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The Reproducibility of Multivariable Systems*

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INTRODUCTION

When studying large complex systems it is important to be able to characterize a given system by referring to the existence of certain properties which reveal, in a simple way, the potential capabilities and limitations of the system. One such property which is relevant for a broad class of control problems is that of reproducibility as it is discussed herein.

Briefly described, reproducibility refers to the ability of a system to achieve, with its outputs, something which is desired of it. For example, functional reproducibility refers to the capabilities of a system with respect to the generation of specified time functions; asymptotic reproducibility refers to the possibility of approaching a desired behavior with increasing time; pointwise reproducibility refers to the possibility of achieving a desired value of the outputs at some one point in time.

The analytic results presented in this paper refer to systems which can be described by a pair of equations of the form $\dot{Z}(t) = F(Z(t), X(t))$; $Y(t) = CZ(t)$; where the input is X, the output is Y, and Z is the state. The basic idea of reproducibility, however, is more easily motivated if the system is thought of as an arbitrary transformation which maps the input time-functions into responses. Such a transformation might be written as $T : X \rightarrow Y$. Normally X will not be arbitrary but will be restricted to belong to a given set S_x. Likewise, not every time-function will be a desired response. Denote the set of desired responses by S_y. A system will be said to be reproducible if for any given time-function Y in S_y there is an X in S_x which generates it. Otherwise stated, a system having an input set S_x, a transformation T, and a desired response set S_y will be said to be reprodu-

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cible if there exists an inverse for the transformation T which maps S_y into S_x.

In this paper four different types of reproducibility are defined and the conditions under which various types of linear and nonlinear systems have reproducible responses are derived. Both state variable and transfer matrix representations are used and the relationship between reproducibility and stability is discussed.

Background

The generic problem which leads to the study of reproducibility is this. Given a system, a set of responses which one would like the system to have, and a set of possible inputs, is it possible to generate any output in the desired set using the given set of inputs?

The problem of determining the limitations which the equations of motion impose on the control of a physical process has been studied from many points of view. Thus the works of Smith [1], Eckman [2], Kalman [3], and Antosiewicz [4], to name a few, all contain different approaches to this problem, usually with the term controllable being applied to systems which have some desired property. The term output-controllability was apparently first used by Bertram and Sarachik [5]. In fact, their definition of output controllability is closely related to what is called pointwise reproducibility here.

The work of Marcus and Lee [6] on the existence of optimal controls also treats certain aspects of the problem under consideration here as do the papers of Kalman, Ho, and Narendra [7] and Roxin [8].

The most common definition of controllability in present use refers to a property of the state of the system. The concept of reproducibility used here refers to requirements which are placed on the outputs and although certain of these are similar to those frequently imposed on the state, others are entirely different as, e.g., functional reproducibility. Actually, the approach to reproducibility taken here is an outgrowth of the earlier studies on interaction [9, 10] in multivariable systems. Systems which are not reproducible correspond to those which have been referred to as having unit interaction. For more details see [12].

Notation and Preliminaries

Lower case letters refer to scalars, upper case to vectors and matrices. If A is an arbitrary vector or matrix then A^T is its transpose, if A is square and nonsingular A^{-1} denotes its inverse. The ith element of a vector X will be written as x_i, the ijth element of a matrix A will be written as a_{ij}. If A_1, A_2, $\cdots$ A_n are a set of matrices or vectors all having the same number of rows then $(A_1, A_2, \cdots A_n)$ denotes the matrix consisting of all the columns of $A_1, A_2, \cdots A_n$. Thus, if the A_i are p by p matrices then the matrix $(A_1, A_2, \cdots, A_n)$ is p by np. For scalars $|a|$ denotes the absolute value, for vectors and matrices $|A|$ denotes the sum of the absolute values of the elements of A. The determinant of A will be written as det A.

To avoid confusion it is imperative to distinguish between a function, that is, a complete description of how the variable varies with time and the value of a function at some particular time t. Symbols such as X, Y, and Z, written without arguments, will be used to denote functions and $X(t)$, $Y(t)$, and $Z(t)$ will denote the values of the functions at time t.

If X is a k-times differentiable function of time then $X^{(k)}$ denotes its kth derivative. Two types of norms will be used. Define $\| X \|$ as

$$\| X \| = \sup_{t \in [0,\infty)} | X(t) | \tag{1}$$

and define

$$\| X \|_p = \max_{0 \leqslant k \leqslant p} \| X^{(k)} \| \tag{2}$$

It is assumed throughout that the inputs and outputs of the system are related by a pair of equations of the form

$$\dot{Z}(t) = F(Z(t), X(t)) \tag{3a}$$

$$Y(t) = CZ(t) \tag{3b}$$

The vectors X, Y, and Z are m, n, and p-dimensional, respectively, and F and X are assumed to be sufficiently smooth to insure that Eq. (3a) has a unique solution corresponding to each initial value β of $Z(t)$. Moreover, it is assumed that F does not depend explicitly on time. The solution of (3a) which results from a forcing function X being applied at $t = 0$ when the state is β will be written as $Z(\beta, X, t)$.

As it is to be studied here, reproducibility refers to the attainability of solutions which are near an unforced solution. However, it is always possible to make a change of variables which allows one to regard a given response as being unforced. That is, if $Y' = CZ(\beta, X', t)$ is a forced response then define W as $X' - X$. Notice that the equations of motion when written in terms of W are

$$\dot{Z}(t) = F'(Z(t), W(t)) \tag{4a}$$

$$Y(t) = CZ(t) \tag{4b}$$

where now Y' is an unforced solution. It is true, however, that the equation (4a) may be time-varying even though the original system was not.

Principal Definitions

To avoid complications at this stage, it will be assumed that the system can be described by the time-invariant equations (3a) and (3b). The question of reproducibility concerns the attainability of some small deviation from a known achievable response. From the previous remarks it follows that without loss of generality, it may be assumed that the known achievable response is a homogeneous response.

Perhaps the least restrictive concept of reproducibility is this: Let $Y'(t) = CZ(\beta, 0, t)$ be a homogeneous response and let $Y(t)$ be any slightly different response. Then if there exists an X which is small and which produces a response $CZ(\beta, X, t)$ which agrees with the desired response $Y(t)$ at least one point in time, the system will be said to be pointwise reproducible. This type of reproducibility is, for example, appropriate for certain types of rendezvous problems. The precise statement is this.

Definition 1. The homogeneous response from an initial state β is said to be pointwise reproducible if for any $\eta > 0$ and $\tau > 0$ there exists a $\delta(\eta, \tau) > 0$ such that corresponding to each Y for which

$$\| Y - CZ(\eta, 0, t) \| < \delta(\eta, \tau)$$

there is an X having the properties: $\| X \| < \eta$; and, $CZ(\eta, X, t) = Y(t)$ for one or more values of t in the interval $[0, \tau]$.

This is not a very strong requirement to impose and many systems have this property and yet are not "controllable" in a practical sense.

Example 1. Consider the one input, two output system. The equations of motion are:

$$\dot{z}_1 = - z_1 + x_1; \qquad y_1 = z_1 \tag{5a}$$

$$\dot{z}_2 = - 2z_2 + x_1; \qquad y_2 = z_2 \tag{5b}$$

It is easily shown that this system is pointwise reproducible even though $y_1(t)$ and $y_2(t)$ are related by the equation

$$\dot{y}_1 + y_1 = \dot{y}_2 + 2y_2 .$$

In most engineering applications one is interested in a stronger type of reproducibility since even systems whose outputs are closely coupled can be pointwise reproducible. One way of strengthening this definition is to require that the actual response and the desired response agree over a finite interval. That is, if the homogeneous response of the system (3) is $CZ(\beta, 0, t)$ and if $Y(t)$ is sufficiently close to $CZ(\beta, 0, t)$, there should be an X which is suitably small and which causes the response of the system to equal $Y(t)$. Here "close" will be interpreted in the sense of the norm defined by Eq. (2) with p taken to be equal to the dimension of $Z(t)$. The reason for choosing this norm is to avoid the necessity of inputs containing impulse functions. The definition to be used here is as follows.

DEFINITION 2. The homogeneous response from an initial state β is said to be *functionally reproducible* if for any $\eta > 0$ and finite $\tau > 0$ there exists a $\delta(\eta, \tau) > 0$ such that corresponding to each Y for which

$$\| Y - CZ(\beta, 0, t) \|_p < \delta(\eta, \tau)$$

there is an X having the properties: $\| X \| < \eta$; and, $CZ(\beta, X, t) = Y(t)$ for all values of t in the interval $[0, \tau]$.

Functional reproducibility, like pointwise reproducibility, is a local concept since only the behavior of the system in a small neighborhood of a known solution is discussed. In general, functional reproducibility implies pointwise reproducibility as may be easily seen.

The concept of uniform functional reproducibility is also of interest. Here it is required that it be possible to find the δ in Definition 2 such that it depends on η only and not on τ.

DEFINITION 3. The homogeneous response from the initial state β is said to be *uniformly functionally reproducible* if for any $\eta > 0$ and all $\tau > 0$ there exists a $\delta(\eta) > 0$ such that corresponding to each Y for which

$$\| Y - CZ(\beta, 0, \tau) \|_p < \delta(\eta)$$

there is an X having the properties: $\| X \| < \eta$; $CZ(\beta, X, t) = Y(t)$ for all t in the interval $[0, \tau]$.

EXAMPLE 2. Consider the second order system whose equations of motion are

$$\dot{z}_1 = -3z_1 + 2x; \qquad \dot{z}_2 = -z_2 - x \tag{6a}$$

$$y = z_1 + z_2 \tag{6b}$$

This system is functionally reproducibile about any homogeneous solution since for any given $y(t)$ such that $\| y(t) - \beta_1 e^{-3t} + \beta_2 e^{-t} \|_2$ is small, it is possible to find an $x(t)$ such that $CZ(\beta, x, t) = y(t)$. In fact, x is given by

$$x(t) = e^{+t} \int_0^t e^{-s}(\ddot{y}(s) + 4\dot{y}(s) + 3y(s))\, ds \tag{7}$$

This system is not uniformly functionally reproducible for if

$$\int_0^{\infty} e^{-s}(\ddot{y}(s) + 4\dot{y}(s) + 3y(s))\, ds$$

is not zero then for any given η and $\delta(\eta)$ it is possible to choose τ so large that the $x(t)$ which produces $y(t)$ on $[0, \tau]$ exceeds $\delta(\tau)$.

A fourth definition of reproducibility, which is of interest in the study of systems which are intended to operate over large periods of time, is that of asymptotic reproducibility. The idea here is that a system should be reproducible in the sense that it may be brought to a new steady-state value by using an input which is small. In general this is a weaker requirement than uniform functional reproducibility.

DEFINITION 4. The homogeneous response from the initial state β is said to be *asymptotically reproducible* if for any $\eta > 0$ there exists a $\delta(\eta) > 0$ such that corresponding to each Y for which

$$\lim_{t\to\infty} | Y(t) - CZ(\beta, 0, t) | < \delta(\eta)$$

there is an X having the properties: $\| X \| < \eta$; and,

$$\lim_{t\to\infty} | CZ(\beta, X, t) - Y(t) | = 0.$$

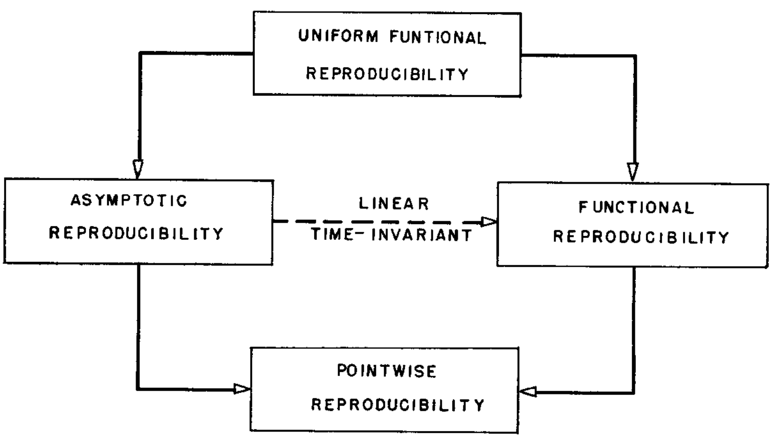

FIG. 1. The relationship between Definitions 1-4

The relationships between these definitions are shown in Fig. 1. It is obvious that uniform functional reproducibility implies both functional reproducibility and asymptotic reproducibility. For linear, time-invariant systems, asymptotic reproducibility implies functional reproducibility but this is not true for nonlinear systems.

EXAMPLE 3. Consider the one-input, two-output, nonlinear system. The equations of motion are

$$\dot{z}_1 = x_1; \qquad y_1 = z_1 \tag{8a}$$

$$\dot{z}_2 = (x_1)^3; \qquad y_2 = z_2 \tag{8b}$$

Clearly,

$$\lim_{t\to\infty} y_1 = \int_0^\infty x_1(t)\, dt \tag{9a}$$

$$\lim_{t\to\infty} y_2 = \int_0^\infty x_1^3(t)\, dt \tag{9b}$$

and by proper choice of X these limits can be made equal to any preassigned constants. Yet, this system is not functionally reproducible since y_1 and y_2 are related by the nonholonomic constraint.

$$(\dot{y}_1)^3 = \dot{y}_2 \tag{10}$$

It should be noted that reproducibility and stability are not related. To see this consider the following example.

EXAMPLE 4. The null solution of the equation

$$\dot{z} = z + x; \qquad y = z \tag{11}$$

is clearly unstable since the homogeneous solutions are of the form βe^t, yet, for any given β and $\eta > 0$ there is a $\delta(\eta) > 0$ such that if $\| y - \beta e^t \|_1 < \delta$ then there is an x such that the response due to x equals y and $\| x \| < \eta$. To see this note that the x which generates y is given by

$$x = -(y - \beta e^t) - (\dot{y} - \beta e^t) \tag{12}$$

and hence if $\| y - \beta e^t \|_1$ is small then x is also. The relationship between stability and reproducibility will be examined in more detail in the next section.

Linear Time-Invariant Systems

Let X, Y, and Z be m, n, and p-dimensional vectors and suppose the equations relating X and Y are

$$\dot{Z}(t) = AZ(t) + BX(t) \tag{13a}$$

$$Y(t) = CZ(t) \tag{13b}$$

where A, B, and C are constant and $p \times p$, $p \times m$, and $n \times p$ respectively. It is well known that if X is a continuous function of time then for each initial value β of $Z(t)$ there is a unique solution of Eq. (13a) and that this solution is given by

$$Z(t) = e^{At}\beta + e^{At} \int_0^t e^{-As}BX(s)\, ds \tag{14}$$

Thus the relationship between $X(t)$ and $Y(t)$ may be expressed as

$$Y(t) = Ce^{At}\beta + Ce^{At} \int_0^t e^{-As}BX(s)\, ds \tag{15}$$

In addition to the time domain equations relating X and Y it is often convenient to make use of their frequency domain equivalents. Let $\mathbf{X}$ and $\mathbf{Y}$ denote the Laplace transforms of X and Y. Then $\mathbf{X}$ and $\mathbf{Y}$ are related by the vector equation

$$\mathbf{Y}(s) = C(Is - A)^{-1}\, B\mathbf{X}(s) + C(Is - A)^{-1}\beta \tag{16}$$

The $n \times m$ matrix $C(Is - A)^{-1} B$ is usually called the transfer matrix. Notice that for all values of s such that $(Is - A)$ is nonsingular $(Is - A)^{-1}$ is bounded and thus the poles of $C(Is - A)^{-1} B$ occur where $\det(Is - A) = 0$. Since the values of s which satisfy this equation are just the eigenvalues of A it follows that if A has eigenvalues with negative real parts then the poles of $C(Is - A)^{-1} B$ lie in the left half-plane. In view of the large body of control system theory which is based on the use of transfer functions and transfer matrices it is appropriate to interpret as many of conditions for reproducibility as possible in terms of it.

An $n \times m$ matrix such as $C(Is - A)^{-1} B$ whose elements are rational functions of a complex variable s will be said to be of rank n if there exists no nonzero row vector $K(s)$ such that $K(s)\, C(IS - A)^{-1} B$ vanishes identically for all s. This does not imply that $C(Is - A)^{-1} B$ is of rank n for all values of s but rather that $C(Is - A)^{-1} B$ is of rank n for all but a finite number of values of s.

The following theorem gives necessary and sufficient conditions for this system to be pointwise and functionally reproducible.

THEOREM 1. *Consider the system (13) with the given assumptions on A, B, and C. All homogeneous responses of this system are:*

(a) *pointwise reproducible if and only if the $n \times mp$ matrix $(CB, CAB, \cdots CA^{p-1}B)$ is of rank n;*

(b) *functionally reproducible if and only if the $np \times (2mp - m)$ matrix*

$$M_p = \begin{Vmatrix} CB & CAB & CA^2B & \cdots & CA^{p-1}B & CA^pB & \cdots & CA^{2p-1}B \\ 0 & CB & CAB & \cdots & CA^{p-2}B & CA^{p-1}B & \cdots & CA^{2p-2}B \\ 0 & 0 & CB & \cdots & CA^{p-3}B & CA^{p-2}B & \cdots & CA^{2p-3}B \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & CB & CAB & \cdots & CA^{p-1}B \end{Vmatrix} \tag{17}$$

is of rank np.

PROOF. (a) Pointwise reproducibility is very closely related to other types of reproducibility which have been studied. The proof of this part of the theorem is based on the work of Kalman [11]. Consider the $n \times n$ matrix $W(\tau)$ which is defined as

$$W(\tau) = \int_0^\tau (Ce^{-At}B)\,(Ce^{-At}B)^T\,dt. \tag{18}$$

Clearly, $W(\tau)$ is at least positive semidefinite since the integrand is positive semidefinite. Suppose there exists a constant, nonzero n-vector R such that $R^TW(\tau)\,R = 0$ for some $\tau > 0$. Then since $R^TCe^{At}B$ is an analytic function of t it follows that at $t = 0$

$$d/dt(R^TCe^{At}B) = d^2/dt^2(R^TCe^{At}B) = \cdots d^p/dt^p(R^TCe^{At}B) = 0 \tag{19}$$

therefore $R^T(CB, CAB, \cdots CA^{p-1}B)$ vanishes. But this is a contradiction since R is nonzero and $(CB, CAB, \cdots CA^{p-1}B)$ is of rank n. It follows that $W(\tau)$ must be positive definite for all $\tau > 0$ and hence that $W(\tau)$ is nonsingular.

For any given homogeneous solution $CZ(\beta, 0, t)$ and any desired response $Y(t)$ which is near $CZ(\beta, 0, t)$ it is possible to find an X such that $CZ(\beta, X, \tau) = Y(\tau)$ provided $\tau > 0$. Define $E(t)$ as $Y(t) - CZ(\beta, 0, t)$. If

$$X(t) = (Ce^{-At}B)^T\,(W(\tau))^{-1}\,E(\tau) \tag{20}$$

then clearly $CZ(\beta, X, \tau) = Y(\tau)$.

Suppose η and τ in Definition 1 are given. Then δ can be taken to be $\eta/mW(\tau)$ where m is the maximum value $|\,Ce^{At}B\,|$ in the interval $[0, \tau]$.

To show that the given condition is also necessary notice that if $(CB, CAB, \cdots CA^{p-1}B)$ is not of rank n then there exists a nonzero, n-vector K such that

$K^T(CB, CAB, \cdots CA^{p-1}B) = 0$. By the Cayley-Hamilton theorem this implies that $K^T Ce^{At}B = 0$ for all t and hence that for any choice of X

$$K^T(CZ(\beta, X, t) - CZ(\beta, 0, t)) = 0 \tag{21}$$

This shows that there exist points arbitrarily close to $CZ(\beta, 0, t)$ which cannot be reached by $CZ(\beta, X, t)$ regardless of the choice of X.

(b) Define E as above. For positive values of i, $X^{(i)}$ denotes the ith derivative of X. For negative values of i let $X^{(i)}$ be defined as

$$X^{(i)} = \int_0^t \frac{(t - s)^{-1-i}\, X(s)}{(-1 - i)!}\, ds \tag{22}$$

Expand e^{At} as $(I + At + A^2t^2/2! \cdots)$ and differentiate Eq. (15) successively to get

$$\begin{bmatrix} E^{(q-1)} \\ E^{(q-2)} \\ \vdots \\ E^{(1)} \\ E \end{bmatrix} = \begin{Vmatrix} CB & CAB & \cdots & CA^{q-1}B & CA^qB & \cdots \\ 0 & CB & \cdots & CA^{q-2}B & CA^{q-1}B & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ 0 & 0 & \cdots & CAB & CA^2B & \cdots \\ 0 & 0 & \cdots & CB & CAB & \cdots \end{Vmatrix} \begin{bmatrix} X^{(q-2)} \\ X^{(q-3)} \\ \vdots \\ X \\ X^{(-1)} \\ \vdots \end{bmatrix} \tag{23}$$

There exists one or more scalar differential equations of the form

$$K_q E^{(q-1)} + K_{q-1} E^{(q-2)} + \cdots K_1 E^{(1)} + K_0 E = 0 \tag{24}$$

if and only if the rank of the matrix in Eq. (23) is less than nq.

The Cayley-Hamilton theorem states that it is possible to express all powers of A in terms of a linear combination of the first $p - 1$ powers of A. From this it follows that the matrix in Eq. (23) is of rank nq if and only if the $nq \times m(p + q - 1)$ matrix M_q defined as

$$M_q = \begin{Vmatrix} CB & CAB & \cdots & CA^{q-1}B & CA^qB & \cdots & CA^{p+q-2} & CA^{p+q-1}B \\ 0 & CB & \cdots & CA^{q-2}B & CA^{q-1}B & \cdots & CA^{p+q-3} & CA^{p+q-2}B \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & CAB & CA^2B & \cdots & CA^{p-3}B & CA^{p-2}B \\ 0 & 0 & \cdots & CB & CAB & \cdots & CA^{p-2}B & CA^{p-1}B \end{Vmatrix} \tag{25}$$

is of rank nq.

It will now be shown that if q exceeds p then M_q is of rank nq if and only if M_p is of rank np. Let q exceed p; then M_q contains the rows of M_p and hence M_p being of rank np is a necessary condition for M_q to be of rank nq. An inductive proof will be used to show that this condition is also sufficient.

Assume $s^p + \alpha_{p-1}s^{p-1} + \cdots \alpha_0$ is the characteristic polynomial of the matrix A and define a $nq \times nq$ matrix T_q as

$$T_q = \begin{Vmatrix} I & \alpha_{p-1}I & \alpha_{p-2}I & \cdots & \alpha_0 I & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & I \end{Vmatrix} \tag{26}$$

Clearly T_q is nonsingular and hence T_qM_q is of rank nq if and only if M_q is of rank nq. Notice that all rows of T_qM_q except the first n are the same as those of M_q. That is, T_qM_q is of the form

$$T_qM_q = \begin{Vmatrix} T_{11} & T_{12} & \cdots & T_{1p} & 0 & \cdots & 0 \\ 0 & CB & \cdots & CA^{p-2}B & CA^{p-1}B & \cdots & CA^{2q-2}B \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & CA^{q-1}B \end{Vmatrix} \tag{27}$$

If M_{q-1} is of rank $n(q-1)$ then the first $n(q-1)$ rows of T_qM_q are linearly independent and the last $n(q-1)$ rows are linearly independent. From the form of T_qM_q it is obvious that there exists no nonzero nq-dimensional row vector K such that $KT_qM_q = 0$. Consequently M_q is of rank nq if M_{q-1} is of rank $n(q-1)$ and hence M_q is of rank nq if and only if M_q is of rank np.

It will now be shown that $C(Is - A)^{-1} B$ is of rank n if and only if M_p is of rank np. The $n \times m$ matrix $C(Is - A)^{-1} B$ is of rank n if and only if there exists no nonzero n-dimensional row vector $K(s)$ such that

$$K(s)\, C(Is - A)^{-1} B \equiv 0.$$

Since $C(Is - A)^{-1} B$ is rational such a $K(s)$ can, without loss of generality, be assumed to be of the form $K_0 + K_1 s + \cdots K_{q-1}s^{q-1}$. From Eq. (16) it follows that such a $K(s)$ exists if and only if $K(s)\, E(s) = 0$ for some $K(s)$. That is, such a $K(s)$ exists if and only if there exists a differential equation of the form of Eq. (24). This shows that $C(Is - A)^{-1} B$ is of rank n if and only if M_p is of rank np.

Let β be an arbitrary initial state and let Y be some desired response which is close to the homogeneous response corresponding to β. Then, if $C(Is - A)^{-1} B$ is of rank n, an $X'(t)$ which generates $Y(t)$ on $[0, \infty)$ can be obtained by solving the equation

$$\mathbf{E}(s) = C(Is - A)^{-1} B\mathbf{X}'(s) \tag{28}$$

for $\mathbf{X}'(s)$ and then computing its inverse transform. An $X'(t)$ thus computed will be free of impulse functions if $\| E \|_p$ is finite. However, it may tend to infinity as $t \to \infty$. Since the τ in Definition 2 is restricted to be finite $X(t)$ be given by

$$X(t) = \begin{cases} X'(t); & 0 < t < \tau \\ 0; & t > \tau \end{cases} \tag{29}$$

Such an $X(t)$ is finite for all t.

This shows that if $\| E \|_p$ is finite then there exists a finite $X(t)$ which generates $Y(t)$ exactly on the interval $[0, \tau]$. Since the system is linear this is equivalent to proving that there exists the $\delta(\eta, \tau)$ called for in Definition 2. Q.E.D.

It should be noted that if Y is a scalar then, at least for linear, constant coefficient systems, pointwise and functional reproducibility are equivalent. Thus it is only for multi-output systems that the distinction becomes important. Also, notice that a linear, constant coefficient system cannot be functionally reproducible if the number of outputs exceeds the number of inputs. For pointwise reproducibility no such relation holds as Example 1 shows.

The following corollary is an easy consequence of this theorem. It is important in that it gives necessary and sufficient conditions for the various types of reproducibility in terms of the transfer matrix.

COROLLARY. *Consider the system (13) with the given assumptions on A, B, and C. All homogeneous responses of this system are:*

(a) *pointwise reproducible if and only if there exists no* constant *n-vector K such that $K^T A(Is - A)^{-1} B$ vanishes identically for all s;*

(b) *functional reproducible if and only if there exists no s-dependent n-vector K such that $K^T A(Is - A)^{-1} B$ vanishes identically for all s.*

PROOF. (a) If $K^T C(Is - A)^{-1} B$ vanishes identically then regardless of the choice of X it follows that $KCZ(\beta, 0, t) - KCZ(\beta, X, t) = 0$ and hence that the system cannot be pointwise reproducible.

Suppose the system is not pointwise reproducible, then from Theorem 1 it follows that $(CB, CAB, \cdots CA^{p-1}B)$ is not of rank n and hence that there exists a nonzero n-vector K such that $K^T(CB, CAB, \cdots CA^{p-1}B) = 0$. From the Cayley-Hamilton theorem it follows that there exist scalars $\alpha_i(s)$ such that

$$C(Is - A)^{-1} B = \alpha_0 CB + \alpha_1 C(Is - A) B + \cdots \alpha_{p-1} C(Is - A)^{p-1} B \tag{30}$$

This expansion of $C(Is - A)^{-1} B$ makes it obvious that $K^T C(Is - A)^{-1} B = 0$ and hence that if the system is not pointwise reproducible there exists a constant n-vector such that $K^T C(Is - A)^{-1} B$ vanishes identically.

(b) It was shown in the proof of the theorem that M_p is of rank np if and only if there exists no nonzero vector $K(s)$ such that $K(s)\, C(Is - A)^{-1} B$

vanishes identically. This makes (b) an immediate consequence of the theorem.

Theorem 1 and its corollary give simple necessary and sufficient conditions for pointwise and functional reproducibility. In general, such conditions for uniform functional reproducibility and asymptotic reproducibility are unknown; however, there are several special cases for which necessary and sufficient conditions for these stronger types of reproducibility can be given. For example, if A is nonsingular then the following theorem gives a necessary and sufficient condition for asymptotic reproducibility.

THEOREM 2. *Consider the system (13) with the given assumptions on A, B, and C. Assume, in addition, that A is nonsingular. Then a necessary and sufficient condition for all homogeneous solutions to be asymptotically reproducible is that $CA^{-1}B$ be of rank n.*

PROOF. If $CA^{-1}B$ is of rank n then the equation

$$CA^{-1}BX = E \tag{31}$$

has a solution for any given E. Suppose $Y(t)$ in Definition 4 is given and that the limit as $t \to \infty$ of $Y(t) - CZ(\beta, 0, t) = E'$. Let X' be a solution of Eq. (31) with E replaced by E' and define $\mathbf{X}(s)$ as

$$\mathbf{X}(s) = -X' \det(Is - A)/s(s+1)^p \det A \tag{32}$$

notice that the poles of $\mathbf{X}(s)$ lie at $s = -1$ and $s = 0$ and that $X(s)$ is free of impulse functions. Also, $C(Is - A)^{-1} B\mathbf{X}(s)$ has its poles at $s = -1$ and $s = 0$ and hence one may use the final value theorem to show that

$$\lim_{t\to\infty} Y(t) - CZ(\beta, 0, t) = \lim_{s\to 0} C(Is - A)^{-1} BX' \det(Is - A)/(s+1)^p \det A$$
$$= CA^{-1}BX'. \tag{33}$$

From the definition of X' it follows that $CZ(\beta, X, t) \to Y(t)$ as $t \to \infty$. Since $X(s)$ has its poles in the left half-plane, $\| X \|$ is finite. Therefore, by making $| E' |$ small one can make X small and the requirements of Definition 4 are fulfilled.

If, on the other hand, $CA^{-1}B$ is not of rank n this construction, and all others, fail because Eq. (31) implies that one can find an X for any given E only if $CA^{-1}B$ is of rank n. Q.E.D.

The conditions for uniform functional reproducibility are more restrictive. In this case it is simplest to state the conditions in terms of $C(Is - A)^{-1}B$ rather than in terms of A, B, and C. The following theorem gives necessary and sufficient conditions for two important cases.

THEOREM 3. *Consider the system (13) with the given assumptions on A, B, and C. Suppose in addition that $m = n$. Then*

(a) *If $m = 1$ a necessary and sufficient condition for all homogeneous responses to be uniformly functionally reproducible is that $C(Is - A)^{-1} B$ have no zeros in the right half-plane or on the imaginary axis; or,*

(b) *If all the eigenvalues of A have negative real parts a necessary and sufficient condition for all homogeneous responses to be uniformly functionally reproducible is that* det $C(Is - A)^{-1} B$ *have no zeros in the right half-plane or on the imaginary axis.*

PROOF. (a) Define $E(t)$ as $Y(t) - CZ(\beta, 0, t)$. Notice that if $\| E \|_p$ is finite and if

$$\mathbf{X}(s) = \mathbf{E}(s)/C(Is - A)^{-1} B \tag{34}$$

then $CZ(\beta, X, t) = Y(t)$ for all t and $X(t)$ is finite for all t if and only if $C(Is - A)^{-1} B$ has no zeros in the right halfplane or on the imaginary axis. This means that there exists a $\delta(\eta)$ which works for any τ if and only if this condition on $C(Is - A)^{-1} B$ is fulfilled.

(b) Define $E(t)$ as above. Note that if $\| E \|_p$ is finite and if

$$\mathbf{X}(s) = (C(Is - A)^{-1} B)^{-1} \mathbf{E}(s) \tag{35}$$

then under the present assumption on the eigenvalues of A, $\mathbf{X}(s)$ has its poles in the left-half-plane if and only if det $C(Is - A)^{-1} B$ has its zeros in the left half-plane. Reasoning as above it follows that the system is uniformly functionally reproducible if and only if no zeros of det $C(Is - A)^{-1} B$ lie in the right half-plane or on the imaginary axis. Q.E.D.

Although stability and reproducibility are independent in that neither implies the other, it should be pointed out that in the linear time-invariant case there exists a close analogy between asymptotic stability and uniform functional reproducibility. It is well known that if the differential equation (13a) is asymptotically stable then any input X such that $\| X \|$ is finite gives rise to a response Y such that $\| Y \|$ is finite. If the system is uniformly functionally reproducible then any Y such that $\| Y \|_p$ is finite can be generated by an X such that $\| X \|$ is finite. In short, stability implies that the transformation from the input set to the output set is continuous while uniform functional reproducibility implies that the inverse transformation which maps the output set into the input set exists and is continuous.

NONLINEAR SYSTEMS

The question of how the previous results are affected by the presence of one type of nonlinear term will now be examined. It will be shown that if the initial state is sufficiently close to the origin then the reproducibility properties of the nonlinear system can be determined by an examination of the linear terms.

Let $N(a)$ denote the point set $\{X(t), Z(t) : | X(t) | < a; | Z(t) | < a\}$. Suppose A, B, and C are constant matrices of dimension $p \times p$, $p \times m$, and $n \times p$ respectively, and suppose that the eigenvalues of A have negative real parts. Consider the system

$$\dot{Z}(t) = AZ(t) + BX(t) + BQ(Z(t), X(t)) \tag{36a}$$

$$Y(t) = CZ(t) \tag{36b}$$

where $Q(X(t), Z(t))$ and its partial derivatives with respect to the components of $Z(t)$ and $X(t)$ are continuous in $N(a)$ and vanish when $X(t)$ and $Z(t)$ vanish. It is well known that for such a system there exists a $b > 0$ such that if $X(t) = 0$ and $| \beta | < b$ then $| Z(\beta, 0, t) | < \delta$, δ being any preassigned positive number.

Also notice that the implicit function theorem implies that under these assumptions on $Q(Z(t), X(t))$ there exists a $b > 0$ such that if $X(t)$ and $Z(t)$ belong to $N(b)$ then the equation

$$X'(t) + Q(Z(t), X'(t)) = X(t) \tag{37}$$

can be solved for $X'(t)$. Let $X'(t) = F(Z(t), X(t))$ be this solution. With these preliminaries it is easy to prove the following theorem.

THEOREM 4. *Consider system (35) with the given assumptions on A, B, C, and $Q(Z(t), X(t))$. Then there exists a $\delta > 0$ such that any homogeneous response of the form $CZ(\beta, 0, t)$ with $| \beta | < \delta$ is* (i) *pointwise,* (ii) *functionally,* (iii) *uniformly functionally, or* (iv) *asymptotically reproducible if all the homogeneous responses of the linearized system*

$$\dot{Z}(t) = AZ(t) + BX(t) \tag{38a}$$

$$Y(t) = CZ(t) \tag{38b}$$

are (i) *pointwise,* (ii) *functionally,* (iii) *uniformly functionally, or* (iv) *asymptotically reproducible.*

PROOF. Suppose the system (37) is reproducible in a given sense. Then for any given $E(t) = Y(t) - CZ(\beta, 0, t)$ there is an $X(t)$ which is small and which at the same time forces $CZ(\beta, X, t)$ to have the characteristics required of it. This X depends on E and system (38): denote it by $X(E, L)$. Now consider system (36). For any given E let $X(t)$ assume the form

$$X(t) = F(Z(\beta, X(E, L), t), X(E, L))$$

where $F(Z(t), X(t))$ is the solution of Eq. (37). With this choice of $X(t)$ the response of system (36) is the same as the response of system (38) to an input $X(E, L)$. This shows that for sufficiently small values of β there exists an input for system (36) which causes it to mimic system (38). Thus, the reproducibility properties of both systems are the same. Q.E.D.

Conclusions

A basic property of those systems which are of interest to automatic control engineers is that their outputs can be altered to meet changing requirements by means of input manipulation. It is also true, however, that for most systems there are definite limitations on the response that can be obtained. These limitations are a result of the initial state of the system, restrictions on the available inputs, and the equations of motion themselves. The purpose of this research has been to study the nature of these limitations and to characterize systems on this basis. The principal results are incorporated in Theorems 1-4 which define conditions under which certain types of systems are reproducible.

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