Surfaces in 4-manifolds and their mapping class groups

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Received 4 April 2005; received in revised form 23 January 2007

Dedicated to Professor Takao Matumoto on his sixtieth birthday

Abstract

A surface in a smooth 4-manifold is called flexible if, for any diffeomorphism $\phi$ on the surface, there is a diffeomorphism on the 4-manifold whose restriction on the surface is $\phi$ and which is isotopic to the identity. We investigate a sufficient condition for a smooth 4-manifold $M$ to include flexible knotted surfaces, and introduce a local operation in simply connected 4-manifolds for obtaining a flexible knotted surface from any knotted surface.

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Keywords: 4-dimensional manifold; Mapping class group; Knotted surface

1. Introduction

There are deformations of embedded surfaces in 4-manifolds which induce isotopically non-trivial diffeomorphisms on surfaces. These deformations define extensions of isotopically non-trivial diffeomorphisms on surfaces. In [5–8,14], extendability of diffeomorphisms on surfaces embedded in 4-manifolds is investigated. In [7], it is shown that any orientation preserving diffeomorphism of some surfaces embedded in the complex projective plane is extendable. In this paper, we investigate embedded surfaces in some other smooth 4-manifolds, for which the same kind of phenomena happen.

\textsuperscript{*}This research was partially supported by Grants-in-Aid for Encouragement of Young Scientists (No. 16740038 and No. 15740030), Ministry of Education, Culture, Sports, Science and Technology, Japan.

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We set up a general formulation. Let $M$ be a simply connected smooth 4-manifold. Let $\Sigma_g$ be the oriented closed surface of genus $g$, and $N_g$ the non-orientable closed surface of genus $g$. For non-orientable surfaces, the genus is defined as the number of projective planes in a connected sum decomposition. We embed a surface $S$ into $M$, and the image of this embedding is denoted by $K$; then the pair $(M, K)$ is called a knotted surface in $M$. If $S = \Sigma_g$, this pair is called a $\Sigma_g$-knot in $M$, and if $S = N_g$, this pair is called an $N_g$-knot in $M$. The group of orientation preserving diffeomorphisms on $\Sigma_g$ is denoted by $\text{Diff}_+ (\Sigma_g)$, and the group of diffeomorphisms on $N_g$ is denoted by $\text{Diff}(N_g)$. The mapping class group of $\Sigma_g$ is $M_g = \pi_0(\text{Diff}_+ (\Sigma_g))$ and the mapping class group of $N_g$ is $N_g = \pi_0(\text{Diff}(N_g))$. For a $\Sigma_g$-knot $(M, S)$, we define

$$\mathcal{F}(M, S) = \left\{ \phi \in M_g \mid \text{there is an orientation preserving diffeomorphism } \phi \text{ on } M \text{ such that } [\phi|_S] = \phi \text{ and } \phi \text{ is isotopic to } id_M \right\},$$

and for $N_g$-knot $(M, N)$, we define

$$\mathcal{F}(M, N) = \left\{ \phi \in N_g \mid \text{there is an orientation preserving diffeomorphism } \phi \text{ on } M \text{ such that } [\phi|_N] = \phi \text{ and } \phi \text{ is isotopic to } id_M \right\}.$$

We note that $\mathcal{F}(M, S)$ is a subgroup of $M_g$ and $\mathcal{F}(M, N)$ is a subgroup of $N_g$. It is natural to ask when $\mathcal{F}(M, S)$ (resp. $\mathcal{F}(M, N)$) is equal to $M_g$ (resp. $N_g$).

A $\Sigma_g$-knot $(M, S)$ is called flexible if $\mathcal{F}(M, S) = M_g$, and an $N_g$-knot $(M, N)$ is called flexible if $\mathcal{F}(M, N) = N_g$. For example, the non-singular plane curves of degree 3 and 4, and trivial $\Sigma_g$-knots in $\mathbb{C}P^2$ are flexible [7]. On the other hand, there is no flexible surface in $S^4$ (see Remark 3.2). In this paper, we investigate a sufficient condition for a smooth 4-manifold $M$ to include flexible knotted surfaces (Section 3), and introduce a local operation called stabilization to obtain a flexible knotted surface from any knotted surface (Section 4).

2. Preliminary: A Hopf band on the boundary of the 4-ball

In this section, we review a proposition from [7] frequently used in this paper. For the convenience of readers, we will prove this result.

A link $L$ in $S^3$ is called a fiber link if there is a map $\phi: S^3 \setminus L \to S^1$ which is a fiber bundle projection. For $t \in S^1$, $\phi^{-1}(t) = F$, which does not depend on $t$, is called the fiber of $L$. Since $\phi$ is a bundle projection, $S^3 \setminus L$ is diffeomorphic to the quotient of $F \times [0, 1]$ by an equivalence $(x, 0) \sim (h(x), 1)$ where $h$ is a diffeomorphism over $F$ and called the monodromy of $L$.

A Hopf band is an annulus embedded in $S^3$ as in Fig. 1. In this picture, there are two types of Hopf bands. In this note, we treat both types of Hopf bands. The boundary of a Hopf band is called a Hopf
link. The Hopf link is a fibered link whose fiber is the Hopf band and whose monodromy is a Dehn twist about the core circle of the Hopf band. Let $B$ be a Hopf band in $S^3 = \partial D^4$. We push the interior of $B$ into the interior of $D^4$ and let $B'$ be the annulus obtained by this deformation and let $c$ be the core circle of $B'$.

**Proposition 2.1.** For the Dehn twist $T_c$ about $c$, there is an orientation preserving diffeomorphism $T$ of $D^4$ such that $T|_{\partial D^4} = \text{id}_{\partial D^4}$, $T|_{B'} = T_c$ and $T$ is isotopic to $\text{id}_{D^4}$.

**Proof.** Since $\partial B$ is a fiber link, whose fiber is $B$ and whose monodromy is $T_c$, there is an orientation preserving diffeomorphism $\psi$ of $S^3$ such that $\psi|_B = T_c$, and there is an isotopy $\psi_t$ ($t \in [0, 1]$) with $\psi_0 = \text{id}_{S^3}$ and $\psi_1 = \psi$, which is defined by shifting fibers. Let $N(\partial D^4)$ be the regular neighborhood of $\partial D^4$ in $D^4$. We parametrize $N(\partial D^4) = S^3 \times [0, 2]$ so that $S^3 \times \{0\} = \partial D^4$ and $B' = \partial B \times [0, 1] \cup B \times \{1\}$. Let $T$ be a diffeomorphism defined as follows:

$$T|_{N(\partial D^4)}(x, t) = \begin{cases} (\psi_t(x), t) & 0 \leq t \leq 1 \\ (\psi_{2-t}(x), t) & 1 \leq t \leq 2 \end{cases}$$

$$T|_{D^4 \setminus N(\partial D^4)} = \text{id}.$$  

This is the diffeomorphism which we need. □

3. Existence theorem

In this section, we will show the following theorem.

**Theorem 3.1.** If $M$ is $\mathbb{C} P^2$, $S^2 \times S^2$, $S^2 \times S^2$, or the elliptic surface $E(n)$, then there are flexible $\Sigma_g$-knots for any $g \geq 0$ and a flexible $N_g$-knot for any $g \geq 1$ in $M$.

**Remark 3.2.** A $\Sigma_g$-knot $(M, K)$ is characteristic if $K$ represents an element of $H_2(M; \mathbb{Z}_2)$ which is a Poincaré dual of the second Stiefel–Whitney class of $M$ (in other words, $K \cdot X \equiv X \cdot X \mod 2$ for any $X \in H_2(M; \mathbb{Z}_2)$). If $(M, K)$ is characteristic, the Rokhlin quadratic form $q_K : H_1(K; \mathbb{Z}_2) \to \mathbb{Z}_2$ is well defined. By the definition of $q_K$, if $\phi \in \mathcal{F}(M, K)$ then $\phi$ must be an element of

$$SP_g(q_K) = \left\{ \phi \in \mathcal{M}_g \mid q_K(\phi_*(x)) = q_K(x) \text{ for any } x \in H_1(K; \mathbb{Z}_2) \right\}.$$  

This group $SP_g(q_K)$ is called a spin mapping class group. If Arf($q_K$) = 0, or Arf($q_K$) = 1 and the genus of $K \geq 2$, then $SP_g(q_K)$ is proper subgroup of $\mathcal{M}_g$, so $(M, K)$ is not flexible. Since $H_2(S^4; \mathbb{Z}_2) = 0$, any $\Sigma_g$-knot $(S^4, K)$ in $S^4$ is characteristic. By the theorem of Rokhlin [16] (see also [3,13]), for any $\Sigma_g$-knot $(S^4, K)$ in $S^4$, Arf($q_K$) = 0. Hence, there is no flexible $\Sigma_g$-knot in $S^4$ for any $g \geq 1$.

**Theorem 3.3.** Let $M$ be a 4-manifold and $D^4$ a 4-ball in $M$. Let $H$ be a Hopf link in $\partial(M \setminus \text{int} D^4)$. If $H$ bounds a disjoint union of two disks in $M \setminus \text{int} D^4$, then, for any $g \geq 0$, there is a flexible $\Sigma_g$-knot in $M$.

**Proof.** Since $\mathcal{M}_0 = \{1\}$, the existence of flexible $\Sigma_0$-knots is trivial.

We construct a $\Sigma_g$-knot $(M, S)$ for $g \geq 1$. Let $H_g$ be a three-dimensional handlebody of genus $g$, which is an orientable 3-manifold constructed from a 3-ball with attaching $g$ 1-handles. We embed $H_g$ into $\partial(M \setminus \text{int} D^4)$ standardly, that is to say, the closure of its complement $\partial(M \setminus \text{int} D^4) \setminus H_g$ is also homeomorphic to a three-dimensional handlebody of genus $g$. We take a point $p$ on the boundary of the
embedded $H_g$ and a regular neighborhood of $p$, which is a 3-ball $B^3$ in $\partial(M \setminus \text{int} D^4)$. We attach a band with one full-twist to the surface $\partial H_g \setminus \text{int} B^3$ in $\partial(M \setminus \text{int} D^4)$ as shown in Fig. 2 and, as a result, we get a surface $S'$ in $\partial(M \setminus \text{int} D^4)$. We remark that its boundary $\partial S'$ is a Hopf link. By the assumption on $M$, there are disks $D_1$ and $D_2$ in $M \setminus \text{int} D^4$ so that $\partial(D_1 \cup D_2) = \partial S'$. The union of $S'$ and these disks $D_1, D_2$ define a surface $S$ in $M$.

We show that $(M, S)$ is flexible. Lickorish [11] showed that Dehn twists about circles in Fig. 3 generate $\mathcal{M}_g$. For each circle $c$ of them, we push $c$ over $D_1$ or $D_2$ as in Fig. 4. The regular neighborhood of the circle obtained as a result of this deformation is a Hopf band. Proposition 2.1 guarantees that the Dehn twist about $c$ is an element of $\mathcal{F}(M, S)$. Therefore, $\mathcal{F}(M, S) = \mathcal{M}_g$. 

**Theorem 3.4.** Let $M$ be a 4-manifold and $D^4$ a 4-ball in $M$. Let $H$ be a Hopf link in $\partial(M \setminus \text{int} D^4)$. If $H$ bounds a disjoint union of two disks in $M \setminus \text{int} D^4$, then, for any $g \geq 1$, there is a flexible $N_g$-knot in $M$.

**Proof.** Since $N_1 = \{1\}$, the existence of flexible $N_1$-knots is trivial.

We construct an $N_g$-knot $(M, N)$ for $g \geq 2$.

First, we consider the case where $g = 2k$ ($k \geq 1$). We parametrize the regular neighborhood of $\partial(M \setminus \text{int} D^4)$ by $S^3 \times [0, 1]$ so that $S^3 \times \{0\} = \partial(M \setminus \text{int} D^4)$. We embed a three-dimensional handlebody $H_{k-1}$ of genus $k - 1$ standardly in $S^3 \times \{\frac{1}{2}\}$. We take three distinct points $p_0, p_1, p_2$ on the boundary of the embedded $H_{k-1}$ and, in $S^3 \times \{\frac{1}{2}\}$, take disjoint regular neighborhoods $B_0, B_1$ and $B_2$ of $p_0, p_1$ and $p_2$ respectively. The surface $N'' = \partial H_{k-1} \setminus (\text{int} B_0 \cup \text{int} B_1 \cup \text{int} B_2)$ has three boundary components
\[ \partial_i N'' = \partial(B_i \cap \partial H_{k-1}) \text{ (} i = 0, 1, 2 \text{)}. \]

We attach a one-full-twist band to \( \partial_0 N'' \), and attach a half-twist band to each one of \( \partial_1 N'' \) and \( \partial_2 N'' \) as shown in Fig. 5; then we get a surface \( N' \) in \( S^3 \times \{ \frac{1}{2} \} \). The boundary of \( N' \) has four components, \( \partial_0 N' \) and \( \partial_2 N' \) originating from \( \partial_0 N'' \), \( \partial_1 N' \) originating from \( \partial_1 N'' \), and \( \partial_2 N' \) originating from \( \partial_2 N'' \). We remark that \( \partial N' \) is a split sum of a Hopf link \( \partial_0 N' \cup \partial_2 N' \) and a trivial link \( \partial_1 N' \cup \partial_2 N' \). This trivial link bounds two disjoint disks \( D_1, D_2 \) properly embedded in \( S^3 \times [0, \frac{1}{2}] \). By the assumption, the Hopf link \( \partial_0 N' \cup \partial_2 N' \) bounds two disjoint disks \( D_01 \cup D_02 \) properly embedded in \( M \setminus \{D^4 \cup S^3 \times [0, \frac{1}{2}] \} \). The union of \( N', D_1 \cup D_2 \) and \( D_01 \cup D_02 \) defines a surface \( N \) in \( M \).

We review some known results on generators of \( \mathcal{N}_g \) (see [1,2,10,12]; for a brief history, see [9]) for the case where \( g = 2k \). Let us define the \( Y \)-homeomorphisms of non-orientable surfaces. We consider a Möbius band \( Mb \) with one hole, and attach a Möbius band \( Mb' \) to \( Mb \) along the boundary of this hole. Then we get a Klein bottle \( K \) with one hole. By moving \( Mb' \) once along the core of \( Mb \) we get a homeomorphism of \( K \) fixing the boundary of \( K \). If we embed \( K \) in \( N \), we can extend this homeomorphism, by the identity on \( N \setminus K \), to a homeomorphism of \( N \). This homeomorphism is called a \( Y \)-homeomorphism about \( K \).

As indicated in Fig. 6, we define circles \( a_1, \ldots, a_{k-1}, b_1, \ldots, b_k, c_1, \ldots, c_{k-1} \), and embed a Klein bottle \( K \) with one hole as the shaded part. Chillingworth [2] showed that \( T_{a_1}, T_{b_1}, T_{c_1}, Y \) (\( 1 \leq i \leq k - 1, 1 \leq j \leq k \)) generate \( \mathcal{N}_g \), where \( T_s \) is the Dehn twist about a simple closed curve \( s \) and \( Y \) is the isotopy class of a \( Y \)-homeomorphism about \( K \).

The same argument as in the proof of Theorem 3.3 shows that \( T_{a_1}, \ldots, T_{a_{k-1}}, T_{b_1}, \ldots, T_{b_{k-1}}, T_{c_1}, \ldots, T_{c_{k-2}} \) are the elements of \( \mathcal{F}(M, N) \). Since each regular neighborhood of \( b_k \) and \( c_{k-1} \) is a Hopf band, \( T_{b_k} \) and \( T_{c_{k-1}} \) are the elements of \( \mathcal{F}(M, N) \).

We show that \( Y \in \mathcal{F}(M, N) \). There is an isotopy \( \phi_t \) (\( t \in [0, 1] \)) of \( S^3 \) bringing \( B_2 \) along the core of the shaded Möbius band in Fig. 6, such that \( \phi_0 = id_{S^3} \) and \( \phi_1 |_N = Y \). We define a diffeomorphism \( \Phi \)
Fig. 7

Fig. 8

Theorem 3.3 shows that... and... into five parts...

We attach a one-full-twist band to...

We get (4) of...

Next, we treat the case where...

For the case where...

As indicated in...

We deform the circle, on which the twisted band is attached, larger...

We slide handles as indicated in (2) and (3) along the lines with arrows...

Then \( \Phi|_N = Y \) and \( \Phi \) is isotopic to the identity of \( M \). Therefore, \( (M, N) \) is flexible.

The same argument as in the proof of... shows that...

For the case where \( g = 2k - 1 \) (\( k \geq 2 \)), we review some known results on generators of \( \mathcal{N}_g \) (for details, see the same references as in the case where \( g = 2k \)).

As indicated in... we define circles... and... embed a Klein bottle... with one hole as the shaded part.

The same argument as in the proof of Theorem 3.3 shows that... are the elements of \( \mathcal{F}(M, N) \).

We show that \( Y \in \mathcal{F}(M, N) \). In Fig. 9, \( K \), the support of \( Y \), is indicated by shaded parts.

We deform the right of Fig. 8 into (1) of Fig. 9. We deform the circle, on which the twisted band is attached, larger, then we obtain (2) of Fig. 9. We slide handles as indicated in (2) and (3) along the lines with arrows, then we get (4) of Fig. 9. The surface \( N \) is constructed from \( N' \) by attaching a 2-disk along \( \partial N' \). We divide the disk \([0, 1] \times [0, 1]\) attached along the boundary in the middle of Fig. 9 into five parts \([i\frac{1}{5}, 2\frac{i}{5}] \times [0, 1]\) (\( i = 1, ..., 5 \)). We push \([i\frac{1}{5}, 2\frac{i}{5}] \times [0, 1]\) and \([3\frac{i}{5}, 4\frac{i}{5}] \times [0, 1]\) into \( S^3 \times \{1\frac{i}{5}\} \) as illustrated in (5) of Fig. 9, where \([1\frac{1}{5}, 2\frac{i}{5}] \times [0, 1]\) is on the left and \([3\frac{i}{5}, 4\frac{i}{5}] \times [0, 1]\) is on the right. We slide the middle twisted band...
along the line with an arrow, and push, along the disk \([0, \frac{1}{5}] \times [0, 1]\), a part of the boundary of the shaded part indicated by thick lines, then we get (6) of Fig. 9, where the shaded part in (5) is deformed into the union of the shaded part in (6) and the disk \([0, 1/5] \times [0, 1]\). The same argument as for the case where \(g = 2k\) shows that \(Y \in \mathcal{F}(M, N)\). Therefore, \((M, N)\) is flexible.

Let \(M\) be a simply connected smooth 4-manifold which is constructed from a 4-ball \(B^4\) by attaching only 2-handles and a 4-handle. The way of attaching 2-handles is described by a framed link in \(\partial B^4\), which is called a Kirby diagram of \(M\). As a corollary of Theorem 3.3 and 3.4, we can see
Corollary 3.5. Let $M$ be a simply connected smooth 4-manifold which has a handle decomposition without 1- or 3-handles. If a Kirby diagram of $M$ has a Hopf link or a framing $\pm 1$ trivial knot, then there are flexible $\Sigma_g$-knots for any $g \geq 0$, and a flexible $N_g$-knot for any $g \geq 1$ in $M$.

From [4], we refer the reader to Kirby diagrams of $\mathbb{C}P^2$ (p. 119, Example 4.4.2), $S^2 \times S^2$ (p. 127, figure 4.30), $S^2 \times S^2$ (p. 144, Example 5.1.3), $E(n)$ (p. 305, figure 8.15(c)). We remark that, in each Kirby diagram, there is a Hopf link or a framing 1-trivial knot. Therefore, Theorem 3.1 follows from the corollary above.

4. Stabilization theorem

Let $(M, K)$ be a knotted surface in a simply connected smooth 4-manifold $M$. Let $p$ be a point in $S^2$; then $(S^2 \times S^2, S^2 \times \{p\})$ is an $S^2$-knot. The connected sum of these knotted surfaces $(M \# S^2 \times S^2, K \# S^2 \times \{p\})$ is called a stabilization of $(M, S)$.

4.1. Orientable case

For the stabilization of orientable knotted surfaces, we show the following theorem.

Theorem 4.1. For any $\Sigma_g$-knot $(M, S)$ in a simply connected smooth 4-manifold $M$, its stabilization $(M \# S^2 \times S^2, S \# S^2 \times \{p\})$ is flexible.

For $(M, S)$, let $D$ be a 2-disk immersed in $M$, whose boundary is on $S$. The normal bundle $v_D$ of $D$ is trivial and its trivialization is unique (see p. 121 of [13]). The normal bundle $v_{\partial D}$ of $\partial D$ in $S$ determines an orientable sub-line bundle in $v_D|_{\partial D}$. Let $\tilde{O}(D)$ be the number of right-handed full twists of $v_{\partial D}$ in $v_D|_{\partial D}$ with respect to the unique trivialization of $v_D$. In [13], $\tilde{O}(D)$ is defined as a modulo 2 of $\tilde{O}(D)$. The following lemma is the main part of a proof of Theorem 4.1.

Lemma 4.2. Let $(M, F)$ be a $\Sigma_g$-knot in an orientable smooth 4-manifold $M$, and $c$ a simple closed curve in $F$. We assume that there is a 2-sphere $S^2$ in $M$, which transversely intersects $F$ at one point, has a trivial normal bundle, and whose exterior in $M$ is simply connected. Then, for any integer $n$, there is a 2-disk $D^2$ embedded in $M$, which satisfies $D^2 \cap F = \partial D^2 \cap F = c$ and $\tilde{O}(D^2) = n$.

Proof. Since $M \setminus S^2$ is simply connected, there is a 2-disk $D$ immersed in $M \setminus S^2$ so that $\partial D = c$ and $\text{int} \; D$ intersects $F$ transversely. By pushing self-intersections of $D$ out of $\partial D$, we may assume that $D$ has no self-intersection. By using a technique introduced by Freedman and Kirby on p. 87 of [3], we change this $D$ into $D'$ so that $\partial D' = \partial D = c$, $\text{int} \; D'$ intersects $F$ transversely, and $\tilde{O}(D') = n$. We eliminate intersections of $\text{int} \; D'$ with $F$ with a technique introduced by Norman [15, proof of Lemma 1] and, implicitly, by Suzuki [17]. We review their technique, since we use a slightly modified one. For the explanation of this technique, we use Fig. 10. Let $p$ be an intersection of $\text{int} \; D'$ with $F$ and $q$ an intersection of $F$ with $S^2$. Let $D_1$ be a small 2-disk neighborhood of $p$ in $D'$ and $D_2$ a small 2-disk neighborhood of $q$ in $S^2$. We take an arc $pq$ in $F$ disjoint from other intersections of $\text{int} \; D'$ with $F$. We can construct a thin tube $[0, 1] \times S^1$, with its axis along $pq$, joining $\partial D_1$ and $\partial D_2$ with the interior disjoint from $D' \cup F \cup S^2$. The disk $(D' \setminus \text{int} \; D_1) \cup ([0, 1] \times S^1) \cup (S^2 \setminus \text{int} \; D_2)$ satisfies the same condition as $D'$ and the number of intersections of this with $F$ is less than that of $D'$ with $F$. We can perform the same method for other intersections. As a result, we get a 2-disk $D^2$, which satisfies $D^2 \cap F = \partial D^2 \cap F = c$. Since $S^2$ has a trivial normal bundle, $\tilde{O}(D^2) = n$. $\square$
**Proof of Theorem 4.1.** Let \( c \) be an any simple closed curve on \( S\#S^2 \times \{p\} \). Since \( S\#S^2 \times \{p\} \) and \( \{p\} \times S^2 \) intersect at one point, and \( M\#S^2 \times S^2 \backslash \{p\} \times S^2 \) is simply connected, we can apply the above lemma. Hence, there is a 2-disk \( D^2 \) embedded in \( M\#S^2 \times S^2 \), which satisfies \( D^2 \cap (S\#S^2 \times \{p\}) = \partial D^2 \cap (S\#S^2 \times \{p\}) = c \) and \( \tilde{O}(D^2) = 1 \). Let \( \nu_{D^2} \) be the normal disk bundle of \( D^2 \); \( \nu_{D^2} \cap (S\#S^2 \times \{p\}) \) is a Hopf band in \( \partial \nu_{D^2} \) whose core is \( c \). By Proposition 2.1, the Dehn twist about \( c \) is an element of \( \mathcal{F}(M\#S^2 \times S^2, S\#S^2 \times \{p\}) \). Since the mapping class group of any orientable surface is generated by Dehn twists, \( (M\#S^2 \times S^2, S\#S^2 \times \{p\}) \) is flexible. \( \square \)

4.2. Non-orientable case

Let \( (M, N) \) be an \( N_g \)-knot in a simply connected smooth 4-manifold \( M \), and let \( D \) be a 2-disk immersed in \( M \), whose boundary is on \( N \). For this 2-disk \( D \), we define \( n(D) \) as an analogue with \( \tilde{O}(D) \) as follows (see p. 132 of [13]). The normal bundle \( \nu_D \) of \( D \) in \( N \) determines a sub-line bundle in \( \nu_D \mid_{\partial D} \). Let \( n(D) \) be the number of right-handed half-twists of \( \nu_D \) in \( \nu_D \mid_{\partial D} \) with respect to the unique trivialization of \( \nu_D \). The same argument as in the proof of Lemma 4.2 shows

**Lemma 4.3.** Let \( (M, N) \) be an \( N_g \)-knot in an orientable smooth 4-manifold \( M \). We assume that there is a 2-sphere \( S^2 \) in \( M \), which transversely intersects \( F \) at one point, has a trivial normal bundle, and whose exterior in \( M \) is simply connected.

1. Let \( c \) be a simple closed curve in \( N \) whose regular neighborhood in \( N \) is a Möbius band. Then, for any integer \( n \), there is a 2-disk \( D^2 \) embedded in \( M \), which satisfies \( D^2 \cap F = \partial D^2 \cap F = c \) and \( n(D^2) = 2n + 1 \).

2. Let \( c \) be a simple closed curve in \( N \) whose regular neighborhood in \( N \) is an annulus. Then, for any integer \( n \), there is a 2-disk \( D^2 \) embedded in \( M \), which satisfies \( D^2 \cap F = \partial D^2 \cap F = c \) and \( n(D^2) = 2n \). \( \square \)

Let \( S^3 \) be the equator of \( S^4 \), and we parametrize the regular neighborhood of this \( S^3 \) by \( S^3 \times [-1, 1] \) so that \( S^3 \times \{0\} = \) the equator \( S^3 \). An \( \mathbb{R}P^2 \)-knot \( (S^4, \mathbb{R}P^2) \) in \( S^4 \), is trivial, if \( \mathbb{R}P^2 \subset S^3 \times [0, 1] \), \( \mathbb{R}P^2 \cap S^3 \times \{0\} \) is a Möbius band as in the left of Fig. 11, \( \mathbb{R}P^2 \cap S^3 \times \{t\} \), for \( 0 < t < 1 \), is a loop as in the middle of Fig. 11, and \( \mathbb{R}P^2 \cap S^3 \times \{1\} \) is a disk as in the right of Fig. 11. An \( N_g \)-knot \( (S^4, N_g) \) is trivial if \( (S^4, N_g) \) is a connected sum of \( g \) trivial \( \mathbb{R}P^2 \)-knots in \( S^4 \).

**Proposition 4.4.** For any \( N_g \)-knot \( (M, N) \) in a simply connected smooth 4-manifold \( M \), its stabilization \( (M\#S^2 \times S^2, N\#S^2 \times \{p\}) \) is a connected sum of an \( S^2 \)-knot in \( M\#S^2 \times S^2 \) with a trivial \( N_g \)-knot in \( S^4 \).

**Proof.** The surface \( N_g \) is constructed from \( S^2 \backslash \) (disjoint \( g \) 2-disks) by attaching \( g \) Möbius bands. We apply Lemma 4.3 to the cores of these Möbius bands; then we get the conclusion. \( \square \)
Theorem 4.5. For any $N_g$-knot $(M, N)$ in a simply connected smooth 4-manifold $M$, its stabilization $(M\#S^2 \times S^2, N\#S^2 \times \{p\})$ is flexible.

Proof. A simple closed curve $c$ on $N_g$ is orientable if the regular neighborhood of $c$ in $N_g$ is an annulus. As we reviewed in the proof of Theorem 3.4, the mapping class group of $N_g$ is generated by Dehn twists about orientable simple closed curves and a $Y$-homeomorphism about a Klein bottle $K$ with one hole embedded in $N_g$. By (2) of Lemma 4.3 and the same argument as in the proof of Theorem 4.1, we see that the Dehn twists about orientable simple closed curves are elements of $\mathcal{F}(M\#S^2 \times S^2, N\#S^2 \times \{p\})$. The Klein bottle $K$ with one hole is constructed from a Möbius band $Mb$ with one hole by attaching a Möbius band $Mb'$ along the boundary of this hole. Under the application of (1) of Lemma 4.3 to the core of $Mb'$, we can use the same argument as in the proof of Theorem 3.4 and conclude that the $Y$-homeomorphism is an element of $\mathcal{F}(M\#S^2 \times S^2, N\#S^2 \times \{p\})$. Therefore, $(M\#S^2 \times S^2, N\#S^2 \times \{p\})$ is flexible. □

Acknowledgements

The authors would like to express their gratitude to Professor Masahico Saito and the referee for their helpful comments.

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