Convergence and Extraction of Bounded Sequences in \( L^1(\mathbb{R}) \)

Heinz-Albrecht Klei

Département de Mathématiques, Université de Reims, Moulin de la Housse, B.P. 347, 51062 Reims Cedex, France

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We present several applications of H. P. Rosenthal's subsequence splitting lemma: Each bounded sequence in \( L^1(\mathbb{R}) \) admits a subsequence satisfying the conclusion of a generalized Fatou's lemma and presenting a concentration of mass phenomenon. We show that the modulus of uniform integrability of a bounded sequence in \( L^1(\mathbb{R}) \) plays a capital role in the convergence in measure of such a sequence. A Cauchy type theorem for the convergence in measure is established. Finally we study the existence of minima of the \( L^1 \)-norm on closed convex subsets of \( L^1(\mathbb{R}) \).

1. INTRODUCTION

The biting lemma (Corollary 2.5) is a subject of increasing interest. It has been proved among others by Gaposhkin [5], Brooks and Chacon [4], and Slaby [13]. It can be derived from H. P. Rosenthal's subsequence splitting lemma [3, 14], which contains some additional information and which is the main tool of the present paper. For some recent applications of the biting lemma we refer to Saadoune and Valadier [12]. Most of the results presented here were announced in [7].

Throughout this paper, \( (\Omega, \Sigma, P) \) will be a fixed probability space. We will consider the Banach space \( L^1(E) \) of Bochner-integrable functions built over the probability space \( (\Omega, \Sigma, P) \) and a separable Banach space \( E \). We are frequently concerned with the case \( E = \mathbb{R} \).

In [14] H. P. Rosenthal defined the modulus of uniform integrability \( \eta(H) \) of a bounded subset \( H \subseteq L^1(\mathbb{R}) \): For \( \varepsilon > 0 \), put

\[
\eta(H, \varepsilon) = \sup \left\{ \int_A |h| \, dP : h \in H, A \in \Sigma, P(A) \leq \varepsilon \right\},
\]

\[
\eta(H) = \lim_{\varepsilon \to 0} \eta(H, \varepsilon).
\]

Thus \( H \) is uniformly integrable if and only if \( \eta(H) = 0 \).
If \((x_n)\) is a sequence of real numbers, then \(x \in \mathbb{R}\) belongs to \(L^s(x_n)\) if and only if there is a subsequence of \((x_n)\) that converges to \(x\). For a subset \(A \subseteq \mathbb{R}\), \(\text{co}(A)\) denotes its convex hull and \(\chi_A\) its characteristic function.

2. RESULTS AND THEIR PROOFS

We start with H. P. Rosenthal’s subsequence splitting lemma [14]. See also [3, p. 68].

**Lemma 2.1.** Let \(f = (f_n)\) be a bounded sequence in \(L^1(\mathbb{R})\). Then there exist a subsequence \((f'_n)\) of \((f_n)\) and a sequence \((A_n)\) of pairwise disjoint measurable sets such that

1. \(\lim_{n \to \infty} \int_{A_n} |f'_n| \, dP = \eta(f)\);
2. the sequence \((\chi_{\Omega \setminus A_n}, f_n)\) converges weakly in \(L^1(\mathbb{R})\).

**Lemma 2.2.** Let \(E\) be a separable Banach space and \(f = (f_n)\) be a bounded sequence in \(L^1(E)\) which converges in measure to an element \(f_\infty \in L^1(E)\). Then the following assertions are equivalent:

1. the sequence \((\|f_n\|_E)\) converges;
2. \(\eta(f') = \eta(f)\) for each subsequence \(f'\) of \(f\);
3. there are subsequences \(f' = (f'_n)\) and \(f'' = (f''_n)\) such that

\[
\lim_{n \to \infty} \|f'_n\|_1 = \lim_{n \to \infty} \|f_n\|_1,
\]

\[
\lim_{n \to \infty} \|f''_n\|_1 = \lim_{n \to \infty} \|f_n\|_1,
\]

\[\eta(f') = \eta(f'');\]

4. \(\lim_{n \to \infty} \|f_n\|_1 = \eta(f') + \int \|f_n\| \, dP\) for each subsequence \(f'\) of \(f\).

**Proof.** We simply apply Lemma 1 of [7] to the sequence \((\|f_n\|)\).

**Theorem 2.3.** Let \(E\) be a separable Banach space and \(f = (f_n)\) a bounded sequence in \(L^1(E)\) that converges in measure to \(f_\infty \in L^1(E)\). Then \((f_n)\) converges in norm if and only if the condition \(\lim\sup_{n \to \infty} \|f_n\|_1 \leq \|f_\infty\|_1\) holds.

**Proof.** Suppose that \(\lim\sup_{n \to \infty} \|f_n\|_1 \leq \|f_\infty\|_1\). From each subsequence \(f'\) of \(f\) we extract another subsequence \(f'' = (f''_n)\) such that \((\|f''_n\|_1)\) converges. We deduce from Lemma 2.2 that \(\eta(f'') = 0\). It is well known that a
measure convergent uniformly integrable sequence is norm convergent. See, for example, [11, Proposition 4.7.5].

Remarks. The preceding theorem is due to Hewitt and Stromberg [6, Theorem 13.47] in the real case. The case $E = \mathbb{R}^n$ is a recent result obtained by Saadoune and Valadier [12, Theorem 4].

**Theorem 2.4.** Let $(f_n)$ be a bounded sequence in $L^2(\mathbb{R})$. Then there exist a subsequence $f' = (f'_n)$, a function $f_\infty \in L^2(\mathbb{R})$, and a sequence $(A_n)$ of pairwise disjoint measurable sets such that for every $A \in \Sigma$ and for every subsequence $f'' = (f''_n)$ of $f'$, we have

1. $\lim_{n \to \infty} \int_A f_n^* dP = \lim_{n \to \infty} \int_{A \cap A_n} f_n^* dP + \int_A f_n dP \leq \eta(\chi_A f^*) + \int_A f_d P$;

2. $\lim_{n \to \infty} \int_A f_n^* dP = \lim_{n \to \infty} \int_{A \cap A_n} f_n^* dP + \int_A f_n dP \geq \eta(\chi_A f^*) + \int_A f_d P$ provided that $\lim_{n \to \infty} \int_{A \cap A_n} f_n^* dP = \lim_{n \to \infty} \int_A f_n^* dP$;

3. $f_\infty(\omega) \in \text{co}(\text{LS}(f'_n(\omega))) \text{ P-a.e.}$;

4. the sequence $C(f''_n) = ((1/n) \sum_{k=1}^n f''_k)_n$ converges P-a.e. to $f_\infty$;

5. $\eta(f^*) = \eta(f) = \eta(C(f''))$;

6. the equivalence of the following statements:
   (i) $\lim_{n \to \infty} \int f''_n dP \leq \int f_\infty dP$;
   (ii) $\eta(f) = 0$;
   (iii) the sequence $f''$ converges weakly to $f_\infty$;
   (iv) the sequence $C(f'')$ converges in norm to $f_\infty$.

Proof. We choose a subsequence $f' = (f'_n)$ of $(f_n)$ and a disjoint sequence $(A_n)$ of measurable sets as in Lemma 2.1. Fix a measurable set $A \in \Sigma$. Let $f_n$ be the weak limit of the sequence $(\chi_{A \setminus A_n} \cdot f'_n)$. It is now easy to verify the equalities in (1) and (2). We know from the proof of the subsequence splitting lemma that $\lim_{n \to \infty} \int_{A \setminus A_n} f'_n dP \leq \eta(\chi_A f')$. Thus the proof of assertion (1) is complete.

Let $f'' = (f''_n)$ be a subsequence of $(f'_n)$ such that $\lim_{n \to \infty} \int_A f''_n dP = \lim_{n \to \infty} \int_{A \cap A_n} f''_n dP$. Note that $\lim_{n \to \infty} \int_A f''_n dP = \lim_{n \to \infty} \int_{A \cap A_n} f''_n dP + \int_A f_n dP$. Applying Lemma 2.2 to the sequence $(\chi_{A \cap A_n} \cdot f''_n)$, we obtain $\lim_{n \to \infty} \int_{A \cap A_n} f''_n dP = \eta(\chi_{A \cap A_n} f''_n)$. It is not difficult to see that the last term is equal to $\eta(\chi_A f''_n)$. The proof of (2) is finished.

Note that $(\chi_{A_n} f''_n)$ converges in measure to 0. Without loss of generality we may assume that it converges P-almost everywhere to 0. It follows that $\operatorname{LS}(f'_n(\omega)) = \operatorname{LS}(\chi_{A_n} \cdot f'_n)$. As $(\chi_{A_n} \cdot f'_n)$ converges weakly to $f_\infty$, we know, e.g., from [8, Proposition 1], that $f_\infty(\omega) \in \text{co}(\operatorname{LS}(\chi_{A_n} \cdot f'_n)) \text{ P-a.e.}$

The combination of Lemma 2.1 with Komlós' theorem [9] allows us to prove (4). There is an element $g \in L^1(\mathbb{R})$ and a subsequence of $(f'_n)$, still
denoted by \((f'_n)\), such that for each further subsequence \((f''_n)\) of \((f'_n)\) the sequence \(C(f'')\) converges \(P\)-a.e. to \(g\). It is not hard to see that the equality \(g = f_n\) holds \(P\)-a.e. Applying Lemma 2.2 to \(C(f'')\), we obtain
\[
\lim_{n \to \infty} \int f'_n \, dP = \eta(C(f'')) + \int f_n \, dP.
\]
It follows from the subsequence splitting lemma that
\[
\lim_{n \to \infty} \int f'_n \, dP = \eta(f') + \int f_n \, dP.
\]
Hence \(\eta(f) \geq \eta(f'') \geq \eta(C(f'')) = \eta(f)\).
The statement (6) follows from Lemma 2.1 and assertions (1)--(5).

The following result is known as the biting lemma.

**Corollary 2.5.** Let \((f_n)\) be a bounded sequence in \(L^1(\mathbb{R})\). Then there exist a subsequence \((f'_n)\) of \((f_n)\) and an integrable selection \(f_n\) of \(\text{co}(Ls(f'_n(\omega)))\) such that the set
\[
H = \{ A \in \Sigma : (\chi_A f'_n) \text{ converges weakly to } (\chi_A f_n) \}
\]
contains, for each \(\varepsilon > 0\), a measurable set \(A_\varepsilon\) with \(P(A_\varepsilon) \geq 1 - \varepsilon\).

**Proof.** Without loss of generality we may assume that the functions \(f_n\) have positive values. Let the notation be the same as in Theorem 2.4. It is easy and sufficient to check that \(H\) contains all the \(A_n\)'s and \((\bigcup_{n \in \mathbb{N}} A_n)'\), where \(c\) stands for complement.

Another application of Theorem 2.4 yields the generalized Fatou's lemma obtained in [8].

**Theorem 2.6.** Let \(f = (f_n)\) be a bounded sequence in \(L^1(\mathbb{R}_+)\). If the sequence \((\int f_n \, dP)\) converges in \(\mathbb{R}_+\), then
\[
\lim_{n \to \infty} \int f_n \, dP \geq \eta(f) + \int \lim_{n \to \infty} f_n \, dP.
\]

**Proof.** Combine assertions (2) and (3) of Theorem 2.4.

**Theorem 2.7.** Let \(f = (f_n)\) be a bounded sequence in \(L^1(\mathbb{R})\) and \(f_n \in L^1(\mathbb{R})\). If \(\lim_{n \to \infty} \| f_n - f \|_1 \leq \eta(f)\), then there exists a subsequence \(f' = (f'_n)\) of \((f_n)\) converging in measure to \(f_n\) such that
\[
\lim_{n \to \infty} \| f'_n - f \|_1 = \eta(f) = \inf(\eta(f'')): f'' \text{ subsequence of } f'.
\]

**Proof.** We proceed as in the proof of Theorem 2.4 which we apply to the sequence \((|f_n - f|)\). Thus we can choose a subsequence \(f' = (f'_n)\) of \((f_n)\) and a sequence of pairwise disjoint measurable sets \((A_n)\) such that \((\chi_{\mathbb{R} \setminus A_n} |f'_n - f_n|)\) converges weakly to an element \(h\) belonging to \(L^1(\mathbb{R}_+)\).
Note that $\eta(f) = \eta(f') = \eta(f'_n - f_n) = \eta(f_n - f_n)$. Assertion (2) of Theorem 2.4 says that $\lim_{n \to \infty} ||f'_n - f_n||_1 \geq \eta(f') + \int h d\mu$. Combine the last inequality with the hypothesis to see that $\lim_{n \to \infty} ||f'_n - f_n||_1 = \eta(f')$ and $h = 0$ $\mu$-a.e. It follows that $(\chi_{A_n} f'_n - f_n)$ converges in norm to 0. Consequently $(f'_n - f_n)$ converges in measure to 0. We know from assertion (5) of Theorem 2.4 that $\eta(f'_n) = \eta(f)$ for each subsequence $f'_n$ of $f$. This completes the proof.

**Corollary 2.8.** Let $f = (f_n)$ be a bounded sequence in $L^1(\mathbb{R})$ and $f_n \in L^1(\mathbb{R})$.

(a) The following assertions are equivalent:

(i) $(f_n)$ converges in measure to $f_n$ and the sequence of reals $(||f'_n - f_n||_1)$ converges.

(ii) $\lim_{n \to \infty} ||f'_n - f_n||_1 \leq \inf \eta(f'_n): f'_n$ subsequence of $f$.

(iii) $\lim_{n \to \infty} ||f'_n - f_n||_1 = \eta(f')$ for each subsequence $f'_n$ of $f$.

(b) Assume that $\eta(f) = \inf \eta(f'_n): f'_n$ subsequence of $f$ and that the sequence of reals $(||f'_n - f_n||_1)$ converges. If $(f_n)$ has a measure convergent subsequence, then $(f_n)$ converges in measure to $f_n$.

**Theorem 2.9.** Let $f = (f_n)$ be a bounded sequence in $L^1(\mathbb{R})$ such that

$$\lim_{n, m \to \infty} ||f'_n - f'_m||_1 \leq 2 \eta(f).$$

Then there exists a measure convergent subsequence $(f'_n)$ of $(f_n)$ such that

$$\lim_{n, m \to \infty} ||f'_n - f'_m||_1 = 2 \eta(f).$$

**Proof.** We apply Rosenthal’s lemma to the sequence $(f_n)$ and choose $(f'_n)$ and $(A_n)$ as in Lemma 2.1. For each $n \in \mathbb{N}$, we define two functions $u_n$ and $d_n$ as

$$u_n = \chi_{\mathbb{R} \setminus A_n} f'_n, \quad d_n = f'_n - u_n.$$ 

We then get the equality

$$|\chi_{A_n \cup A_n} (f'_n - f'_m) - (d_n - d_m)| = |\chi_{A_n} u_n - \chi_{A_n} u_m|.$$ 

Remember that $(u_n)$ is uniformly integrable and that $(P(A_n))$ converges to 0. Therefore we have

$$\lim_{n, m \to \infty} \int |\chi_{A_n} u_m - \chi_{A_n} u_n| d\mu = 0.$$
Note that \( \lim_{n,m \to \infty} \int |d_n - d_m| \, dP = 2\eta(f) \). Summarizing we obtain

\[
\lim_{n,m \to \infty} \int |\chi_{A_n \cup A_m}(f'_n - f'_m)| \, dP = 2\eta(f). \tag{\ast}
\]

In order to finish the proof, it is sufficient to show that \((u_n)\) is a norm Cauchy sequence. Note that

\[
|u_n - u_m| = \chi_{A_n}|f'_n| + \chi_{A_m}|f'_m| + |f'_n - f'_m| - \chi_{A_n \cup A_m}|f'_n - f'_m|.
\]

Hence \( \|u_n - u_m\| = \int \chi_{A_n}|f'_n| \, dP + \int \chi_{A_m}|f'_m| \, dP + \|f'_n - f'_m\|_1 - \int \chi_{A_n \cup A_m}|f'_n - f'_m| \, dP \). Let us consider the four sequences on the right side of the previous equality. The first two converge to 0 when \( n \) and \( m \) tend to infinity, \( n \neq m \). By virtue of the hypothesis we may assume that the third sequence converges to a point less than or equal to \( 2h \). The last sequence converges to \( 2h \) by (\ast).

**Corollary 2.10.** Let \( f = (f_n) \) be a bounded sequence in \( L^2(\mathbb{R}) \) such that \( \lim_{n,m \to \infty} \|f_n - f_m\|_1 \) exists. Then \((f_n)\) has a measure convergent subsequence if and only if

\[
\lim_{n,m \to \infty} \|f_n - f_m\|_1 = 2\eta(f).
\]

**Theorem 2.11.** Let \( C \) be a convex subset of \( L^2(\mathbb{R}_+) \) closed with respect to the topology of convergence in measure. Suppose that \((f_n)\) is a sequence in \( C \) such that

\[
\lim_{n \to \infty} \|f_n\|_1 = \inf\{\|g\|_1 : g \in C\}.
\]

Then \((f_n)\) is uniformly integrable. Furthermore \((f_n)\) admits a subsequence that converges weakly to an element \( f \in C \) such that \( \|f\|_1 = \inf\{\|g\|_1 : g \in C\} \).

**Proof.** Theorem 2.4 yields a subsequence \( f' \) of \( f \) such that the sequence \( C(f') \) of its Cesaro means converges \( P\)-a.e. to an element \( f \in L^1(\mathbb{R}_+) \). Note that \( f \) belongs to \( C \). By virtue of the hypothesis and Lemma 2.2 we get

\[
\inf\{\|g\|_1 : g \in C\} = \lim_{n \to \infty} \|f_n\|_1 = \eta(C(f')) + \|f\|_1.
\]

It follows that \( \eta(C(f')) = 0 \). Assertion (5) of Theorem 2.4 says that \( \eta(C(f')) = \eta(f) = 0 \). An application of assertion (6) of the same theorem completes the proof.
Remarks. The existence of the minimum in the previous theorem is due to Levin [10]. Balder [1] proved Levin's theorem by means of Komlós' theorem. He applied the latter to the study of weak compactness in $L^2$ spaces [2].

Corollary 2.12. Let $C$ denote a convex non-void subset of $L^2(\mathbb{R}_+)$ which is closed with respect to the topology of convergence in measure. Then the set

$$C_{\min} = \{ g \in C : \|g\|_1 = \inf \{ \|h\|_1 : h \in C \} \}$$

is convex, non-void, and weakly compact.

REFERENCES

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