Brane solution with an increasing warp factor

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Abstract

We present a new brane solution in the $(1 + 5)$-dimensional space–time with an increasing warp factor, which localizes the zero modes of all kinds of matter fields and Newtonian gravity. The interesting features of this solution are: (1) In contrast to the Gogberashvili–Singleton case the effective wave function of the zero mode spinor field confined on the brane is finite at a position of brane; (2) There is exactly one zero mode for each matter field and the gravitational field confined on the brane with a finite probability and infinitely many zero modes (labelled by the angular momentum corresponding to extra space) are located in the bulk at different localization radii.

1. Introduction

A new wave of activity in the field of extra dimensions came with the framework of Antoniadis, Arkani-Hamed, Dimopoulos and Dvali [1] who observed that the Higgs mass hierarchy problem can be addressed in models with large extra dimensions. The idea that our world is a three brane embedded in a higher-dimensional space–time with non-factorizable warped geometry has been much investigated since the appearance of papers [2,3]. In this idea, the key observation is that the graviton, which is allowed to be free to propagate in the bulk, is confined to the brane because of the warped geometry, thereby implying that the gravitational law on the brane obeys the usual four-dimensional Newton's law as desired. A first particle physics application of the brane world idea was put forward by Rubakov and Shaposhnikov [4] and independently by Akama [5].

On the other hand, the other local fields except the gravitational field are not always localized on the brane even in the warped geometry. Indeed, in the Randall–Sundrum model in five dimensions [2], the following facts are well known: spin 0 field is lo-

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calized on a brane with positive tension which also localizes the graviton while the spin $1/2$ and $3/2$ fields are localized not on a brane with positive tension but on a brane with negative tension [6]. Spin 1 field is not localized neither on a brane with positive tension nor on a brane with negative tension [7]. In six space–time dimensions, the spin 1 gauge field is also localized on the brane [8]. Thus, in order to fulfill the localization of Standard Model particles on a brane with positive tension, it seems that some additional interactions except the gravitational interaction must be also introduced in the bulk. There is a lot of papers devoted to the different localization mechanisms of the bulk fields in various brane world models.

The brane solution with an increasing warp factor constructed by Midodashvili, Gogberashvili and Singleton [9–11] has a remarkable feature that it can localize the zero modes of all kinds of matter fields ranging from spin 0 scalar field to the spin 2 gravitational field by the gravitational interaction. To this brane the effective wave function of the zero mode spinor field that is confined on the brane becomes infinite at a position of the brane. At first glance this singularity looks to be absolutely harmless since it is integrable, but the main problem it causes is related with the localization of a field theory model including the interaction of matter fields with the fermion field in some width of a brane.

In this Letter we present a new brane solution to the Einstein equations in $(1 + 5)$-dimensional space–time with an increasing warp factor, which localizes the gravitational and matter fields with the regular wave functions. In addition this brane localizes infinitely many zero modes in the bulk at the different localization radii from the brane. The brane solution is derived in Section 2. In Section 3 we consider the localization problem of zero modes of matter and gravitational fields on the brane. Section 4 contains some concluding remarks.

2. Brane solution

In this section we closely follow the derivation of the brane solution presented in papers [11]. The general form of action of the gravitating system in six

dimensions is

$$S = \int d^6 x \sqrt{-^{(6)}g}\left(\frac{M^4}{2}{}^{(6)}R + \Lambda + \mathcal{L}\right), \qquad (1)$$

where M is the fundamental scale, $^{(6)}R$ is the scalar curvature, Λ is the cosmological constant and $\mathcal{L}$ is the Lagrangian of matter fields. All of these quantities are six-dimensional. Throughout this Letter the Greek indices refer to the coordinates on the brane and take the range 0–3, the lower case Latin indices that correspond to the coordinates of a transverse space take values 5, 6, and the capital Latin indices refer to the coordinates in the bulk $A, B, \ldots = 0$–3, 5, 6.

The 6-dimensional Einstein equations with stress-energy tensor T_{AB} are

$$^{(6)}R_{AB} - \frac{1}{2}g_{AB}\,^{(6)}R = \frac{1}{M^4}(\Lambda g_{AB} + T_{AB}). \qquad (2)$$

The bulk space–time metric has the form

$$ds^2 = \phi^2(r)\bar{g}_{\alpha\beta}(x^\nu)\,dx^\alpha\,dx^\beta - \lambda(r)\delta_{ik}\,dx^i\,dx^k, \quad (3)$$

where the metric of ordinary 4-space, $\bar{g}_{\alpha\beta}(x^\nu)$, has the signature $(+, -, -, -)$, δ_{ik} is the Kronecker symbol, $r = \sqrt{\delta_{ik}x^i x^k}$, the summation convention is used over repeated indices and all coordinates x^A run in the interval $(-\infty, \infty)$. So that the extra part of Eq. (3) is a conformally flat 2-dimensional space.

The energy–momentum tensor is assumed to have the form

$$\begin{aligned} T_{\mu\nu} &= -g_{\mu\nu}F(r), & T_{\theta\theta} &= r^2\lambda(r)K(r), \\ T_{rr} &= \lambda(r)K(r), & T_{r\mu} &= T_{\theta\mu} = 0, \end{aligned} \qquad (4)$$

where r, θ are the polar coordinates in the x^5, x^6 plane. The energy–momentum conservation gives the following relation between the source functions

$$K' + 4\frac{\phi'}{\phi}(K - F) = 0, \qquad (5)$$

where the prime denotes differentiation with respect to r. Eqs. (2) are solved under assumption that the 4-dimensional Einstein equations have the ordinary form without a cosmological term

$$R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = 0. \qquad (6)$$

The Ricci tensor in four dimensions $R_{\alpha\beta}$ is constructed from the 4-dimensional metric tensor $\bar{g}_{\alpha\beta}(x^\nu)$ in the standard way. Then with the ansätze (3) and (4) the

Einstein field equations (2) become

$$3\frac{\phi''}{\phi} + 3\frac{\phi'}{r\phi} + 3\frac{(\phi')^2}{\phi^2} + \frac{1}{2}\frac{\lambda''}{\lambda}$$
$$- \frac{1}{2}\frac{(\lambda')^2}{\lambda^2} + \frac{1}{2}\frac{\lambda'}{r\lambda} = \frac{\lambda}{M^4}[F(r) - \Lambda],$$
$$\frac{\phi'\lambda'}{\phi\lambda} + 2\frac{\phi'}{r\phi} + 3\frac{(\phi')^2}{\phi^2} = \frac{\lambda}{2M^4}[K(r) - \Lambda],$$
$$2\frac{\phi''}{\phi} - \frac{\phi'\lambda'}{\phi\lambda} + 3\frac{(\phi')^2}{\phi^2} = \frac{\lambda}{2M^4}[K(r) - \Lambda]. \tag{7}$$

These equations are for the $\alpha\alpha$, rr, and $\theta\theta$ components, respectively. Subtracting the rr from the $\theta\theta$ equation and multiplying by ϕ/ϕ' we arrive at

$$\frac{\phi''}{\phi'} - \frac{\lambda'}{\lambda} - \frac{1}{r} = 0.$$

This equation has the solution

$$\lambda(r) = \frac{\rho^2\phi'}{r}, \tag{8}$$

where ρ is an integration constant with units of length. By taking into account Eq. (8), the system of equations (7) reduces to one independent equation. Taking either the rr, or $\theta\theta$ component of these equations and multiplying it by $r\phi^4$ gives

$$r\phi^3\phi'' + \phi^3\phi' + 3r\phi^2(\phi')^2 = \frac{\rho^2\phi^4\phi'}{2M^4}[K(r) - \Lambda]. \tag{9}$$

Notice that the left-hand side of Eq. (9) is equal to $(r\phi^3\phi')'$. Eq. (7) admits the solution of the form

$$\phi = 1 + a\tanh(r^2/\epsilon^2),$$
$$\lambda = \frac{2a\rho^2}{\epsilon^2\cosh^2(r^2/\epsilon^2)}, \tag{10}$$

where ϵ is an integration constant with units of length, which corresponds to the following rather complicated source functions expressed in term of ϕ as

$$K(\phi) = \Lambda + \frac{4M^4}{\rho^2\phi}\left\{1 - \frac{1}{2}\ln\left(\frac{a + \phi - 1}{a + 1 - \phi}\right)\right.$$
$$\left. \times \left[2\frac{\phi - 1}{a} + \frac{3a}{\phi}\left(1 - \frac{(\phi - 1)^2}{a^2}\right)\right]\right\},$$
$$F(\phi) = K(\phi) + \frac{\phi}{4}\frac{dK}{d\phi}, \tag{11}$$

where a is an arbitrary dimensionless positive parameter. As it is seen from Eq. (10), the warp factor ϕ monotonically increases from 1 to $1 + a$ as r goes from 0 to ∞. The brane (10) is located at $x^5 = x^6 = 0$ and its width is characterized by ϵ. It is straightforward to generalize this solution to the case when the bulk space–time dimension is grater than 6 and the brane codimension $\geqslant 2$ [12].

The brane solution constructed in [9–11], which will be refereed to as a GS brane in what follows, corresponds to the following choice of the source functions

$$K(\phi) = \frac{c_1}{\phi^2} + \frac{c_2}{\phi}, \qquad F(\phi) = \frac{c_1}{2\phi^2} + \frac{3c_2}{4\phi},$$

where c_1, c_2 are constants, and has the form

$$\phi = \frac{3\epsilon^2 + br^2}{3\epsilon^2 + r^2}, \qquad \lambda = \frac{9\epsilon^4}{(3\epsilon^2 + r^2)^2}, \tag{12}$$

where constant $b > 1$. The extension of this solution to the case when the codimension of the brane is greater than 2 is given in [13].

3. Localization of matter and gravitational fields

Now let us consider the localization problem of different kinds of matter fields on the brane (10). We treat the dependence of $\sqrt{-{}^{(6)}g}\mathcal{L}$ on the extra dimensions as (effective) wave functions for the Kaluza–Klein modes.

3.1. Scalar field

The transverse equation for the scalar field $\Phi(x^\mu) \times \varphi(r)\exp(in\theta)$, where n is an arbitrary integer number, reads

$$\varphi'' + \left(\frac{1}{r} + \frac{4\phi'}{\phi}\right)\varphi' - \frac{n^2}{r^2}\varphi = 0. \tag{13}$$

The wave function of scalar nth zero mode has the form

$$\Delta_n^{(0)}(x^5, x^6) \propto \varphi^2\phi^2\lambda. \tag{14}$$

The zero mode field with zero angular momentum $n = 0$, $\varphi = \text{const}$ is located on the brane with the probability $\Delta_0^{(0)}(0,0) \propto a\rho^2/\epsilon^2$ and the localization width $\sim \epsilon$. The second solution of Eq. (13) with the

zero angular momentum for $r \to 0, \infty$ behaves as $\varphi \sim \ln(r/c)$, where c is integration constant with units of length. This solution gives the normalizable wave function too. Taking $c \ll \epsilon$ one concludes that since for this solution $\varphi = 0$ at $r = c$ this zero mode is located either on the brane (with an infinite probability) or in the region $r > c$. The regular solutions at $r = 0$ of Eq. (13) for $n \neq 0$ are localized in the bulk at some radii from the brane. Namely, by taking into account Eq. (10), one finds that the regular solutions of Eq. (13) at the origin ($r = 0$) behave as $r^{|n|}$ for $r \to 0, \infty$. As it is clear from Eq. (14) these solutions give the normalizable wave functions. The greater $|n|$, the larger the localization radius. The GS brane (12) localizes only one, $n = 0$, mode exactly on the brane. The corresponding wave function obtained from Eq. (12) has the form

$$\Delta_{\text{GS}}^{(0)}(x^5, x^6) \propto \frac{(3\epsilon^2 + br^2)^2}{(3\epsilon^2 + r^2)^4}.$$

3.2. Spinor field

Now we turn to the spinor field localization problem on the brane (10). Due to Fock's formalism [14], for constructing the Dirac operator in the gravitational background one has to introduce the local Minkowskian frame at each point of space.[1] Introducing the following *sechsbein*

$$e_A{}^B = \left(\delta_\mu^B \phi^{-1}, \delta_5^B \lambda^{-1/2}, \delta_6^B \lambda^{-1/2}\right), \tag{15}$$

where the first index corresponds to the flat tangent 6d Minkowski space, the transverse Dirac operator takes the form

$$D = \Gamma^k \left\{ \partial_k + \frac{\delta_{km} x^m}{r} \left(2\frac{\phi'}{\phi} + \frac{\partial_r \sqrt{\lambda}}{2\sqrt{\lambda}} \right) \right\}. \tag{16}$$

The Dirac Γ-matrices used in the paper are given in Appendix A. Taking the zero mode spinor field to be

$$\Psi = \begin{pmatrix} \Psi_1(x^\mu)\psi_1(r) \\ \Psi_2(x^\mu)\psi_2(r) \\ \Psi_3(x^\mu)\psi_3(r) \\ \Psi_4(x^\mu)\psi_4(r) \end{pmatrix} e^{in\theta}, \tag{17}$$

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[1] For the diagonal metric there is a simple prescription to construct the Dirac operator immediately [14].

where Ψ_1, Ψ_2, Ψ_3, Ψ_4 are two-component spinors, the equation for zero mode $D\Psi = 0$ reduces to

$$(\partial_r + 2\phi'/\phi + \partial_r \sqrt{\lambda}/2\sqrt{\lambda} + n/r)\psi_{1,3} = 0,$$
$$(\partial_r + 2\phi'/\phi + \partial_r \sqrt{\lambda}/2\sqrt{\lambda} - n/r)\psi_{2,4} = 0.$$

Thus

$$\psi_{1,3} \propto \phi^{-2}\lambda^{-1/4}r^{-n}, \qquad \psi_{2,4} \propto \phi^{-2}\lambda^{-1/4}r^n.$$

The wave function for the nth zero mode of $\Psi_{2,4}$ spinor fields takes the form

$$\Delta_n^{(1/2)}(x^5, x^6) \propto \phi^{-1}\lambda^{1/2}r^{2n}. \tag{18}$$

Analogously, for the wave function of nth zero mode of $\Psi_{1,3}$ spinor fields one finds

$$\Delta_n^{(1/2)}(x^5, x^6) \propto \phi^{-1}\lambda^{1/2}r^{-2n}. \tag{19}$$

For $n = 0$ the fields $\Psi_{1,3} = \Psi_{2,4}$ are localized on the brane with probability $\Delta_0^{(1/2)}(0, 0) \propto \sqrt{a}\rho/\epsilon$. As it is seen from Eqs. (18), (19) the fields $\Psi_{2,4}$ ($\Psi_{1,3}$) for $n > 0$ ($n < 0$) are localized in the bulk. The GS brane localizes the spinor zero mode provided n is a half-integer number with the wave function

$$\Delta_{\text{GS}}^{(1/2)}(x^5, x^6) \propto \frac{1}{r(3\epsilon^2 + br^2)}. \tag{20}$$

3.3. Vector field

As a next step we consider spin 1 gauge field. The transverse equation for the $U(1)$ vector field $A_\nu(x^\mu)\chi(r)\exp(in\theta)$, $A_5 = A_6 = 0$ takes the form

$$\chi'' + \left(\frac{1}{r} + 4\frac{\phi'}{\phi} - \frac{\lambda'}{\lambda} \right)\chi' - \frac{n^2}{r^2}\chi = 0. \tag{21}$$

For the zero mode wave function one obtains

$$\Delta_n^{(1)}(x^5, x^6) \propto \chi^2\lambda. \tag{22}$$

The $n = 0$ mode $\chi = \text{const}$ is located on the brane with a finite probability. The second solution, which for $r \to 0$ behaves as $\chi \sim \ln(r/c)$ and for $r \to \infty$ as $\chi \sim \int_0^r \exp(-2x^2/\epsilon^2)\,dx$, under assumption $c \ll \epsilon$ is located either on the brane (with an infinite probability) or in the region $r > c$. For $n \neq 0$ the asymptotic behavior of χ that corresponds to the convergent wave function (22) at the origin ($r = 0$) is given by $r^{|n|}$ when $r \to 0$. To find the asymptotic behavior when $r \to \infty$ we transform Eq. (21) by the transformation

$\chi = vr^{-1/2}\exp(-r^2/\epsilon^2)$ to the form

$$v'' - \left[\frac{4}{\epsilon^2} + \left(n^2 - \frac{1}{4}\right)r^{-1} + \frac{4}{\epsilon^4}r^2\right]v = 0,$$

and use the WKB approximation which is valid to this equation for large values of r, see [15]. The asymptotic solutions obtained in this way read

$$\chi \sim r^{-1}\exp\left(-r^2/\epsilon^2\right)\exp\left(\pm r^2/\epsilon^2\right).$$

Both these solutions give the normalizable wave function (22) for $r \to \infty$. Thus, the zero mode vector fields $n \neq 0$ are located in the bulk. For the zero mode $n = 0$ wave function in the case of GS brane one obtains

$$\Delta_{\mathrm{GS}}^{(1)}\left(x^5, x^6\right) \propto \frac{1}{(3\epsilon^2 + r^2)^2}.$$

3.4. Gravitational field

For the graviton fluctuations

$$g_{\mu\nu} = \phi^2(r)\left\{\eta_{\mu\nu} + h_{\mu\nu}\left(x^\alpha\right)\sigma(r)\exp(in\theta)\right\},$$

where $\eta_{\mu\nu} = \mathrm{diag}(+, -, -, -)$, the transverse equation for the zero modes is analogous to Eq. (13) with φ replaced by σ [11]. Thus the localization property of the gravitational field is the same as for the scalar field.

4. Conclusions

A new brane solution in the $(1 + 5)$-dimensional space–time with an increasing warp factor is presented. This brane localizes the zero modes of all kinds of matter fields and Newtonian gravity by the gravitational interaction. Besides the zero mode that is located on the brane with a finite probability, there is an additional zero mode with zero angular momentum, which behaves as $\sim \ln(r/c)$ in the vicinity of $r = 0$. But one needs not to worry about this mode because the wave function corresponding to this solution becomes zero at $r = c$ (we assume that $c \ll \epsilon$) and one can merely assume that this mode is confined in the region $r > c$. The zero modes $n \neq 0$ are located in the bulk at different radii from the brane. The greater $|n|$ the larger the localization radius. The fact that for $n = -1, 0, 1$ the brane localizes two two-component spinors of the opposite chirality in the width of a brane may be interesting for explanation of the fermionic family replication and the fermionic mass hierarchy. For instance, localization of different species of fermions at different points of a thick brane was used to solve the hierarchy problem in the split fermion model [16]. Notice that in the case of GS brane all matter fields are located exactly on the brane and the different localization radii for the different kinds of matter fields indicated in [11] are due to fact that the radial variable r coming in polar coordinates from the integration measure is included in the wave function [17]. The fact that the wave function (20) becomes infinity at a position of brane prevents the localization of the field theory model including the interaction of matter fields with the fermion field in some neighborhood of the brane. For the presented brane solution this problem is absent because the fermion wave function is finite on the brane. We note that the localization of fermion field on the GS brane requires the fermion field to satisfy the antiperiodic boundary condition with respect to θ. In the present approach, the presence of a solution to Einstein's equations heavily depends on the form of the source functions. For obtaining of the present brane solution we have used the ugly source functions (11). It is desirable to construct the source functions from fundamental matter fields such that to insure the stability of the brane.

Appendix A. Γ matrices

The chiral representation of the 6d flat Dirac Γ-matrices we use in the Letter is

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^0\gamma^\mu \\ \eta_{\mu\nu}\gamma^0\gamma^\nu & 0 \end{pmatrix},$$

$$\Gamma^5 = \begin{pmatrix} 0 & i\gamma^0\gamma^5 \\ -i\gamma^0\gamma^5 & 0 \end{pmatrix},$$

$$\Gamma^6 = \begin{pmatrix} 0 & \gamma^0 \\ \gamma^0 & 0 \end{pmatrix}, \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \eta_{\mu\nu}\sigma^\nu & 0 \end{pmatrix},$$

where $\gamma^5 = \mathrm{diag}(1, -1)$, σ^0 is unit matrix and σ^1, σ^2, σ^3 are the Pauli matrices. For the eight component spinor (17) one simply finds that spinors $\Psi_{1,4}$ and $\Psi_{2,3}$ have the opposite chirality from the four-dimensional point of view.

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