_{ij}
(1, 2)	7	(1, 3)	8
(1, 4)	9	(2, 3)	2
(2, 6)	$\mathcal{Z}(9, 10, 11)$	(3, 5)	$\mathcal{Z}(7, 9, 10)$
(4, 5)	$\mathcal{Z}(10, 11, 12)$	(4, 7)	$\mathcal{Z}(8, 9, 10)$
(5, 6)	3	(5, 7)	3
(5, 8)	$\mathcal{Z}(12, 13, 15)$	(6, 8)	$\mathcal{Z}(12, 13, 14)$
(7, 8)	$\mathcal{Z}(10, 12, 14)$	–	–

Table 2
List of $\Phi_{ij}^{-1}(0.95)$.

arc(i, j)	$\Phi_{ij}^{-1}(0.95)$	arc(i, j)	$\Phi_{ij}^{-1}(0.95)$
(1, 2)	7	(1, 3)	8
(1, 4)	9	(2, 3)	2
(2, 6)	10.9	(3, 5)	9.9
(4, 5)	11.9	(4, 7)	9.9
(5, 6)	3	(5, 7)	3
(5, 8)	14.8	(6, 8)	13.9
(7, 8)	13.8	–	–

For any $\alpha > \alpha'$, $\alpha \in (0, 1)$, it is obvious that $\Phi_{ij}^{-1}(\alpha) > \Phi_{ij}^{-1}(\alpha)$, $(i, j) \in A$. Then

$$f_{SP}(\Phi_{ij}^{-1}(\alpha) \mid (i, j) \in A) > f_{SP}(\Phi_{ij}^{-1}(\alpha') \mid (i, j) \in A) = T_0,$$

which means that, for any path P from source node 1 to destination node n,

$$\sum_{(i,j)\in P} \Phi_{ij}^{-1}(\alpha) \geq f_{SP}(\Phi_{ij}^{-1}(\alpha) \mid (i, j) \in A) > T_0, \tag{5}$$

since $f_{SP}(\Phi_{ij}^{-1}(\alpha) \mid (i, j) \in A)$ is the shortest path length in network $\bar{N} = (\bar{V}, \bar{A})$. Expression (5) leads to a contradiction with the constraints in model (4).

It follows that $\alpha' = \Psi(T_0)$ is the optimal objective of model (4), and path P' is the most shortest path; that is, $P_0 = P'$. The theorem is proved. □

7. An example

In this section, we will give an example to illustrate the conclusions presented above.

Example 4. The network $N = (V, A, \xi)$ is shown in Fig. 5. The length of the shortest path is f_{SP}, with uncertainty distribution Ψ. The length of each arc is listed in Table 1. We will obtain the following:

(1) the α-shortest path when $\alpha = 0.95$;
(2) the most shortest path when $T_0 = 30$; and
(3) the uncertainty distribution of $f_{SP}(\xi)$.

If $\xi_{ij} = c$ is a constant, we set $\Phi_{ij}^{-1}(\alpha) = c$, for any $\alpha \in (0, 1)$. When $\alpha = 0.95$, we can calculate $\Phi_{ij}^{-1}(0.95)$ for each ξ_{ij}. The values are listed in Table 2. Using the data in Table 2, we construct a deterministic network $\bar{N} = (\bar{V}, \bar{A})$; $\bar{V} = V$, $\bar{A} = A$, and the length of arc$(i, j) \in \bar{A}$ is $\Phi_{ij}^{-1}(0.95)$. We employ the Dijkstra algorithm to get the shortest path in network $\bar{N}$; that is,

$$\bar{P}_0 : 1 \to 2 \to 6 \to 8,$$

Table 3
List of α-shortest paths.

α	α-shortest path	$\Psi^{-1}(\alpha)$, T_0
0.95	$1 \to 2 \to 6 \to 8$	31.8
0.90	$1 \to 2 \to 6 \to 8$	31.6
0.80	$1 \to 2 \to 6 \to 8$	31.2
0.70	$1 \to 2 \to 6 \to 8$	30.8
0.60	$1 \to 2 \to 6 \to 8$	30.4
0.50	$1 \to 3 \to 5 \to 8$	30
0.40	$1 \to 4 \to 7 \to 8$	29.4
0.30	$1 \to 3 \to 5 \to 8$	28.8
0.20	$1 \to 3 \to 5 \to 8$	28.2

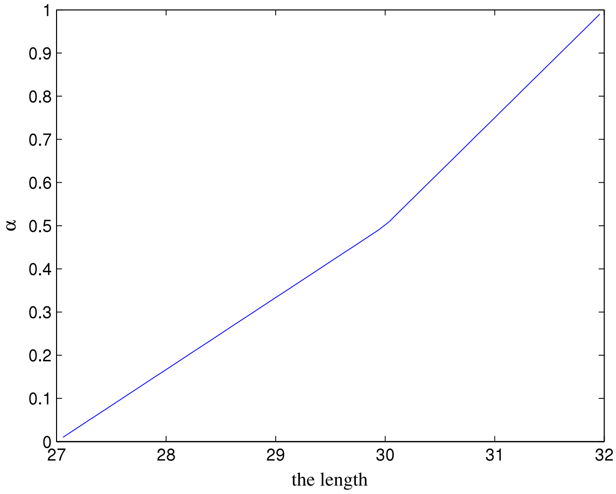

Fig. 6. Uncertainty distribution of f_{SP}.

and the length of $\bar{P}_0$ is 31.8. According to Theorem 4, the 0.95-shortest path in uncertain network $N = (V, A, \xi)$ is

$$P_0 : 1 \to 2 \to 6 \to 8,$$

and $\Psi^{-1}(0.95) = 31.8$.

Choosing different α, we obtain Table 3.

Given a length $T_0 = 30$, from Table 3, we get $\Psi(30) = 0.5$. Then Theorem 5 tells us that the most shortest path is $1 \to 3 \to 5 \to 8$.

Repeating this process, we obtain the uncertainty distribution of f_{SP} in a numerical sense, which is drawn by MATLAB in Fig. 6.

Generally speaking, we can obtain the α-shortest path in uncertain network $N = (V, A, \xi)$ by using the following three steps. *Step* 1: Calculate $\Phi_{ij}^{-1}(\alpha)$, for each arc $(i, j) \in A$. *Step* 2: Construct a deterministic network $\bar{N} = (\bar{V}, \bar{A})$; $\bar{V} = V, \bar{A} = A$, and the length of arc$(i, j) \in \bar{A}$ is $\Phi_{ij}^{-1}(\alpha)$. *Step* 3: Employ the Dijkstra algorithm to get the shortest path in network $\bar{N}$.

The path we obtain in Step 3 is just the α-shortest path we want. Repeating this process gives the uncertainty distribution of f_{SP} and the most shortest path at given length T_0.

8. Conclusion

Nondeterministic factors often appear in programming problems. In the past, probability or fuzzy theory has been employed to deal with these nondeterministic factors. Uncertainty theory provides a new approach to deal with nondeterministic factors. In this paper, we have investigated the uncertainty distribution of the shortest path length, the α-shortest path, and the most shortest path in networks with uncertain arc lengths.

Under the framework of uncertainty theory, the uncertainty distribution of the shortest path length is derived, and it is proved that there exists an equivalence relation between the α-shortest path of an uncertain network and the shortest path of the corresponding deterministic network. This equivalence relation leads to a stable and global optimal method to find the α-shortest path. As a dual problem to the α-shortest path, we can transform the most shortest path problem to an α-shortest path problem, and then solve it.

Acknowledgments

This work was supported by the National Natural Science Foundation of China Grant Nos. 60874067 and 91024032.

References

[1] E. Bellman, On a routing problem, Quarterly of Applied Mathematics 16 (1) (1958) 87–90.
[2] E.W. Dijkstra, A note on two problems in connection with graphs, Numerical Mathematics 1 (1) (1959) 269–271.
[3] S. Dreyfus, An appraisal of some shortest path algorithms, Operations Research 17 (3) (1969) 395–412.
[4] R.W. Floyd, Algorithm-97-shortest path, Communications of the ACM 5 (6) (1962) 345.
[5] H. Frank, Shortest paths in probability graphs, Operations Research 17 (4) (1969) 583–599.
[6] R. Hall, The fastest path through a network with random time-dependent travel time, Transportation Science 20 (3) (1986) 182–188.
[7] P. Loui, Optimal paths in graphs with stochastic or multidimensional weights, Communications of the ACM 26 (9) (1983) 670–676.
[8] P.B. Mirchandani, Shortest distance and reliability of probabilistic networks, Computers & Operations Research 3 (4) (1976) 347–676.
[9] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980.
[10] C.M. Klein, Fuzzy shortest paths, Fuzzy Sets and Systems 39 (1) (1991) 27–41.
[11] X. Ji, K. Iwamura, New models for shortest path problem with fuzzy arc lengths, Applied Mathematical Modelling 31 (2) (2007) 259–269.
[12] K. Lin, M. Chen, The fuzzy shortest path problem and its most vital arcs, Fuzzy Sets and Systems 58 (3) (1994) 343–353.
[13] S. Okada, Fuzzy shortest path problems incorporating interactivity among paths, Fuzzy Sets and Systems 142 (3) (2004) 335–357.
[14] B. Liu, Uncertainty Theory, 2nd ed., Springer-Verlag, Berlin, 2007.
[15] W. Liu, Uncertain programming models for shortest path problem with uncertain arc lengths, in: Proceedings of the First International Conference on Uncertainty Theory, Urumchi, China, August 11–19, 2010, pp. 148–153.
[16] B. Liu, Fuzzy process, hybrid process and uncertain process, Journal of Uncertain Systems 2 (1) (2008) 3–16.
[17] X.W. Chen, B. Liu, Existence and uniqueness theorem for uncertain differential equations, Fuzzy Optimization and Decision Making 9 (1) (2010) 69–81.
[18] B. Liu, Uncertain set theory and uncertain inference rule with application to uncertain control, Journal of Uncertain Systems 4 (2) (2010) 83–98.
[19] B. Liu, Theory and Practice of Uncertain Programming, 2nd ed., Springer-Verlag, Berlin, 2009.
[20] X. Gao, Y. Gao, D.A. Ralescu, On Liu's inference rule for uncertain systems, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 18 (1) (2010) 1–11.
[21] B. Liu, Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty, Springer-Verlag, Berlin, 2010.
[22] Y. Zhu, Uncertain optimal control with application to a portfolio selection model, Cybernetics and Systems 41 (7) (2010) 535–547.
[23] B. Liu, Some research problems in uncertainty theory, Journal of Uncertain Systems 3 (1) (2009) 3–10.