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Product decomposition of loop spaces of configuration spaces

Miguel A. Xicoténcatl [a,b]

[a] *Max-Planck-Institut für Mathematik, P.O. Box 7280, D-53072 Bonn, Germany*
[b] *Departamento de Matemáticas, CINVESTAV–IPN, Apdo. Postal 14-740, México, DF 07300, Mexico*

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Abstract

The configuration space of k points in $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$ are studied. In this article we show that after looping once, they split as a product of spheres and the loop space of certain orbit configuration spaces. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction and main result

Let M be a topological space and G a group acting freely on M. Let Gm denote the orbit of an element $m \in M$ under the action of G. Define the orbit configuration space of k points in M by

$$F_G(M,k) = \left\{(m_1,\ldots,m_k) \in M^k \mid Gm_i \neq Gm_j \text{ for } i \neq j\right\}.$$

Such spaces were introduced in [6] as generalizations of the ordinary configuration spaces defined by Fadell and Neuwirth [5]. Indeed, if G is the trivial group, the space $F_G(M,k)$ coincides with the usual configuration space

$$F(M,k) = \left\{(m_1,\ldots,m_k) \in M^k \mid m_i \neq m_j \text{ for } i \neq j\right\}.$$

The purpose of this note is to exhibit product decompositions of the loop space of $F(M,k)$ for $M = \mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$, in terms of loop spaces of orbit configuration spaces and known spaces.

E-mail address: xico@mpim-bonn.mpg.de, xico@math.cinvestav.mx (M.A. Xicoténcatl).

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An equivalent description of $F_G(M,k)$ can be given in terms of ordinary configuration spaces. Let $f : M/G \to BG$ be the map which classifies the principal bundle $G \to M \to M/G$. The following theorem was proven in [7]:

Theorem 1.1. *The space $F_G(M,k)$ is homeomorphic to the total space of the pull-back of the principal fibration $G^k \to (EG)^k \to (BG)^k$ along the composition*

$$F(M/G,k) \hookrightarrow (M/G)^k \xrightarrow{f^k} (BG)^k.$$

Therefore, there is a principal G^k-bundle, $F_G(M,k) \to F(M/G,k)$. More precisely, the group G^k acts coordinatewise on the space $F_G(M,k)$ and the quotient $F_G(M,k)/G^k$ is homeomorphic to $F(M/G,k)$. In particular there is a fibration (up to homotopy):

$$F_G(M,k) \to F(M/G,k) \to BG^k. \tag{1}$$

For the rest of the article, we will only be concerned with the following special cases:
(i) the $\mathbb{Z}_2$ action on S^n given by the antipodal map,
(ii) the S^1 action on S^{2n+1} given by complex multiplication, and
(iii) the S^3 action on S^{4n+3} given by quaternionic multiplication,
having as orbit spaces the projective spaces $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$, respectively. It will be shown in Section 2 that in cases (i), (ii) and (iii), the corresponding fibrations (1) admit a section after looping and they split as products. Thus the main result can be stated as follows:

Theorem 1.2. *For every $n, k \geqslant 1$ there are homotopy equivalences*:
(a) $\Omega F(\mathbb{R}\mathrm{P}^n,k) \simeq (\mathbb{Z}_2)^k \times \Omega F_{\mathbb{Z}_2}(S^n,k)$ *if* $n \geqslant 3$.
(b) $\Omega F(\mathbb{C}\mathrm{P}^n,k) \simeq (S^1)^k \times \Omega F_{S^1}(S^{2n+1},k)$ *if* $n \geqslant 2$.
(c) $\Omega F(\mathbb{H}\mathrm{P}^n,k) \simeq (S^3)^k \times \Omega F_{S^3}(S^{4n+3},k)$ *if* $n \geqslant 2$.

It is important to mention here that loop spaces of configuration spaces have received a lot of attention recently. This is due to some interesting relations between their homology and the Vassiliev invariants of braids [3], the Lie algebra associated to the descending central series of braid groups, [3,6] and the n-body problem [4], among other topics.

2. Some auxiliary lemmas

The proof of Therorem 1.2 is given after the following lemmas:

Lemma 2.1. *For $n \geqslant 3$, the natural inclusion $F(\mathbb{R}\mathrm{P}^n,k) \to (\mathbb{R}\mathrm{P}^\infty)^k$ is an isomorphism on fundamental groups.*

Proof. The statement of the lemma is clear for $k = 1$. By induction, assume this is true for $k-1$ and consider the map of fibrations induced by the inclusion

$$\begin{tikzcd}
(\mathbb{R}\mathrm{P}^n - Q_{k-1}) \arrow[r] \arrow[d] & \mathbb{R}\mathrm{P}^\infty \arrow[d] \\
F(\mathbb{R}\mathrm{P}^n, k) \arrow[r] \arrow[d] & (\mathbb{R}\mathrm{P}^\infty)^k \arrow[d] \\
F(\mathbb{R}\mathrm{P}^n, k-1) \arrow[r] & (\mathbb{R}\mathrm{P}^\infty)^{k-1}
\end{tikzcd}$$

where both fibrations are projection onto the first $k-1$ coordinates, and Q_{k-1} denotes a set of $k-1$ distinct points in $\mathbb{R}\mathrm{P}^n$. Notice that $(\mathbb{R}\mathrm{P}^n - Q_{k-1})$ homotopy equivalent to $\mathbb{R}\mathrm{P}^{n-1} \vee (\bigvee_{k-2} S^{n-1})$ and so the natural inclusion $(\mathbb{R}\mathrm{P}^n - Q_{k-1}) \to \mathbb{R}\mathrm{P}^\infty$ gives an isomorphism of fundamental groups (both isomorphic to $\mathbb{Z}_2$). By induction hypothesis, the map between the base spaces also induces an isomorphism on π_1. The lemma follows. $\square$

Similarly we have:

Lemma 2.2. *For $n \geqslant 2$, the natural inclusion $F(\mathbb{C}\mathrm{P}^n, k) \to (\mathbb{C}\mathrm{P}^\infty)^k$ is an isomorphism on π_2.*

Lemma 2.3. *For $n \geqslant 2$, the natural inclusion $F(\mathbb{H}\mathrm{P}^n, k) \to (\mathbb{H}\mathrm{P}^\infty)^k$ is an isomorphism on π_4.*

Thus after looping the fibration $F_{\mathbb{Z}_2}(S^n, k) \to F(\mathbb{R}\mathrm{P}^n, k) \to (B\mathbb{Z}_2)^k$ we obtain a principal fibration

$$\Omega F_{\mathbb{Z}_2}\big(S^n, k\big) \to \Omega F\big(\mathbb{R}\mathrm{P}^n, k\big) \to (\mathbb{Z}_2)^k$$

where the projection induces an isomorphism on π_0. Therefore $\Omega F(\mathbb{R}\mathrm{P}^n, k)$ is indeed the disjoint union of 2^k copies of $\Omega F_{\mathbb{Z}_2}(S^n, k)$. Thus we have $\Omega F(\mathbb{R}\mathrm{P}^n, k) \simeq (\mathbb{Z}_2)^k \times \Omega F_{\mathbb{Z}_2}(S^n, k)$. Part (a) follows.

For case (b), consider the fibration: $F_{S^1}(S^{2n+1}, k) \to F(\mathbb{C}\mathrm{P}^n, k) \to (BS^1)^k$. Since the projection induces an isomorphism on π_2, then after looping, the projection induces an isomorphism on π_1 and there is a lifting

$$\begin{tikzcd}
 & \Omega F(\mathbb{C}\mathrm{P}^n, k) \arrow[d] \\
(\bigvee_k S^1) \arrow[r] \arrow[ur, dashed] & \Omega (BS^1)^k
\end{tikzcd}$$

Thus by Lemma 2.4 below there exists a section for the multiplicative fibration $\Omega F_{S^1}(S^{2n+1}, k) \to \Omega F(\mathbb{C}\mathrm{P}^n, k) \to (S^1)^k$ and part (b) follows.

For case (c), consider the fibration: $F_{S^3}(S^{4n+3},k) \to F(\mathbb{H}\mathrm{P}^n,k) \to (BS^3)^k$. Since the projection induces an isomorphism on π_4, then after looping, the projection induces an isomorphism on π_3 and there is a lifting

$$\begin{tikzcd} & \Omega F(\mathbb{H}\mathrm{P}^n,k) \arrow[d] \\ (\bigvee_k S^3) \arrow[r] \arrow[ur, dashed] & \Omega(BS^3)^k \end{tikzcd}$$

Thus by Lemma 2.4 there exists a section for the multiplicative fibration

$$\Omega F_{S^3}\big(S^{4n+3},k\big) \to \Omega F\big(\mathbb{H}\mathrm{P}^n,k\big) \to \big(S^3\big)^k$$

and part (c) follows.

We prove now the lemma needed in cases (b) and (c). Let i denote the inclusion $i : \bigvee_k G \hookrightarrow G^k = \Omega(BG)^k$.

Lemma 2.4. *Let $\alpha : Z \to (BG)^k$ be a fibration. Assume that after looping, the map $i : \bigvee_k G \to \Omega(BG)^k$ admits a homotopy lifting to ΩZ that is, there is a map $f : \bigvee_k G \to \Omega Z$ such that $\Omega\alpha \circ f \simeq i$. Then, f extends to a homotopy section, that is there is a map $F : G^k \to \Omega Z$ such that the following diagram homotopy commutes*

$$\begin{tikzcd} G^k \arrow[r, dashed, "F"] & \Omega Z \arrow[d, "\Omega\alpha"] \\ \bigvee_k G \arrow[u, hook] \arrow[ur, "f"] \arrow[r, "i"] & \Omega(BG)^k \end{tikzcd}$$

and the composite: $G^k \xrightarrow{F} \Omega Z \xrightarrow{\Omega\alpha} \Omega(BG)^k$ *is homotopic to the identity.*

We prove the lemma in the case $k=2$. The proof for general k is similar and requires the generalization of the statements below to k factors. As preparation we need the following facts. Spaces are assumed to be in $\mathcal{T}_*$, the category of compactly generated, weak Hausdorff spaces, with non-degenerate base points.

Recall from [1] that there are maps

$$f_i : X \times Y \to \Omega\Sigma(X \vee Y), \quad i = 1, 2, 3,$$

given as follows. The maps f_1 and f_2 are given by the compositions:

$$X \times Y \xrightarrow{\text{proj}} X \xrightarrow{\sigma_X} \Omega\Sigma X \xrightarrow{j_1} \Omega\Sigma(X \vee Y),$$

$$X \times Y \xrightarrow{\text{proj}} Y \xrightarrow{\sigma_Y} \Omega\Sigma Y \xrightarrow{j_2} \Omega\Sigma(X \vee Y),$$

where σ_X (respectively σ_Y) is the adjoint to the identity map $\Sigma X \to \Sigma X$ (respectively $\Sigma Y \to \Sigma Y$), j_1, j_2 are the natural inclusions and f_3 is given by the Samelson product map:

$$X \times Y \to X \wedge Y \xrightarrow{i_X \wedge i_Y} (X \vee Y) \wedge (X \vee Y) \xrightarrow{[\,,\,]} \Omega\Sigma(X \vee Y).$$

The difference $\gamma = f_1 + f_2 - f_3$ in the group $[X \times Y,\ \Omega\Sigma(X \vee Y)]$ satisfies the following properties:

(1) When restricted to $X \vee Y$, γ is homotopic to the suspension $\sigma : X \vee Y \to \Omega\Sigma(X \vee Y)$.

(2) The map γ composed with the loops on the projection $\Omega\Sigma(X \vee Y) \to \Omega\Sigma X$ is homotopic to the map obtained as the composite

$$X \times Y \xrightarrow{\text{proj}} X \xrightarrow{\sigma_X} \Omega\Sigma X.$$

(3) The map γ composed with the loops on the projection $\Omega\Sigma(X \vee Y) \to \Omega\Sigma Y$ is homotopic to the map obtained as the composite

$$X \times Y \xrightarrow{\text{proj}} Y \xrightarrow{\sigma_Y} \Omega\Sigma Y.$$

The next lemma follows at once:

Lemma 2.5. *The map γ considered above is an extension of the suspension $\alpha : X \vee Y \to \Omega\Sigma(X \vee Y)$ over the product $X \times Y$. Furthermore, the composite*

$$X \times Y \xrightarrow{\gamma} \Omega\Sigma(X \vee Y) \xrightarrow{\Omega(\theta)} \Omega\Sigma X \times \Omega\Sigma Y$$

is homotopic to the product $\sigma_X \times \sigma_Y$, where $\theta : \Sigma X \vee \Sigma Y \to \Sigma X \times \Sigma Y$ is the natural inclusion.

Lemma 2.6. *Let $\alpha : A \to \Sigma(X \vee Y)$ which admits a section. Taking adjoints this implies that the suspension $\sigma : X \vee Y \to \Omega\Sigma(X \vee Y)$ admits a homotopy lifting*

$$\begin{tikzcd}
 & \Omega(A) \arrow[d, "\Omega(\alpha)"] \\
X \vee Y \arrow[ur, dashed] \arrow[r, "\sigma"] & \Omega\Sigma(X \vee Y)
\end{tikzcd}$$

Then there exists an extension $X \times Y \to \Omega(A)$ such that the composite:

$$X \times Y \to \Omega(A) \to \Omega\Sigma(X \vee Y) \to \Omega\Sigma X \times \Omega\Sigma Y$$

is homotopic to $\sigma_X \times \sigma_Y$.

Proof. We indicate the construction of a lifting. The map $X \vee Y \to \Omega(A)$ can be expressed as $g_1 \vee g_2 : X \vee Y \to \Omega(A)$ where $g_1 : X \to \Omega(A)$ and $g_2 : Y \to \Omega(A)$. Then the following diagram is homotopy commutative:

$$\begin{tikzcd}
X \times Y \arrow[r, "g_1 \times g_2"] \arrow[dr] & \Omega(A) \times \Omega(A) \arrow[r, "\text{mult}"] \arrow[d] & \Omega(A) \arrow[d] & \\
 & \Omega\Sigma(X \vee Y)^2 \arrow[r] & \Omega\Sigma(X \vee Y) \arrow[r] & \Omega\Sigma X \times \Omega\Sigma Y
\end{tikzcd}$$

The desired map is given by the composite on the top. The second assertion follows now from the previous lemma. $\square$

Proof of Lemma 2.4. Let $\alpha : Z \to BG \times BH$ and assume that after looping the inclusion $G \vee H \to \Omega(BG \times BH)$ admits a lifting $G \vee H \to \Omega Z$. If we take $X = G$ and $Y = H$ in the previous lemmas, we get a homotopy commutative diagram:

$$\begin{tikzcd}
G \times H \arrow[r, "\gamma"] & \Omega\Sigma(G \vee H) \arrow[rr, "\text{lift}"] \arrow[d, equal] & & \Omega Z \arrow[d, "\Omega(\alpha)"] \\
G \vee H \arrow[u] \arrow[r] & \Omega\Sigma(G \vee H) \arrow[r] & \Omega(BG \vee BH) \arrow[r] & \Omega(BG \times BH)
\end{tikzcd}$$

where

(1) the left hand square homotopy commutes by Lemma 2.5,

(2) the right hand square homotopy commutes by hypothesis,

(3) the composite:

$$G \times H \to \Omega\Sigma(BG \vee BH) \to \Omega\Sigma G \times \Omega\Sigma H \to \Omega BG \times \Omega BH$$

is a homotopy equivalence. This is a direct consequence of the fact that the natural composite $G \to \Omega\Sigma G \to \Omega BG$ is a homotopy equivalence.

This gives the desired section and the lemma follows. □

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