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Interpolating d-r.e. and REA degrees between r.e. degrees

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Abstract

We provide three new results about interpolating 2-r.e. (i.e. d-r.e.) or 2-REA (recursively enumerable in and above) degrees between given r.e. degrees:

Proposition 1.13. *If* $\mathbf{c} < \mathbf{h}$ *are r.e.,* $\mathbf{c}$ *is low and* $\mathbf{h}$ *is high, then there is an* $\mathbf{a} < \mathbf{h}$ *which is REA in* $\mathbf{c}$ *but not r.e.*

Theorem 2.1. *For all high r.e. degrees* $\mathbf{h} < \mathbf{g}$ *there is a properly d-r.e. degree* $\mathbf{a}$ *such that* $\mathbf{h} < \mathbf{a} < \mathbf{g}$ *and* $\mathbf{a}$ *is r.e. in* $\mathbf{h}$.

Theorem 3.1. *There is an incomplete nonrecursive r.e. A such that every set REA in A and recursive in* $0'$ *is of r.e. degree.*

The first proof is a variation on the construction of Soare and Stob (1982). The second combines highness with a modified version of the proof strategy of Cooper et al. (1989). The third theorem is a rather surprising result with a somewhat unusual proof strategy. Its proof is a $0'''$ argument that at times moves left in the tree so that the accessible nodes are not linearly ordered at each stage. Thus the construction lacks a true path in the usual sense. Two substitute notions fill this role: The *true nodes* are the leftmost ones accessible infinitely often; the *semitrue nodes* are the leftmost ones such that there are infinitely many stages at which some extension is accessible. Another unusual feature of the construction is that it involves using distinct priority orderings to control the interactions of different parts of the construction.

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1. Introduction

This paper is a contribution to the investigation of the relationship between the r.e. degrees (the complexity classes under Turing reducibility of sets which can be effectively enumerated) and those of two important generalizations of recursive (effective) enumerability. The first generalization starts with the characterization of the r.e. sets as those sets A that can be effectively approximated with at most one change in the approximation: We begin by guessing that x is not in A and we may change our mind at most once to put x into A (when it is enumerated in A in the usual definition of an r.e. set). The natural generalization of this property (introduced by Putnam [28] and Gold [20]) is to allow the approximation to change more often.

Definition 1.1. A set A is n-r.e. if there is a recursive function $f(x,s)$ such that for every x

1. $f(x,0) = 0$.
2. $\lim_s f(x,s) = A(x)$.
3. $|\{s\colon f(x,s) \neq f(x,s+1)\}| \leqslant n$.

So, in particular, the 1-r.e. sets are precisely the r.e. sets. The 2-r.e. sets are also known as the d-r.e. sets as they are the differences of r.e. sets, i.e. the ones of the form $B-C$ with both B and C r.e. Similarly the n-r.e. sets are those given by starting with r.e. sets and alternating the Boolean operations of difference and union. These sets form a true hierarchy even in terms of degree: There are, for each $n > 0$, $(n+1)$-r.e. sets which are not of n-r.e. degree, i.e. not of the same degree as any n-r.e. set (Cooper [5]). This hierarchy can be carried into the transfinite (Ershov [17–19]) to define α-r.e. sets for recursive ordinals α by associating the changes allowed in the recursive approximation with elements of a recursive system of notations for α. Two remarkable facts here are: first the ω-r.e. sets are precisely those truth table reducible to the complete r.e. set K and also those A for which there are recursive functions f, g with f as in the above definition except that (3) becomes $|\{s\colon f(x,s) \neq f(x,s+1)\}| \leqslant g(n)$. Second, if we fix any path through Kleene's $\mathcal{O}$ (i.e. any system of notations for all the recursive ordinals) then the union of the classes of α-r.e. sets for all α in this system is precisely the class of sets recursive in K, i.e. the Δ^0_2 sets (Ershov [19]). The appropriate definitions for the α-r.e. sets and proofs of all of these results and more can be found in the paper by Epstein et al. [16] which is a fine introduction to the α-r.e. degrees.

The second generalization of recursive enumerability that we want to consider is the hierarchy of REA sets and operators introduced by Jockusch and Shore [22]. Here the motivating ideas were the jump operator and relative recursive enumerability.

Definition 1.2. We define the sets *REA* in X by induction.

1. X is *0-REA* in X.
2. If Y is n-REA in X and $e \in \omega$, then $Y \oplus W_e^Y$ is $(n+1)$-REA in X. The *n-REA* sets are those which are n-REA in the empty set.

Once again, the 1-REA degrees are precisely the r.e. degrees; the hierarchy is non-degenerate even in terms of degrees and it can be extended into the transfinite along notations for recursive ordinals. The ω-REA in X sets, for example, are the ones of the form $\bigoplus A_i$, where $A_0 = X$ and $A_i = A_{i-1} \oplus W_{f(i)}^{A_{i-1}}$ for some recursive function f. This generalization is strictly stronger than the first: every n-r.e. degree is n-REA for each n, but there is a 2-REA set recursive in K which is not of n-r.e. degree for any n. (Indeed, given a fixed notation system for any recursive ordinal α, there is a 2-REA set recursive in K which is not of α-r.e. degree.) (All these results are in [22].)

There are a number of applications of REA operators to other questions in degree theory in [22] but clearly the most striking is the natural definition in the structure $\mathscr{D}$ (the Turing degrees of all sets with just the relation of Turing reducibility) of the binary relation "$\mathbf{c}$ is arithmetic in $\mathbf{a}$". For example, $\mathbf{c}$ is arithmetic (i.e. $\mathbf{c} < \mathbf{0}^{(n)}$ for some $n < \omega$) if and only if $\exists y > \mathbf{c} \forall z (z \vee y$ is not a minimal cover of $z)$. ($\mathbf{x}$ is a minimal cover of $\mathbf{z} < \mathbf{x}$ iff there is no $\mathbf{w}$ strictly between them. The relativization of this definition to the degrees above $\mathbf{a}$ defines when a degree $\mathbf{c} \geqslant \mathbf{a}$ is arithmetic in $\mathbf{a}$. As an arbitrary $\mathbf{c}$ is arithmetic in $\mathbf{a}$ iff $\mathbf{c} \vee \mathbf{a}$ is, we have the desired definition of "$\mathbf{c}$ is arithmetic in $\mathbf{a}$".) Combining this result with general definability arguments from [26, 27] and [33] gives many corollaries on definability and automorphisms of $\mathscr{D}$. For example, every relation on degrees above $\mathbf{0}^{(\omega)}$ which is definable in second order arithmetic is definable in $\mathscr{D}$ and every automorphisms of $\mathscr{D}$ is the identity on every degree above $\mathbf{0}^{(\omega)}$.

An even more remarkable application of these hierarchies is Cooper's natural definitions of the binary relations "$\mathbf{c}$ is recursively enumerable in $\mathbf{a}$" (Cooper [10]) and "$\mathbf{c}$ is the Turing jump of $\mathbf{a}$" (Cooper [6, 9, 11]). For example, $\mathbf{0}'$ is the largest degree $\mathbf{x}$ such that $\neg\exists \mathbf{a}, \mathbf{b}(\mathbf{x} \vee a$ is unsplittable over $\mathbf{a}$ avoiding $\mathbf{b})$. (We say that $\mathbf{c}$ is *unsplittable over* $\mathbf{a}$ *avoiding* $\mathbf{b}$ if $\mathbf{c} > \mathbf{a}, \mathbf{b}$ but there do not exist $\mathbf{c}_0, \mathbf{c}_1$ such that $\mathbf{a} < \mathbf{c}_0, \mathbf{c}_1 < \mathbf{c}, \mathbf{c} = \mathbf{c}_0 \vee \mathbf{c}_1$ and $\mathbf{b} \nleqslant \mathbf{c}_0, \mathbf{c}_1$.) Once again applying the results of Nerode and Shore [27] and Shore [33], these results have immediate corollaries for definability and automorphisms strengthening the ones above. For example, every relation on degrees above $\mathbf{0}'''$ definable in second order arithmetic is definable in $\mathscr{D}$ and every automorphism of $\mathscr{D}$ is the identity on degrees above $\mathbf{0}'''$. (With some additional care (or by work of Slaman and Woodin [34]) one can replace $\mathbf{0}'''$ by $\mathbf{0}''$ in these results.)

Both of these basic definability results are proved in the same style. First some local structural property P_α of α-r.e. sets is isolated which distinguishes them from β-REA sets for $\beta < \alpha$ in the sense that every β-REA set has property P_α (even relative to any degree below it) for $\beta < \alpha$ but there is an α-r.e. set which does not have P_α. Then a generalization of both the Friedberg completeness theorem and the Posner–Robinson cupping theorem for α-REA operators derived from α-r.e. ones proved in [22] is applied to see that every $X \nleqslant 0^{(\beta)}$ for any $\beta < \alpha$ joins $0^{(\alpha)}$ up to a degree which has the property P_α relative to some degree below it. For defining "arithmetic in", $\alpha = \omega$ and P_ω is the property of not being a minimal degree. For the definition of $\mathbf{0}'$, $\alpha = 2$ and P_2 is the property of having a splitting which avoids any given smaller degree $\mathbf{b}$.

That, for any $\beta < \alpha$, every β-REA set A has property P_α (even relative to $C \leqslant_T A$) follows in each case from one of the basic structural properties of the r.e. degrees:

Theorem 1.3 (Density Theorem, Sacks [32]). *If* $\mathbf{a} < \mathbf{c}$ *are r.e. degrees then there is an r.e. degree* $\mathbf{c}$ *such that* $\mathbf{a} < \mathbf{b} < \mathbf{c}$.

Theorem 1.4 (Splitting Theorem with cone avoiding, Sacks [31]). *If* $\mathbf{c} \not\geqslant \mathbf{b}$ *and* $\mathbf{c}$ *is r.e. then there are r.e. degrees* $\mathbf{c}_0, \mathbf{c}_1 < \mathbf{c}$ *such that neither is above* $\mathbf{b}$ *and their join is equal to* $\mathbf{c}$.

The structural results on the α-r.e. side are the following:

Theorem 1.5 (Minimal degrees, Sacks [30]). *There is an* ω*-r.e. set* M *of minimal degree.*

Theorem 1.6 (Unsplittable degrees, Cooper [6, 11]). *There is a* 2*-r.e. degree* $\mathbf{c}$ *and a degree* $\mathbf{b} < \mathbf{c}$ *such that* $\mathbf{c}$ *is unsplittable* (*over* $\mathbf{0}$) *avoiding* $\mathbf{b}$.

Now even without the striking applications to definability, these basic properties of the r.e. degrees, particularly density, have been the center of structural investigations of the generalizations to n-r.e. and n-REA degrees. An early unpublished result of Lachlan showed that no 2-r.e. degree could be minimal. (See also [16] for a direct proof of the nonminimality of n-r.e. degrees.) Of course, from our current vantage point, this follows directly from the facts that the 2-r.e. degrees are 2-REA and that the n-REA degrees are dense (for each n separately and for the union over all n). This density result in turn is an easy corollary of the Density Theorem for the r.e. degrees. Much work was devoted to the questions of density and splitting in the 2-r.e. degrees themselves. Partial positive results can be found in [1–3] and [21]. Important related results on branching and nonbranching degrees in the 2-r.e. degrees can be found in [23, 24]. We also mention the following specific theorems:

Theorem 1.7 (Weak Density Theorem, Cooper et al. [8]). *Given any r.e. degrees* $\mathbf{a} < \mathbf{c}$ *there is a properly* 2*-r.e. degree* $\mathbf{b}$ *between them.* ($\mathbf{b}$ *is properly* 2*-r.e. if it is* 2*-r.e. but not r.e.*)

Theorem 1.8 (Splitting Theorem, Cooper [8]). *If* $\mathbf{c}$ *is a* 2*-r.e. degree then there are incomparable* 2*-r.e. degrees* $\mathbf{c}_0, \mathbf{c}_1$ *such that* $\mathbf{c}_0 \vee \mathbf{c}_1 = \mathbf{c}$.

Theorem 1.9 (Low$_2$ density and splitting, Cooper [7]). *The* low$_2$ 2*-r.e. degrees are dense and each is splittable above any lower* 2*-r.e. degree.*

As described above, the failure of density in the REA hierarchy occurs at level ω (Theorem 1.5) and of splitting with cone avoiding at level 2 of the r.e. hierarchy

(Theorem 1.6). A long awaited and difficult result was the failure of density for the 2-r.e. degrees.

Theorem 1.10 (Nondensity, Cooper et al. [12]). *There is a 2-r.e. degree* $\mathbf{d} < \mathbf{0}'$ *such that there is no 2-r.e. (or even ω-r.e.) degree* $\mathbf{e}$ *with* $\mathbf{d} < \mathbf{e} < \mathbf{0}'$.

On the other hand, there is an older important result dealing with a version of the density problem combining both the n-REA and n-r.e. hierarchies.

Theorem 1.11 (Soare and Stob [36]). *If* $\mathbf{c} > \mathbf{0}$ *is r.e. then there is an* $\mathbf{a}$ *REA in* $\mathbf{c}$ *which is not of r.e. degree. Of course, if* $\mathbf{c}$ *is low then* $\mathbf{a} < \mathbf{0}'$.

Soare and Stob [36] also claimed that a modification of their strategy for low $\mathbf{c}$ would make $\mathbf{a}$ 2-r.e. They have since withdrawn this claim (personal communication) but it and other results mentioned above suggest a general question about density and the r.e., 2-r.e. and 2-REA degrees which we address in this paper:

Question 1.12. *When, given two r.e. degrees* $\mathbf{a} < \mathbf{c}$, *can we find a 2-r.e. degree* $\mathbf{b}$ *which is both REA in* $\mathbf{a}$ *and below* $\mathbf{c}$?

Now several of the previously cited results give partial answers to this question. In particular, Theorem 1.7 says that we can always do it if we give up the requirement that $\mathbf{b}$ be REA in $\mathbf{a}$. Indeed, by Cooper and Yi [14], there is always a 2-r.e. degree $\mathbf{b}$ between $\mathbf{a}$ and $\mathbf{c}$ as long as $\mathbf{a}$ is r.e. and $\mathbf{c}$ is 2-r.e. Theorem 1.11 says that the answer is yes if we give up the requirement that $\mathbf{b}$ be 2-r.e. but assume that $\mathbf{a}$ is low and $\mathbf{c}$ is $\mathbf{0}'$. We do not know if it is possible to also make $\mathbf{b}$ 2-r.e. We can instead describe an argument that will produce $\mathbf{b}$ REA in $\mathbf{a}$ and below $\mathbf{c}$ if $\mathbf{a}$ is low and $\mathbf{c}$ is high:

Proposition 1.13. *If* $\mathbf{c} < \mathbf{h}$ *are r.e.,* $\mathbf{c}$ *is low and* $\mathbf{h}$ *is high, then there is an* $\mathbf{a} < \mathbf{h}$ *which is REA in* $\mathbf{c}$ *but not r.e.*

Proof. We describe the modifications needed in the construction of Soare and Stob [36]. First note that $\mathbf{c}'' = \mathbf{h}'$. Thus there is a function k recursive in $\mathbf{h}$ such that k dominates every function recursive in $\mathbf{c}$. Let H be an r.e. set of degree $\mathbf{h}$ and e be such that $\Phi_e^H = k$. Let $g(x,s) = \Phi_e(H;x)[s]$, if it is convergent and 0 otherwise (with the usual convention that $\Phi_e(H;x)[s] < s$). Of course, g is recursive, $\lim_s g(x,s) = k(x)$, the limit is reached only after $k(x)$ (i.e. $\mu s(\forall t > s\{g(x,s) = g(x,t)\}) > k(x)$) and, recursively in H, we can find a stage s after which $g(x,s)$ never changes. We adjust the construction as follows. When we seem to have a situation in which we would want to put x_{i-1}^s into $A(B)$ (remember, Soare and Stob construct two sets, A and B, one of which is of the desired degree $\mathbf{a}$) with some associated axiom, we preserve $A(B)$ on the axiom use and wait for $g(x_{i-1}^s, t)$ to change. If it changes before C changes on the axiom, we put x_{i-1}^s into $A(B)$. Otherwise, we proceed as in [36]. To verify that the construction works, suppose each oracle question about getting C-correct computations as needed to trigger our wanting to put each x_{i-1}^s into $A(B)$ is eventually answered yes. (If not then we satisfy the requirement by some finite action or a divergence attested to by this answer.) In this case, we argue that C is recursive for a contradiction. The function of i giving the stages at which we get C-correct computations for wanting to put x_{i-1}^s into $A(B)$ is recursive in C. Thus for almost every i, we actually do put x_{i-1}^s into $A(B)$ after the associated axiom is C-correct. Thus we can argue as in the original paper that C is recursive except that we begin at the point after which every x_{i-1}^s gets into $A(B)$ after the previous use is correct. (The inductive argument proceeds by showing that, once C is correct on the interval determined by x_{i-1}, x_i, the next stage at which $W(V)$ changes on the interval determined by x_i, x_{i+1} gives a stage after which C itself cannot change on the interval determined by x_i, x_{i+1}.) Of course, the sets A, B constructed are recursive in H by the permitting restriction on enumerating numbers into them as (uniformly in x) H can compute a stage after which $g(x,s)$ never changes. □

In the two remaining sections of the paper, we provide two other pieces of information about this question. The first says that the answer is yes if **a** is high. The second says in a very strong way that, in general, the answer is no. Indeed, it is no even if we drop the requirement that **b** be 2-r.e. and even if we fix **c** to be $\mathbf{0}'$.

Theorem 2.1. *For all high r.e. degrees* $\mathbf{h} < \mathbf{g}$ *there is a properly d-r.e. degree* $\mathbf{a}$ *such that* $\mathbf{h} < \mathbf{a} < \mathbf{g}$ *and* $\mathbf{a}$ *is r.e. in* $\mathbf{h}$.

Theorem 3.1. *There is an incomplete nonrecursive r.e. A such that every set REA in A and recursive in* $0'$ *is of r.e. degree.*

The proof of the first of these results combines highness with a modified version of the proof strategy of Cooper et al. [13]. A description of the needed modifications is given in Section 2. The second theorem is a rather surprising result with a somewhat unusual proof strategy. It is a $0'''$ argument that at times moves left in the tree so that the accessible nodes are not linearly ordered at each stage. Thus the construction lacks a true path in the usual sense. Two substitute notions fill this role: The *true nodes* are the leftmost ones accessible infinitely often; the *semitrue nodes* are the leftmost ones such that there are infinitely many stages at which some extension is accessible. Another unusual feature of the construction is that it involves using distinct priority orderings to control the interactions of different parts of the construction.

An intuitive description of the construction and a description of these orderings along with a formal definition of the construction and full proof is in given in Section 3. We just note here that by Proposition 1.13 and Theorem 2.1 the set A constructed in Theorem 3.1 cannot be either low_2 or high, so in particular $\emptyset' <_T A' <_T \emptyset''$. (The theorem immediately rules out the possibility that A could be high. On the other hand, if $A'' \equiv_T 0''$, choose a $D <_T A$ with $D' \equiv_T A'$. Then $0'$ is high over D and A is low over it. By the Proposition relativized to D, there would be a set B REA in A and below $0'$ but not of degree REA in D and so certainly not of r.e. degree.)

It is tempting to suggest that Theorem 1.11 might be improved by changing the top degree $\mathbf{0}'$ to any r.e. degree $\mathbf{b} > \mathbf{a}$ (as long as $\mathbf{a}$ is low) in analogy with Theorem 2.1 where we only require that the bottom degree $\mathbf{h}$ be high. Proposition 1.13 does this for $\mathbf{b}$ high but it does not seem possible to make $\mathbf{b}$ an arbitrary r.e. degree above $\mathbf{a}$. More precisely, we can use Theorem 3.1 to prove that this proposal fails relative to some degree: Let $\mathbf{a}$ be the degree of the r.e. set A of Theorem 3.1 and let $\mathbf{c} < \mathbf{a}$ be such that $\mathbf{c}' = \mathbf{a}'$. Thus $\mathbf{a}$ is low with respect to $\mathbf{c}$ and $\mathbf{a} < \mathbf{0}'$ but there is no degree $\mathbf{d}$ which is REA in $\mathbf{a}$ and below $\mathbf{0}'$ which is not of r.e. degree and so, of course, r.e. in $\mathbf{c}$.

Another interesting notion related to density connecting the r.e. and 2-r.e. degrees has been introduced by Cooper and Yi [14]:

Definition 1.14. A 2-r.e. degree $\mathbf{d}$ is isolated by the r.e. degree $\mathbf{a}$ if $\mathbf{a} < \mathbf{d}$ and every r.e. $\mathbf{b} < \mathbf{d}$ is also less than or equal to $\mathbf{a}$.

Cooper and Yi [14] prove that there are such degrees and that there are 2-r.e. degrees $\mathbf{d}$ which are not isolated by any r.e. degree $\mathbf{a}$. They also raise a number of interesting questions about the isolated and isolating degrees which are answered in forthcoming papers by LaForte [25], Ding and Qian [15] and Arslanov et al. [4].

Our notation is generally standard and follows Soare [35]. We note, however, that we append $[s]$ to various functionals such as $\Phi_e(A;x)[s]$ to indicate the state of affairs at stage s. In particular, if A is r.e. (or otherwise being approximated) we mean by this notation the result of running the eth Turing machine for s steps on input x with oracle $A[s]$, the subset of A enumerated by stage s (the approximation to A at stage s). We take the use of this computation to be the greatest number about which it queries the oracle and denote it by $\phi_e(A;x)[s]$; so changing the oracle at $\phi_e(A;x)[s]$ destroys the computation. In particular, if A is r.e. we may assume that $\phi_e(A;x)[s]$ is not in $A[s]$ and so putting it in destroys the computation. We also use a modified version of the restriction notation for functions to mesh with this definition of the use: $f\lceil x$ means the restriction of the function f to numbers $y \leqslant x$. Thus, if $\Phi_e(A;x)$ is convergent, then the use is $A\lceil\phi_e(A;x)$ and changing A at $\phi_e(A;x)$ destroys this computation (and similarly for computations and approximations at stage s of a construction).

2. Interpolation between high degrees

Theorem 2.1. *For all high r.e. degrees* $\mathbf{h} < \mathbf{g}$ *there is a properly d-r.e. degree* $\mathbf{a}$ *such that* $\mathbf{h} < \mathbf{a} < \mathbf{g}$ *and* $\mathbf{a}$ *is r.e. in* $\mathbf{h}$.

Proof. Let $H \in \mathbf{h}$ and $G \in \mathbf{g}$ be fixed r.e. sets. We will construct a d-r.e. set D so that $A = H \oplus D$ has the desired properties, namely, A is r.e. in H, $A \leqslant_T G$ and A does not have r.e. degree.

To satisfy the last property we meet the following requirements for all e,

$$R_e\colon\ D \neq \Phi_e^{W_e} \vee W_e \neq \Psi_e^{H\oplus D},$$

where $\{(W_e, \Phi_e, \Psi_e)\}_{e\in\omega}$ is some enumeration of all possible triples consisting of an r.e. set W and partial recursive functionals Φ and Ψ. In addition, we will ensure that $A \leqslant_T G$ by a permitting argument.

To meet these requirements we use the strategy for the Weak Density Theorem from [13] with some modifications.

The basic strategy for R_e without the requirement $A \leqslant_T G$ and in the absence of any H-changes is the one developed by Cooper [15] to prove the existence of a properly d-r.e. degree. To attack R_e we choose an unused witness x and wait for a stage s such that

$$D_s(x) = \Phi_e^{W_e}\lceil \varphi_e(x)[s] \wedge W_e\lceil \varphi_e(x)[s] = \Psi_e^{(H\oplus D)\lceil \psi_e\varphi_e(x)}\lceil \varphi_e(x)[s];$$

preserve $D\lceil \psi_{e,s}\varphi_{e,s}(x)$ from injury by other strategies; put x into D and wait for a stage s' at which

$$D_{s'}(x) = \Phi_e^{W_e}\lceil \varphi_e(x)[s'] \wedge W_e\lceil \varphi_e(x)[s'] = \Psi_e^{(H\oplus D)\lceil \psi_e\varphi_e(x)}\lceil \varphi_e(x)[s'].$$

We then remove x from D and preserve $D\lceil \psi_{e,s'}\varphi_{e,s'}(x)$.

If $H\lceil \psi_{e,s}\varphi_{e,s}(x)$ does not change after stage s then x is a witness to the success of R_e. As in [13], we now impose "indirect" restraint on H by threatening to show that $G \leqslant_T H$ via a functional Γ_e. We make infinitely many such attacks on R_e by an ω-sequence of "cycles", where each cycle k proceeds as follows:

1. Choose an unused candidate x_k greater than any number mentioned thus far in the construction.
2. Wait for a stage s at which

$$D(x_k) = \Phi_e^{W_e}\lceil \varphi_e(x_k) \wedge W_e\lceil \varphi_e(x_k) = \Psi_e^{(H\oplus D)\lceil \psi_e\varphi_e(x_k)}\lceil \varphi_e(x_k).$$

   (If this never happens then x_k is a witness to the success of R_e.)
3. Preserve $D\lceil \psi_{e,s}\varphi_{e,s}(x_k)$.
4. Set $\Gamma_e^H(k) = G_s(k)$ with *use* $\gamma_e(k) = \psi_{e,s}\varphi_{e,s}(x_k)$, and start cycle $k+1$ to run simultaneously with cycle k.
5. Wait for $G(k)$ to change (at a stage s', say).
6. Stop cycles $k' > k$, put x_k into D.
7. Wait for a stage s'' at which

$$D(x_k) = \Phi_e^{W_e}\lceil \varphi_e(x_k) \wedge W_e\lceil \varphi_e(x_k) = \Psi_e^{(H\oplus D)\lceil \psi_e\varphi_e(x_k)}\lceil \varphi_e(x_k).$$

8. Remove x_k from D and preserve $D\lceil \psi_{e,s''}\varphi_{e,s''}(x_k)$.

Whenever some cycle sees an $H\lceil \psi_{e,s}\varphi_{e,s}(x_k)$-change after stage s, it will kill the cycles $k' > k$, make their functionals undefined, and go back to step 2.

The module has the following possible outcomes:

(A) Eventually each cycle k gets stuck at step 5 waiting for a $G(k)$-change, or gets an $H\lceil\psi_{e,s}\varphi_{e,s}(x_k)$-change after step 6. In this case, $\Gamma_e^H = G$, contrary to hypothesis.

(B) Some (least) cycle k_0 gets stuck at step 2, 7, or 8. Then we were successful in restraining H and satisfy R_e through cycle k_0.

(C) Some (least) cycle k_0 gets infinitely many H-changes after step 2. Then $\Phi_e^{W_e}$ or $\Psi_e^{H\oplus D}$ is partial, and R_e is again satisfied by cycle k_0.

Therefore, either we were successful in satisfying R_e through outcomes (B) or (C), or there are infinitely many cycles with a G-change such that $H\lceil\psi_{e,s}\varphi_{e,s}(x_k)$ does not change after step 6. Keeping this in mind let us now turn to the requirement that A is r.e. in H.

To ensure this result we use a common method which works as follows. When a witness x_k is enumerated into D at stage s we appoint a certain marker $\alpha(x_k)$. Then we allow x_k to be removed from A at a later stage t only if $H\lceil\alpha(x_k) \neq H_t\lceil\alpha(x_k)$.

Obviously, this ensures that A is r.e. in H. But now the difficulty is that the $H\lceil\alpha(x_k)$-change may entail an $H\lceil\psi_{e,s}\varphi_{e,s}(x_k)$-change after stage s' and so after step 6 (if $\alpha(x_k)\leqslant\psi_{e,s}\varphi_{e,s}(x_k)$) which ruins our attack of R_e by the witness x_k.

As we saw before, if we are not successful via outcome (B) or (C), then we must have infinitely many cycles k such that $G(k)$ changes after stage s but $H\lceil\psi_{e,s}\varphi_{e,s}(x_k)$ does not change after step 6. We define a partial recursive function α such that in this case, by a characterization of high degrees, beginning with some k_0, any cycle $k > k_0$ gets an $H\lceil\alpha(x_k)$-change after the stage $m = s''$ of step 7.

Therefore, for some cycle $k > k_0$ we will have a $G(k)$-change at step 5, no $H\lceil\psi_{e,s}\varphi_{e,s}(x_k)$-change after step 6, and an $H\lceil\alpha(x_k)$-change after step 7. This will be sufficient to win R_e through cycle k.

By a theorem of Robinson [29], we may choose a r.e. set $H \in \mathbf{h}$ and an effective enumeration $\{H_s\}_{s\in\omega}$ of H so that the computation function

$$c_H(x) = (\mu s > x)[H_s\lceil x = H\lceil x]$$

dominates all recursive functions.

Now we define functions α and m in the following way: Each cycle k proceeds as above but with the following step inserted after step 6:

$6\frac{1}{2}$. (a) Let $\alpha(x_k)$ be a number greater than any mentioned thus far in the construction, in particular greater than the maximum of all current Ψ_e-uses.

(b) Suppose p is the least integer such that $m(p)$ is undefined. We will define $m(p)$ to be the first stage $t > s$ (if there is one) such that either

$$H_t\lceil\psi_{e,s}\varphi_{e,s}(x_k) \neq H_{t-1}\lceil\psi_{e,s}\varphi_{e,s}(x_k),$$

or

$$D_t(x) = \Phi_e^{W_e\lceil\varphi_e(x)}(x)[t] \wedge W_e\lceil\varphi_e(x)[t] = \Psi_e^{(H\oplus D)\lceil\psi_e\varphi_e(x)}\lceil\varphi_e(x)[t].$$

(which is step 7 of the k-cycle).

Clearly, $\alpha(x_k) \geqslant p$. Notice also that if $m(p)$ is not defined for some (least) p, then the requirement R_e is satisfied by the cycle k at which the search for $m(p)$ was begun.

If m is total then $c_H(p) > m(p)$ for all $p \geqslant$ some p_0. For any such p we have $H_{m(p)}\lceil p \neq H\lceil p$. If $m(p)$ was defined by cycle k then $\alpha(x_k) \geqslant p$. It follows that $H_{m(p)}\lceil \alpha(x_k) \neq H\lceil \alpha(x_k)$ for all $k \geqslant$ some k_0. We have already mentioned that $H\lceil \psi_{e,s}\varphi_{e,s}(x_k)$ does not change after step 6 for infinitely many k. For any such k we have $m(p) = s''$ (the stage of step 7). This means that all these cycles receive the desired $H\lceil \alpha(x_k)$-change after step 7.

Now each cycle k proceeds as above but with step $6\frac{1}{2}$ inserted after step 6 and the following step inserted after step 7:

$7\frac{1}{2}$. Wait for $H\lceil \alpha(x_k)$ to change and then proceed.

This ensures that A is r.e. in H.

Now we have to ensure that $A \leqslant_T G$ through a permitting argument. The strategy again is essentially the same as in [13].

We need G to permit x to enter D at step 6 as well as to leave D at step 8. The former permission is already given by the $G(k)$-change at step (6). As in [13], the latter has to be built into the strategy (by asking for permission j many times for larger and larger j).

The basic module for the R_e-strategy consists of an $(\omega \times \omega)$-sequence of cycles (j,k) for $j,k \in \omega$. Cycle $(0,0)$ starts first, and each cycle (j,k) can start cycles $(j,k+1)$ or $(j+1,0)$ and stop, or cancel, cycles (j',k') for $(j,k) < (j',k')$ (in the lexicographical ordering). Each cycle (j,k) can define $\Gamma_j^H(k)$ and $\Delta^H(j)$. (Γ_j and Δ are functionals that are threatening to compute G from H.) We also define functions m and α. Each cycle (j,k) may define values $\alpha(x)$ and $m(p)$ for the current witness x and the least p such that $m(p)$ is undefined, respectively. The cycle proceeds as follows:

1. Choose an unused candidate x greater than any number mentioned thus far in the construction.
2. Wait for a stage s at which
$$D(x) = \Phi_e^{W_e\lceil \varphi_e(x)}(x) \wedge W_{e,s}\lceil \varphi_e(x) = \Psi_e^{(H\oplus D)\lceil \psi_e f \varphi_e(x)}\lceil \varphi_e(x).$$
(If this never happens then x is a witness to the success of R_e.)
3. Preserve $D\lceil \psi_{e,s}\varphi_{e,s}(x)$ from injury by other strategies from now on.
4. Set $\Gamma_j^H(k) = G_s(k)$ with *use* $\gamma_j(k) = \psi_{e,s}\varphi_{e,s}(x)$, and start cycle $(j,k+1)$ to run simultaneously with cycle (j,k).
5. Wait for $H\lceil \psi_{e,s}\varphi_{e,s}(x)$ or $G(k)$ to change (at a stage s', say). If H changes first then cancel cycles $(j',k') > (j,k)$, drop the D-restraint of cycle (j,k) to 0, and go back to step 2. If G changes first then stop cycles $(j',k') > (j,k)$ and proceed to step 6.
6. Put x into D.
7. Let $\alpha_s(x)$ be a number greater than all mentioned thus far in the construction, in particular greater than the maximum of all current Ψ_e-uses. Suppose p is the least integer such that $m(p)$ is undefined. Define $m(p)$ to be the first stage $t > s$ (if

one exists) such that either $H_t\lceil\psi_{e,s}\varphi_{e,s}(x) \neq H_{t-1}\lceil\psi_{e,s}\varphi_{e,s}(x)$, or

$$D_t(x) = \Phi_e^{W_e\lceil\varphi_e(x)}(x)[t] \wedge W_e\lceil\varphi_e(x) = \Psi_e^{(H\oplus D)\lceil\psi_e\varphi_e(x)}\lceil\varphi_e(x)[t].$$

8. Wait for a stage s'' at which

$$D(x) = \Phi_e^{W_e\lceil\varphi_e(x)}(x) \wedge W_{e,s''}\lceil\varphi_e(x) = \Psi_e^{(H\oplus D)\lceil\psi_e\varphi_e(x)}\lceil\varphi_e(x).$$

9. Preserve $D\lceil\psi_{e,s''}\varphi_{e,s''}(x)$ from injury by other strategies from now on.
10. Set $\Delta^H(j) = G_{s''}(j)$ with *use* $\delta(j) = \psi_e\varphi_e(x)$ and start cycle $(j+1,0)$ to run simultaneously with the (j,k) cycles now running.
11. Wait for $H\lceil\psi_{e,s''}\varphi_{e,s''}(x)$ or $G(j)$ to change. If H changes first then cancel cycles $(j',k') \geqslant (j+1,0)$, drop the D-restraint of cycle (j,k) to $\psi_{e,s}\varphi_{e,s}(x)$, and go back to step 8. If G changes first then stop cycles $(j',k') \geqslant (j+1,0)$ and proceed to step 12.
12. Wait for $H\lceil\alpha_s(x)$ to change.
13. Remove x from D.
14. Wait for

$$H\lceil\psi_{e,s}\varphi_{e,s}(x) \neq H\lceil\psi_e\varphi_e(x)[s] \text{ or } H\lceil\psi_{e,s''}\varphi_{e,s''}(x) \neq H\lceil\psi_e\varphi_e(x)[s''].$$

    Proceed to step 15 or 16, respectively.
15. Reset $\Gamma_j^H(k) = G(k)$, cancel cycles $(j',k') > (j,k)$, start cycle $(j,k+1)$, and halt cycle (j,k).
16. Reset $\Delta^H(j) = G(j)$, cancel cycles $(j',k') \geqslant (j+1,0)$, start cycle $(j+1,0)$, and halt cycle (j,k).

Whenever a cycle (j,k) is started, any previous version of it has been cancelled and its functionals have become undefined through H-changes. Therefore Γ_j and Δ are defined consistently.

The explicit construction and the remaining parts of the proof of Theorem 2.1 are now essentially the same as in [13] with only obvious changes. So we will not give them here except for the proof of the claim that $A \leqslant_T G$ which now is a little more delicate.

Lemma 2.2. $D \leqslant_T G$.

Proof. To G-recursively compute whether $x \in D$, first find a stage s such that $G_s\lceil x = G\lceil x$. If $\alpha_s(x)$ is not defined then $x \notin D$. Otherwise, find a stage t such that $H_t\lceil\alpha_s(x) = H\lceil\alpha_s(x)$. (Remember, $H \leqslant_T G$.) Now $x \in D$ if and only if $x \in D_t$. □

3. A noninterpolation result

Theorem 3.1. *There is an incomplete nonrecursive r.e. A such that every set REA in A and recursive in $0'$ is of r.e. degree.*

We will build the desired r.e. set A along with an auxiliary r.e. set C and various r.e. sets B_e. There are three types of requirements for our construction.

- P_e: $\Phi_e \neq A$ (for each partial recursive functional Φ_e).
- N_e: $\Phi_e^A \neq C$ (for each partial recursive functional Φ_e).
- R_e: If $W_e^A = \Psi_e^K$ then $W_e^A \leqslant_T B_e \oplus A$ & $B_e \leqslant_T W_e^A \oplus A$ (for each partial recursive functional Ψ_e and each r.e. in A set $W_e^A = \mathrm{dom}(\Phi_e^A)$ we build an associated r.e. set B_e).

The first two types of requirements are handled in the usual way. For P_e we will choose a follower x from the column associated with the requirement which is larger than all higher priority restraints. When the follower is realized ($\Phi_e(x)\downarrow = 0$), we will put x into A. For N_e we will choose a follower x from the column associated with the requirement, wait for $\Phi_e(A;x)$ to converge and then put x into C and attempt to preserve A on $\phi_e(x)$, the use of the computation. (This preservation will be interconnected with the actions for requirements related to various R_i.)

The basic plan for R_e is that, when the length of agreement between W_e^A and Ψ_e^K becomes larger than y, we will appoint markers $b_{e,y}$ and $a_{e,y}$ targeted for B_e and A, respectively. If, at a later stage s, it appears that $y \in W_e^A$ and $\Psi_e^K \lceil y = W_e^A \lceil y$, we would expect to put $b_{e,y}$ into B_e and protect the use $\phi_e(A;y)[s]$ of the computation putting y into W_e^A. With an eye towards showing that $W_e^A \leqslant_T A \oplus B$, we would then promise to put $a_{e,y}$ into A if y leaves W_e^A because of a change in $A\lceil\phi_e(y)[s]$ to record, in $A\oplus B$, the fact that y does not seem to be in W_e^A. Of course, this would immediately impose infinitary restraint on the construction and prevent us from satisfying the positive requirements. The natural procedure now is to break R_e up into subrequirements. We phrase them so as to also make our intended reductions between $W_e^A \oplus A$ and $B_e \oplus A$ explicit:

- $R_{e,y}$: If $W_e^A(y) = \Psi_e(K;y) = 1$ then [there is eventually a pair of markers such that] $b_{e,y} \in B_e$ and $a_{e,y} \notin A$. If $W_e^A(y) = \Psi_e(K;y) = 0$ then [there is eventually a marker] $b_{e,y} \notin B_e$.

Thus our procedure for R_e will measure the length of agreement between W_e^A and Ψ_e^K and appoint markers $a_{e,y}$, $b_{e,y}$, but it will be $R_{e,y}$ that starts our action by putting $b_{e,y}$ into B_e when appropriate. $R_{e,y}$ will then impose restraint on $A\lceil\phi_e(y)[s]$. It is the interaction of these restraints, and that of the overtly negative requirements N_i, with our overarching commitment to put other $a_{e',y'}$ into A if $b_{e',y'}$ is put into $B_{e'}$ at s' and A later changes, say at s'', on $\phi_{e'}(y')[s']$ that is the source of the real difficulty in satisfying the requirements. For example, suppose $a_{e',y'} < \phi_e(y)[s]$ but $s' > s$. At s'' we would have to put $a_{e',y'}$ into A and so injure $R_{e,y}$. If we attempt to simply increase the restraint imposed by $R_{e,y}$ to prevent the A change on $\phi_{e'}(y')[s']$, we will eventually impose larger and larger restraint in this effort: When we put $b_{e',y'}$ into $B_{e'}$ at s', $R_{e,y}$ will impose restraint on $\phi_{e'}(y')[s']$, but then some new $a_{e'',y''}$ may be smaller than $\phi_{e'}(y')[s']$. If we then must put $b_{e'',y''}$ into $B_{e''}$ at $s'' > s'$ we will have to impose restraint $\phi_{e''}(y'')[s'']$. For, if not, when some lower priority P_i puts some $x < \phi_{e''}(y'')[s'']$ into A we will have to put $a_{e'',y''}$ into A. This will force us to put $a_{e',y'}$

into A and injure $R_{e,y}$'s restraint. (We will call this sequence of numbers that we are successively forced to put into A because of x's entry the *cascade* (of elements into A) initiated by x's entry into A.) Of course, the positive requirements cannot live with the infinitary restraint that would be imposed in this way by even a single subrequirement $R_{e,y}$.

The solution to this conflict has two components. On the one hand, we allow the restraint to grow as described above but only for $a_{e',y'}$ of "higher priority" than $R_{e,y}$. On the other hand, before putting $b_{e,y}$ into B_e at s and imposing our restraint, we act to preempt the possible actions of "lower priority" $a_{e',y'}$ that might later injure $\phi_e(y)[s]$. We do this by immediately putting these markers into A ourselves. In this case, we ourselves may destroy the computation of $\Phi_e(A;y)$ and so obviate the need to put $b_{e,y}$ into B_e and impose restraint. The price we pay for this security is that we may be forced to do this infinitely often (y may enter and leave W_e^A infinitely often) and so $R_{e,y}$ or N_i may become an infinitary positive requirement.

Initially, we deal with this in the usual way by employing a tree argument with nodes α assigned to the various requirements $P_e, N_e, R_e, R_{e,y}$. On each path of our priority tree T we will have a node ε assigned to R_e before any assigned to an $R_{e,y}$. It is at such nodes ε that we assign markers $b_{\varepsilon,y}$ and $a_{\varepsilon,y}$. If α is assigned to some $R_{e,y}$ then it works on the associated set B_ε being built at the last (i.e. longest) node $\varepsilon \subset \alpha$ assigned to R_e by dealing with the markers $a_{\varepsilon,y}$, $b_{\varepsilon,y}$. In this situation, we will say that α is *associated* with ε, y. We begin the construction by associating with $\emptyset$, the root of T, any ω-type ordering of these requirements, $<_\emptyset$, such that R_e precedes every $R_{e,y}$ and the requirements P_i occupy every other place in this ordering. [The second condition is a technical convenience that prevents two similar types of requirements from being assigned to successive nodes on the tree.]

The crucial point about our action for $R_{e,y}$ is that if we do actually act positively for it infinitely often, then the hypotheses of R_e fail: $y \notin W_e^A$ but $\Psi_e(K;y)[t] = 1$ for infinitely many t and so $W_e^A(y) \neq \Psi_e(K;y)$. Thus we satisfy the overall requirement R_e. We will then restart all requirements $R_{e'}$ of lower priority than R_e below this outcome in the usual $0'''$ fashion. We phrase this in terms of defining a priority ordering $<_{\alpha\hat{}z}$ associated with outcome z of node α and assigning the first element of this ordering to $\alpha\hat{}z$. One somewhat unusual point to keep in mind is that the preemptive positive action for $R_{e,y}$ may well be directed by some higher priority requirement wishing to keep $a_{\varepsilon,y}$ out of A. In this case, we assign the outcome corresponding to the infinitary positive action that shows that R_e is satisfied to the node of highest priority restraining $a_{\varepsilon,y}$. It is this procedure that at times forces us to jump to the left in the priority tree when determining the next accessible node.

The final issue to be considered is the appropriate priority ordering to be used to decide if action for a node α assigned to $R_{e,y}$ and associated with some incarnation of R_e at some earlier node ε can preempt another requirement assigned to some $R_{e',y'}$ by putting $a_{\varepsilon',y'}$ into A. The ordering that correctly takes into account the idea that actions for α assigned to $R_{e,y}$ cannot ruin the intended reductions for $R_{e'}$ of higher priority (for example, by sending the markers $a_{\varepsilon,y}$ and $b_{\varepsilon,y}$ to infinity) and still manages to spread the restraint out in such a way as to keep it finite is the lexicographic ordering of pairs $\langle \varepsilon, y \rangle$ with which α is associated. Here, the first coordinates are themselves nodes on the tree and are given the usual priority ordering of a tree construction. The second coordinates are just numbers with the usual ordering on ω. We can now describe the formal construction.

3.1. Construction

We will define a tree construction priority argument that is somewhat different from the standard arguments like those in [35, Ch. XIV]. We use $<$ to denote the usual priority ordering on the sequences (of outcomes) which are the nodes of our priority tree T. We use $<_L$ to denote the usual left-to-right ordering on the priority tree that corresponds to the lexicographic ordering on nodes incomparable in the subsequence relation. At each stage s we will proceed through a sequence of substages u at each of which we will define an *accessible* node α. (When it is necessary to distinguish the substage u of stage s at which we are acting, as for example, to indicate the current value of the restraint function for α, we write $r(\alpha, u)$ in place of $r(\alpha, s)$. In such cases, the stage s of which u is a substage will be determined by the context.) If α is accessible at some substage u of s, we call s an α-stage as usual. However, the accessible nodes will not necessarily be nested in the subsequence ordering $\subseteq$; there may be jumps to the left. We terminate stage s when we reach a node of length s. Until such a substage, we act for each node α when it becomes accessible in some way which may include adding to the possible outcomes of a node of higher priority. We will also declare some node β to be the next accessible node and define an ordering $<_\beta$ of (a subset of the) initial requirements and assign the first requirement in $<_\beta$ to this node. The other specific actions for an accessible node α at substage u of stage s are determined by the type of requirement assigned to α and are specified below. [Remarks in square brackets [ ] are to help explain the construction. They are not part of the formal procedure.] Before defining the specific actions for each type of requirement we give some general rules for our construction.

We will put a marker $b_{\varepsilon,y}$ into B_ε only at a stage s when some γ assigned to $R_{e,y}$ and associated with ε, y is accessible and $\Phi_e(A;y)\downarrow[s]$. We will then put $a_{\varepsilon,y}$ into A whenever A later changes on $\phi_e(A;y)[s]$. Thus, when any number z is put into A, we immediately check to see if this action necessitates putting any markers $a_{\varepsilon,y}$ into A and then continue this process until it stops (as it must, as there are only finitely many markers defined at any stage). We call this the *cascade* (of elements into A) initiated by z's entering A. The markers $b_{\varepsilon,y}$ and $a_{\varepsilon,y}$, once defined, become undefined if and only if $a_{\varepsilon,y}$ enters A or ε is initialized. Initialization of a node ε assigned to a requirement R_e consists of canceling all markers $a_{\varepsilon,y}$, $b_{\varepsilon,y}$. Such a node ε is initialized whenever a node $\gamma <_L \varepsilon$ becomes accessible and at certain other times described in the construction. Initialization for a node α assigned to a requirement N_e or $R_{e,y}$ at substage u of stage s consists of canceling the current follower (for N_e), setting the associated restraint $r(\alpha,u) = 0$, and so the auxiliary set

$S(\alpha, u) = \emptyset$ (but not cancelling the markers $a_{\varepsilon,y}$, $b_{\varepsilon,y}$ for the $\langle \varepsilon, y \rangle$ associated with α). The auxiliary set $S(\alpha, u)$ is introduced for notational convenience and is defined as $\{\langle \varepsilon, y \rangle \mid a_{\varepsilon,y} \downarrow [u] < r(\alpha, u)\}$. These nodes are initialized whenever a $\gamma <_L \alpha$ becomes accessible and only then. We now describe the actions at the node α which becomes accessible at substage u of stage s according to the type of requirement assigned to α.

R_e: For notational convenience, we denote by ε the node assigned to R_e that has just been declared accessible. At this node we measure the length of agreement ℓ between W_e^A and Ψ_e^K. To do this appropriately for the ε-stages, we incorporate the idea of the "hat trick" into the definition of the versions $\Phi_\varepsilon^A, W_\varepsilon^A$ and Ψ_ε^K of Φ_e^A, W_e^A and Ψ_e^K, respectively, that we use at ε. Let t be the last ε-stage before s (0 if s is the first ε-stage). We define $\Phi_\varepsilon^A, W_\varepsilon^A$ and Ψ_ε^K as follows:

If $K_s \lceil \psi_{e,s}(x) = K_t \lceil \psi_{e,s}(x)$ then $\Psi_\varepsilon(K;x)[s] = \Psi_e(K;x)[s]$;
otherwise $\Psi_\varepsilon(K;x)[s]$ is divergent.

If $A_s \lceil \phi_{e,s}(x) = A_t \lceil \phi_{e,s}(x)$, then $\Phi_\varepsilon(A;x)[s] = \Phi_e(A;x)[s]$;
otherwise $\Phi_\varepsilon(A;x)[s]$ is divergent.

$$x \in W_\varepsilon^A[s] \Leftrightarrow \Phi_\varepsilon(A;x)[s] \downarrow .$$

We define the length of agreement function as usual:

$$\ell(\varepsilon, s) = \mu x \neg (W_\varepsilon^A(x)[s] = \Psi_\varepsilon(K;x)[s]).$$

The possible outcomes for ε are ∞ and 0 (in left to right order). If $\ell(\varepsilon, s)$ has reached a new maximum, i. e. $\ell(\varepsilon, s) > \ell(\varepsilon, t)$ for every previous ε-stage t, then the outcome of ε is ∞ and we declare $\varepsilon\hat{}\infty$ accessible. Its associated priority ordering $<_{\varepsilon\hat{}\infty}$ is the same as that for ε with R_e removed from the beginning of the ordering. If any of the markers $b_{\varepsilon,y}$, $a_{\varepsilon,y}$ are undefined for $y < \ell(\varepsilon, s)$, we define them to be new distinct large numbers in $\omega^{[\varepsilon]}$. [This happens only when $\ell(\varepsilon, s) > y$ for the first time or we have put $a_{\varepsilon,y}$ into A or initialized ε since the last ε-stage. The actions enumerating elements $b_{\varepsilon,y}$ into B_ε take place after we reach a node below $\varepsilon\hat{}\infty$ associated with the subrequirements $R_{e,y}$.] Otherwise, ε's outcome is 0; its associated priority ordering $<_{\varepsilon\hat{}0}$ is the same as that for ε except that R_e and all its subrequirements $R_{e,y}$ are removed from the list.

$R_{e,y}$: Suppose ε is the longest node $\subset \alpha$ assigned to R_e. We say that α is *associated* with $\langle \varepsilon, y \rangle$. [We shall see that for α to be accessible at u, $\varepsilon\hat{}\infty$ must have already have been accessible at some previous substage of s.] The initial possible outcomes of α are 1,0 (in left-to-right order). At any later point t of the construction the set of possible outcomes will be $S(\alpha, t) \cup \{1, 0\}$. (Remember that $S(\alpha, t) = \{\langle \varepsilon, y \rangle \mid a_{\varepsilon,y} \downarrow [t] < r(\alpha, t)\}$.) The elements of $S(\alpha, t)$ are ordered from left to right by the lexicographic ordering on pairs $\langle \varepsilon', y' \rangle$ (where the first coordinates are ordered by the tree priority ordering and the second by the natural ordering on ω). The outcomes 1,0 are then added in order to the (right-hand) end of this ordering. Our action depends on the status of the markers $b_{\varepsilon,y}$ and $a_{\varepsilon,y}$ and whether $y \in W_\varepsilon^A$.

(1) If $a_{\varepsilon,y}$ is undefined then the outcome of α is 0; $\alpha\hat{}0$ is accessible and $<_{\alpha\hat{}0}$ is the final segment of $<_\alpha$ with $R_{e,y}$ removed. [This situation cannot "really" occur infinitely often if the hypotheses of R_e are satisfied and so the outcome is not essential except for the completeness of our description of the construction.]

(2) If $y \notin W_\varepsilon^A$ at s (i.e. $\Phi_\varepsilon(A;y) \uparrow [s]$) [and so $\Psi_\varepsilon(K;y) = \Psi_e(K;y) = 0$] then the outcome of α is 0 [the expected value of $\Psi_e(K;y)$], $\alpha\hat{}0$ is accessible and $<_{\alpha\hat{}0}$ is $<_\alpha$ with $R_{e,y}$ removed. [Note that if $a_{\varepsilon,y}$ is defined then $y < \ell(\varepsilon,s)$ as we are at an ε-expansionary stage and $a_{\varepsilon,y'}$ gets defined only for $y' < \ell(\varepsilon,s)$.]

(3) $y \in W_\varepsilon^A$ at s (i.e. $\Phi_\varepsilon(A;y)\downarrow[s] = \Phi_e(A;y)\downarrow[s]$) [and so $\Psi_\varepsilon(K;y) = \Psi_e(K;y) = 1$] but $b_{\varepsilon,y} \notin B_\varepsilon$ (at u).

We first see if, for the sake of some requirement of higher priority, we need to try to force y out of W_e^A and preempt future actions that might make us put $a_{\varepsilon,y}$ into A. If not, i.e. there is no $\beta < \alpha$ such that $a_{\varepsilon,y} \in S(\beta,u)$, we put $b_{\varepsilon,y}$ into B_ε and let $r(\alpha,u)$ be a new large number. [The purpose of this restraint will be to keep y in W_e^A and so the computation associated with β convergent.] The outcome of α is 1 [for convergent], $\alpha\hat{}1$ is accessible and $<_{\alpha\hat{}1}$ is just $<_\alpha$ with its first element, $R_{e,y}$, removed. On the other hand, if there is a $\beta < \alpha$ such that $a_{\varepsilon,y} \in S(\beta,u)$, we let $\beta < \alpha$ be the highest priority requirement such that $a_{\varepsilon,y} \in S(\beta,u)$ [and so $\langle\varepsilon,y\rangle$ is a possible outcome of β]. We put into A every $a_{\varepsilon',y'}$ such that $\langle\varepsilon,y\rangle < \langle\varepsilon',y'\rangle$ unless

(i) there is a $\gamma \leqslant \beta$ such that $r(\gamma,u) > a_{\varepsilon',y'}$, or

(ii) $\varepsilon' \supseteq \beta\hat{}\langle\eta,z\rangle$ for some $\langle\eta,z\rangle \leqslant \langle\varepsilon,y\rangle$.

If the cascade initiated by putting all of these elements into A includes a number less than $\phi_e(A;y)[s]$ [and so kills the computation], we *jump* to $\beta\hat{}\langle\varepsilon,y\rangle$ and declare it to be accessible. We restart all requirements $R_i >_\varepsilon R_e$ by defining $<_{\beta\hat{}\langle\varepsilon,y\rangle}$ to be the final segment of $<_\varepsilon$ beginning immediately after R_e with all requirements $R_{e,y'}$ removed. If not, the outcome of α is again 1 [for convergent], $\alpha\hat{}1$ is accessible and $<_{\alpha\hat{}1}$ is just $<_\alpha$ with its first element, $R_{e,y}$, removed. In this case, we initialize all ε' with $\varepsilon' \supseteq \beta\hat{}\langle\eta,z\rangle$ for any $\langle\eta,z\rangle$. Next, we redefine the restraint function $r(\beta,u)$ to be a number larger than any used so far in the construction. [The purpose of this restraint will be to preserve the computation associated with β by keeping y in W_e^A.] If the original computation associated with $\langle\varepsilon,y\rangle$ is ever injured, i.e. a number $z \leqslant \phi_e(A;y)[s]$ enters A at some later point t, we put $a_{\varepsilon,y}$ into A and declare the markers $b_{\varepsilon,y}$ and $a_{\varepsilon,y}$ to be undefined, as described by the general rules of our construction.

(4) $y \in W_\varepsilon^A$ at s (i.e. $\Phi_\varepsilon(A;y)\downarrow[s] = \Phi_e(A;y)\downarrow[s]$) [so $\Psi_\varepsilon(K;y) = \Psi_e(K;y) = 1$] and $b_{\varepsilon,y}$ is defined and in B_ε.

We maintain the situation initiated when we put $b_{\varepsilon,y}$ into B_ε: The outcome of α is 1; $\alpha\hat{}1$ is accessible and $<_{\alpha\hat{}1}$ is $<_\alpha$ with $R_{e,y}$ removed. If $r(\alpha,u)$ is not already defined, we let $r(\alpha,u)$ be a new large number and so let $S(\alpha,u)$ consist of all $\langle\varepsilon',y'\rangle$ such that $a_{\varepsilon',y'}$ is defined.

N_e: If α has no current follower [this is the first α-stage or its follower has been canceled by initialization or injury since the last α-stage], we appoint a large number from $\omega^{[\alpha]}$ as the current follower of α and let 1, 0, in left-to-right order, be the possible outcomes of α. Now suppose x is the current follower of α. If $\Phi_e(A;x)\downarrow = 0$ and

$x \notin C$, then we put x into C. In this case, we impose *restraint* $r(\alpha, u)$ on A equal to a new large number. The possible outcomes for α are $S(\alpha, u) \cup \{1, 0\}$ ordered as for $R_{e,y}$ in case (3). The outcome of α is 1, $\alpha\hat{\ }1$ is accessible and $<_{\alpha\hat{\ }1}$ is just $<_\alpha$ with the first element, N_e, removed. If $r(\alpha, t)$ is injured at some later point t, i.e. some $z < r(\alpha, t)$ enters A, we cancel α's current follower. If, at s, $\neg(\Phi_e(A;x)\downarrow = 0)$, then the outcome of α is 0, $\alpha\hat{\ }0$ is accessible and $<_{\alpha\hat{\ }0}$ is also just $<_\alpha$ with the first element, N_e, removed. [$S(\alpha, t)$ may have new pairs added in and the outcomes in $S(\alpha, t)$ may become accessible when we consider jumping to them from a node γ associated with some $\langle \varepsilon, y\rangle$ with $\langle \varepsilon, y\rangle \in S(\alpha, t)$.]

P_e: The possible outcomes of α in left-to-right order are 1, 0. Let x be the least element of $\omega^{[\alpha]}$ larger than all restraints $r(\beta, u)$ for $\beta \leqslant \alpha$. We say that α is satisfied if there is a z such that $\Phi_e(z) = 0$ and $z \in A$. If α is not satisfied and $\Phi_e(x) = 0$, we put x into A. Now, if α is satisfied then its outcome is 1; otherwise, it is 0. In either case, α concatenated with its outcome is accessible and the associated ordering is $<_\alpha$ with its first element, P_e, removed.

3.2. Verifications

We must now verify that the construction satisfies the requirements. As the construction is somewhat unusual, there are a number of auxiliary lemmas that must be proven to show that it behaves at all the way we might expect.

Lemma 3.2. *If some $\beta \supseteq \alpha\hat{\ }\langle \varepsilon, y\rangle$ is accessible at t, then there is an $s \leqslant t$ at which $\alpha\hat{\ }\langle \varepsilon, y\rangle$ becomes accessible by jumping to it from a node γ to its right.*

Proof. The only way to get below $\alpha\hat{\ }\langle \varepsilon, y\rangle$ without first going through it is to jump to some node of the form $\delta\hat{\ }\langle v, z\rangle \supset \alpha\hat{\ }\langle \varepsilon, y\rangle$. However, no δ can have an outcome of the form $\langle v, z\rangle$ before it is accessible. Thus there is a stage at which $\alpha\hat{\ }\langle \varepsilon, y\rangle$ first becomes accessible. It can do so only by our jumping to it from its right. □

Lemma 3.3. (i) *If $\langle \varepsilon, y\rangle \in S(\alpha, s)$, then $\varepsilon < \alpha$.*

(ii) *Moreover, if any $\beta \supseteq \alpha\hat{\ }\langle \varepsilon, y\rangle$ is ever accessible, then $\varepsilon \subset \alpha$.*

Proof. (i) We prove the first assertion of the lemma for all α, ε simultaneously by induction on the (sub)stages of the construction. Suppose s is the first time we produce a counterexample. If $r(\alpha, s)$ is now defined for the first time since α was last initialized, say at t, then if any successor of α has been accessible since t, it must be an extension of $\alpha\hat{\ }0$. Thus when $r(\alpha, s)$ is defined, $\alpha\hat{\ }1$ becomes accessible and all nodes to its right are initialized. In particular, no marker $a_{\varepsilon', y'}$ remains defined for $\alpha < \varepsilon'$. (Markers with ε' to the right of $\alpha\hat{\ }1$ are initialized now; markers with $\varepsilon' \supseteq \alpha\hat{\ }1$ were initialized at t and have not been accessible since then; and $\alpha \neq \varepsilon'$ as they are assigned different requirements.) Thus no such pair is put into $S(\alpha, s)$ contrary to the assumption that α becomes a counterexample at s. If $r(\alpha, s)$ increases at s, then we considered jumping to

$\alpha\hat{}\langle\varepsilon',y'\rangle$ from some γ associated with a $\langle\varepsilon',y'\rangle$ already in $S(\alpha,s)$. By the minimality of s, $\varepsilon' < \alpha$. The construction now directs us to put every $a_{\varepsilon,y}$ with $\langle\varepsilon,y\rangle > \langle\varepsilon',y'\rangle$ into A (and so make these markers undefined) unless $a_{\varepsilon,y} < r(\gamma,s)$ for some $\gamma \leqslant \alpha$ or $\varepsilon \supseteq \alpha\hat{}\langle\eta,z\rangle$ for some $\langle\eta,z\rangle \leqslant \langle\varepsilon',y'\rangle$. If $a_{\varepsilon,y}$ satisfies the first restriction, $\varepsilon < \gamma$ by induction and so $\varepsilon < \alpha$ as required. If $a_{\varepsilon,y}$ satisfies the second restriction, then ε is initialized before we redefine $r(\alpha,s)$ and so again $\langle\varepsilon,y\rangle$ is not eligible to be put into $S(\alpha,s)$ and we have no counterexample to the lemma for α at s.

(ii) By Lemma 3.2, there is a stage s at which $\alpha\hat{}\langle\varepsilon,y\rangle$ becomes accessible by jumping to it from a node γ associated with $\langle\varepsilon,y\rangle$ which is to the right of $\alpha\hat{}\langle\varepsilon,y\rangle$. As γ is associated with $\langle\varepsilon,y\rangle$, $\varepsilon \subset \gamma$ by definition, but, by (i) of our lemma, $\varepsilon < \alpha$ and so $\varepsilon \subset \alpha$ as required. □

Lemma 3.4. (i) *If* $\alpha \supseteq \beta\hat{}\langle\varepsilon,y\rangle, \langle v,z\rangle \in S(\alpha,s)$ *and* $\langle v,z\rangle \notin S(\gamma,s)$ *for any* $\gamma \leqslant \beta$, *then* $\langle v,z\rangle \leqslant \langle\varepsilon,y\rangle$ *or* $v \supseteq \beta\hat{}\langle\varepsilon,y\rangle$.

(ii) *Moreover, if any* $\delta \supseteq \alpha\hat{}\langle v,z\rangle$ *is ever accessible then* $v \subseteq \varepsilon$ *or* $v \supseteq \beta\hat{}\langle\varepsilon,y\rangle$.

Proof. (i) Let $\beta\hat{}\langle\varepsilon,y\rangle$ be fixed. We prove part (i) of the lemma for all $\alpha \supseteq \beta\hat{}\langle\varepsilon,y\rangle$ by induction on the (sub)stages of the construction. Suppose for the sake of a contradiction that substage u of stage s is the first point at which a counterexample occurs and it does so by $\langle v,z\rangle$ entering $S(\alpha,u)$ for $\alpha \supseteq \beta\hat{}\langle\varepsilon,y\rangle$ with $\langle v,z\rangle \nleq \langle\varepsilon,y\rangle$ and $v \not\supseteq \beta\hat{}\langle\varepsilon,y\rangle$. By our convention on the priority ordering $<_\emptyset$, $\alpha \supset \beta\hat{}\langle\varepsilon,y\rangle$ since $\beta\hat{}\langle\varepsilon,y\rangle$ is assigned a requirement of the form P_i. In order for any pair to enter $S(\alpha,u)$ at u, α must be accessible or we must be considering a jump to an immediate extension $\alpha\hat{}\langle\varepsilon',y'\rangle$ of α from some accessible node $\gamma > \alpha$ associated with $\langle\varepsilon',y'\rangle \in S(\alpha,u)$. Thus there must be a substage $v < u$ of stage s at which we actually jump to a node $\delta\hat{}\langle\varepsilon',y'\rangle \supseteq \beta\hat{}\langle\varepsilon,y\rangle$ from a γ associated with $\langle\varepsilon,y\rangle$ which is to the right of $\beta\hat{}\langle\varepsilon,y\rangle$ or we consider jumping to some $\alpha\hat{}\langle\varepsilon',y'\rangle$ from a γ associated with some $\langle\varepsilon',y'\rangle$ which is to the right of $\beta\hat{}\langle\varepsilon,y\rangle$ at u itself.

We first deal with the case that we jump to $\beta\hat{}\langle\varepsilon,y\rangle$ at $v < u$. If $a_{v,z}$ is undefined when we jump to $\beta\hat{}\langle\varepsilon,y\rangle$, it cannot be (re)defined as long as we remain below $\beta\hat{}\langle\varepsilon,y\rangle$, as $v \not\supseteq \beta\hat{}\langle\varepsilon,y\rangle$ and so we cannot produce the assumed counterexample at u. If we do not remain below $\beta\hat{}\langle\varepsilon,y\rangle$ for the rest of stage s, we must move to its left. Once we have moved to the left of $\beta\hat{}\langle\varepsilon,y\rangle$ and so of α, α can never again become accessible or have $S(\alpha,w)$ increase at any later substage of stage s for a contradiction. If $a_{v,z}$ is still defined when we jump to $\beta\hat{}\langle\varepsilon,y\rangle$, $\langle v,z\rangle$ is put into $S(\beta,v)$. It can only leave $S(\beta,w)$ by β or v being initialized. If v is initialized the marker $a_{v,z}$ becomes undefined and we are in the situation just analyzed. On the other hand, β can be initialized only by our moving to its left which again prevents $S(\alpha,w)$ from increasing at any later substage of stage s.

We next deal with the case that, at $v < u$, we jump to some $\delta\hat{}\langle\varepsilon',y'\rangle \supset \beta\hat{}\langle\varepsilon,y\rangle$ from a γ associated with $\langle\varepsilon',y'\rangle$ which is to the right of $\beta\hat{}\langle\varepsilon,y\rangle$. As γ is to the right of $\beta\hat{}\langle\varepsilon,y\rangle$ and $\gamma \supseteq \varepsilon'$, $\varepsilon' \not\supseteq \beta\hat{}\langle\varepsilon,y\rangle$. Moreover, $\langle\varepsilon',y'\rangle$ cannot be in $S(\gamma',v)$ for any $\gamma' \leqslant \beta$ or we would have considered jumping to $\gamma'\hat{}\langle\varepsilon',y'\rangle \neq \delta\hat{}\langle\varepsilon',y'\rangle$ instead. Thus, by our inductive hypothesis at v (with $\alpha = \delta$ and $\langle v,z\rangle = \langle\varepsilon',y'\rangle$), $\langle\varepsilon',y'\rangle \leqslant \langle\varepsilon,y\rangle$.

Now we show that, in this case, if $a_{v,z}$ is defined at v, and not restrained with priority at least β, it enters A and so the marker becomes undefined. By our assumption that $\langle v,z\rangle$ supplies the assumed counterexample, $\langle \varepsilon, y\rangle < \langle v,z\rangle$ and $v \not\supseteq \beta\hat{}\langle \varepsilon, y\rangle$. The first inequality, together with the established fact that $\langle \varepsilon', y'\rangle \leqslant \langle \varepsilon, y\rangle$, implies that $\langle \varepsilon', y'\rangle < \langle v,z\rangle$. The second together with the fact that $\delta\hat{}\langle \varepsilon', y'\rangle \supset \beta\hat{}\langle \varepsilon, y\rangle$, implies that $v \not\supseteq \delta\hat{}\langle \eta, w\rangle$ for any $\langle \eta, w\rangle \leqslant \langle \varepsilon', y'\rangle$. (Indeed, they guarantee that $v \not\supseteq \delta$ at all.) Thus the instructions for the construction at substage v direct us to put $a_{v,z}$ into A and so make the marker undefined unless it is restrained with priority at least that of δ. Of course, if $a_{v,z}$ is restrained with priority at least that of δ, $\langle v,z\rangle \in S(\eta, v)$ for some $\eta \leqslant \delta$ where we may assume that η is the highest priority node such that $\langle v,z\rangle \in S(\eta, v)$. If $\eta \not\leqslant \beta$ then, as $\delta\hat{}\langle \varepsilon', y'\rangle \supset \beta\hat{}\langle \varepsilon, y\rangle$, $\eta \supseteq \beta\hat{}\langle \varepsilon, y\rangle$ which would contradict the induction hypothesis with η for α. Thus if $a_{v,z}$ is defined at v and not restrained with priority at least β, it is put into A and so the marker becomes undefined.

We are now in the same situation as when we jumped to $\beta\hat{}\langle \varepsilon, y\rangle$: a node extending $\beta\hat{}\langle \varepsilon, y\rangle$ is accessible and either $a_{v,z}$ is undefined or restrained with priority at least β. As before, this produces a contradiction to the assumption that a counterexample is produced at substage u.

Finally, we deal with the case that, at u, we consider jumping to some $\alpha\hat{}\langle \varepsilon', y'\rangle$ from a γ associated with some $\langle \varepsilon', y'\rangle$ which is to the right of $\beta\hat{}\langle \varepsilon, y\rangle$. The argument proceeds as in the previous case (with α for δ and u for v) until we reach the conclusion that (at u) $a_{v,z}$ is undefined when we are about to redefine α's restraint and so its S set or it is restrained with priority at least β for the required contradiction.

(ii) By Lemma 3.2, there is a point t at which $\alpha\hat{}\langle v,z\rangle$ becomes accessible by jumping to it from a node γ associated with $\langle v,z\rangle$ which is to the right of $\alpha\hat{}\langle v,z\rangle$. As γ is associated with $\langle v,z\rangle$, $v \subset \gamma$ by definition, and so if $\langle v,z\rangle \leqslant \langle \varepsilon, y\rangle$, $v \subseteq \varepsilon$ as required. Of course, $\langle v,z\rangle \notin S(\gamma, t)$ for any $\gamma < \alpha$ (and so, *a fortiori*, for any $\gamma < \beta\hat{}\langle \varepsilon, y\rangle \leqslant \alpha$) or we could not have jumped to $\alpha\hat{}\langle v,z\rangle$. Thus $v \subseteq \varepsilon$ or $v \supseteq \beta\hat{}\langle \varepsilon, y\rangle$ as required. □

Lemma 3.5. *If $\alpha \supseteq \beta\hat{}\langle \varepsilon, y\rangle$ and α is assigned to requirement $R_{e',y'}$ and is associated with $\langle \varepsilon', y'\rangle$ then either $\varepsilon' \subset \varepsilon$ or $\varepsilon' \supseteq \beta\hat{}\langle \varepsilon, y\rangle$.*

Proof. Suppose ε and ε' are assigned to R_e and $R_{e'}$, respectively. By the definition of the pair associated with α, $\varepsilon' \subseteq \alpha$. By Lemma 3.3(ii), $\varepsilon \subset \beta$ (there is a stage at which $\beta\hat{}\langle \varepsilon, y\rangle$ is accessible since $\alpha \supseteq \beta\hat{}\langle \varepsilon, y\rangle$ is assigned a requirement.) Thus the only concern is that $\varepsilon \subseteq \varepsilon' \subset \beta$. Suppose first that $e' \leqslant e$. By construction, a node $\varepsilon' \supset \varepsilon$ can be assigned to $R_{e'}$ with $e' \leqslant e$ only if there is some $\delta\hat{}\langle v,z\rangle$ with $\varepsilon \subset \delta\hat{}\langle v,z\rangle \subset \varepsilon'$ such that v is assigned to a requirement R_n with $n < e$. Let $\delta\hat{}\langle v,z\rangle$ be the first such node. Let t be a stage at which $\beta\hat{}\langle \varepsilon, y\rangle$ is accessible. Now, by Lemma 3.4, $\varepsilon \subseteq v$ or $\varepsilon \supseteq \delta\hat{}\langle v,z\rangle$. The latter possibility contradicts our assumption that $\varepsilon \subset \delta\hat{}\langle v,z\rangle$. The former contradicts our assumption that $\delta\hat{}\langle v,z\rangle$ is the first instance of the phenomenon that would allow us to have a node between ε and β assigned to an R_i with $i < e$ (since $v \subset \delta$ by Lemma 3.3 applied to $\langle v,z\rangle \in S(\delta, s)$).

Now consider the possibility that $e < e'$ and let $\beta\hat{}\langle\varepsilon, y\rangle$ be the longest node contained in α which provides a counterexample to the lemma. By definition, $\leqslant_{\beta\hat{}\langle\varepsilon,y\rangle}$ is the final segment of $\leqslant_\varepsilon$ starting after R_e with all $R_{e,y'}$ removed. Of course, $R_{e'}$ precedes any $R_{e',y'}$ by our conventions on the original priority ordering. (Changes in the ordering always produce a final segment of a previous ordering with perhaps some R_i and all its subrequirements removed.) Thus as long as we stay inside this ordering, we must have a node δ assigned to $R_{e'}$ before α and so α would be associated with $\langle\delta, y'\rangle$ and not $\langle\varepsilon', y'\rangle$. The only way we can get outside this ordering is for us to restart at a higher point, i.e. to get a $\delta\hat{}\langle v,z\rangle$ with $\beta\hat{}\langle\varepsilon, y\rangle \subset \delta\hat{}\langle v,z\rangle \subset \alpha$ with $v \subset \beta\hat{}\langle\varepsilon, y\rangle$. In this case, Lemma 3.4 would tell us that $v \subseteq \varepsilon$. This would then contradict the choice of $\beta\hat{}\langle\varepsilon, y\rangle$ as the last node contained in α providing a counterexample with α. $\square$

Lemma 3.6. *If we jump from a node α to a node $\beta\hat{}\langle\varepsilon', y'\rangle$ at stage s, then $\beta\hat{}\langle\varepsilon', y'\rangle <_L \alpha$. Thus if δ is accessible after γ at stage s, then $\delta <_L \gamma$ or $\delta \supset \gamma$.*

Proof. Of course, we can only jump to $\beta\hat{}\langle\varepsilon', y'\rangle$ from α if α is associated with $\langle\varepsilon', y'\rangle$. Now, the second claim is immediate from the first which we now prove. If $\beta <_L \alpha$, the first assertion is clear. Otherwise, $\beta \subset \alpha$ as we only jump to successors of nodes of higher priority. If $\beta\hat{}1 \subseteq \alpha$ or $\beta\hat{}0 \subseteq \alpha$, then the claim is again obvious as $\beta\hat{}\langle\varepsilon', y'\rangle <_L \beta\hat{}1 <_L \beta\hat{}0$. The only other possibility is that $\beta\hat{}\langle\varepsilon, y\rangle \subseteq \alpha$ for some $\langle\varepsilon, y\rangle$. By Lemma 3.5, $\varepsilon' \subset \varepsilon$ or $\varepsilon' \supseteq \beta\hat{}\langle\varepsilon, y\rangle$. Lemma 3.3, however, tells us that $\varepsilon' < \beta$ and so $\varepsilon' \subset \varepsilon$. Thus $\langle\varepsilon', y'\rangle < \langle\varepsilon, y\rangle$ and so by definition of the ordering $\beta\hat{}\langle\varepsilon', y'\rangle <_L \beta\hat{}\langle\varepsilon, y\rangle \subseteq \alpha$ as required. $\square$

Lemma 3.7. *Each stage s of the construction eventually terminates.*

Proof. We proceed by induction on s. Assume we have finished every stage less than s and the current priority tree is T which is necessarily finite. As all new nodes added to the priority tree during stage s must be below nodes in T by construction, there is a leftmost node α_0 in T which is ever accessible at stage s. Suppose it is accessible at substage v_0. By our choice of α_0 and Lemma 3.6, all nodes accessible at substages $v > v_0$ must extend α_0. Let T_0 be the finite priority tree constructed by substage v_0. Again all nodes added after v_0 must extend nodes in T_0 and so there is a leftmost node α_1 in T_0 that is ever accessible during stage s. If we did not terminate the construction upon reaching α_0, $\alpha_1 \supset \alpha_0$. Continuing on by induction we must either terminate stage s or build a strictly increasing sequence of accessible nodes. Of course, we must then terminate stage s as well as the nodes must eventually become longer than s. $\square$

Lemma 3.8. *A node α associated with $\langle\varepsilon, y\rangle$ can be accessible at some substage u of stage s only if $\varepsilon\hat{}\infty$ (and so also ε) was previously accessible at s.*

Proof. First, we note that it is clear from the construction that $\varepsilon\hat{}\infty$ can become accessible only immediately after ε becomes accessible. We now prove the lemma by induction on the (sub)stages of the construction. Suppose for the sake of a contradiction that substage u of stage s is the first point at which a counterexample occurs. By definition, $\alpha \supset \varepsilon$. If $\alpha \supseteq \varepsilon\hat{}0$, then α could be assigned to requirement $R_{e,y}$ only if we first restart the priority ordering at some point before R_e. In this case, some node δ would be assigned to R_e before any to $R_{e,y}$ and so α would be associated with $\langle \delta, y \rangle$. Thus $\alpha \supseteq \varepsilon\hat{}\infty$. Consider the first substage $v \leqslant u$ of stage s at which some $\beta \supset \varepsilon\hat{}\infty$ becomes accessible before $\varepsilon\hat{}\infty$ has become accessible. We must have jumped to $\beta \supset \varepsilon\hat{}\infty$ from a node γ to the right of $\varepsilon\hat{}\infty$. γ is associated with some $\langle \varepsilon', y' \rangle \in S(\delta, v)$ and $\beta = \delta\hat{}\langle \varepsilon', y' \rangle$ for some $\delta \supseteq \varepsilon\hat{}\infty$. So by Lemma 3.3, $\varepsilon' \subset \delta$. As no node extending $\varepsilon\hat{}\infty$ has been previously accessible at s by our choice of $\beta \supset \varepsilon$, and ε' has been accessible by our inductive hypothesis (after all γ which is associated with $\langle \varepsilon', y' \rangle$ is already accessible), $\varepsilon' \subset \varepsilon$. Now, by Lemma 3.5, no node extending β is associated with $\langle \varepsilon, y \rangle$. Thus, as long as the accessible nodes continue to be extensions of β, we cannot produce the assumed counterexample. If any node $\beta' = \delta'\hat{}\langle \varepsilon'', y'' \rangle$ not extending β ever later becomes accessible at a substage t of stage s, it does so because we jumped to it from some $\gamma' \supseteq \beta$ with γ' associated with $\langle \varepsilon'', y'' \rangle$ and $\langle \varepsilon'', y'' \rangle \in S(\delta', t)$. Now, by Lemma 3.3 again, $\varepsilon'' \subset \varepsilon'$ and we are in the same situation as with β and ε'. The Lemma now follows by induction (on the number of such jumps). $\square$

Lemma 3.9. *Suppose ρ is assigned to a requirement of the form N_e or $R_{e,y}$ and there is a stage s_0 of the construction after which no node to the left of ρ is ever accessible. Then there is a stage $s_1 \geqslant s_0$ after which ρ is never initialized or injured and a stage $s_2 \geqslant s_1$ after which both $r(\rho,t)$ and $S(\rho,t)$ are constant.*

Proof. We begin by noting that no node $\gamma < \rho$ assigned to any requirement P_i can act to put a number into A after s_0 as that would make $\gamma\hat{}1$ accessible when it had not been so before (we act at most once for any P_i by construction).

Next, we note that there are only finitely many nodes $\alpha <_L \rho$ on T by s_0. Although we may add immediate successors to these nodes α after s_0 none of these successors are ever accessible and so none of them define restraints or auxiliary sets or get successors of their own. Of course, there are also only finitely many nodes $\alpha \subseteq \rho$. Thus we may prove the lemma for the nodes $\alpha < \rho$ by induction on the priority ordering. Suppose therefore that we have established the lemma for all $\gamma < \alpha \leqslant \rho$ with the point t' (after s_0) of the construction as the least witness to the fact that never again is such a γ initialized or its restraint $r(\gamma, v)$ injured or increased (and so its auxiliary set $S(\gamma, v)$ also never changes again). We now wish to prove the lemma for α which is assigned a requirement N_e or $R_{e,y}$.

First of all, α can be initialized only when some node to the left of α is accessible and so never after s_0. We next prove that $r(\alpha, v)$ is never injured after t'.

If there is no point in the construction after t' at which $r(\alpha, v)$ is defined, there is nothing to prove. So suppose substage v_0 of stage t_0 is the first point after t' at which $r(\alpha, v)$ is defined. As no node of higher priority than α which is assigned to a requirement of the form P_i ever acts again and all of lower priority ones are prohibited from putting a number less than $r(\alpha, v)$ into A, no action for any requirement P_i can be the first to directly injure $r(\alpha, v)$ (i.e. by putting in a follower less than this restraint). The only other nodes γ that initiate putting elements into A are nodes associated with some $\langle \varepsilon', y' \rangle$. Suppose such a node γ is accessible at a substage v_1 (of stage $t_1 \geqslant t_0$) after v_0. Let β be the highest priority node such that $\langle \varepsilon', y' \rangle \in S(\beta, v_1)$. If $\alpha \leqslant \beta$ then the dumping action for γ cannot directly put any number less than $r(\alpha, v_1)$ into A by construction. Suppose $\beta < \alpha$. In this case, we either increase $r(\beta, v)$ or declare $\beta\hat{\ }\langle \varepsilon', y' \rangle$ accessible. The former is not possible by our inductive assumption that the β restraint has settled down for all $\beta < \alpha$. In the latter case, $\beta\hat{\ }\langle \varepsilon'', y'' \rangle \subseteq \alpha$ for some $\langle \varepsilon'', y'' \rangle \leqslant \langle \varepsilon', y' \rangle$ since no node to the left of α can be accessible. Remember that we are concerned that some $a_{\varepsilon,y} \leqslant r(\alpha, v_1)$ is about to be put into A by our immediate action for γ. Thus by definition, $\langle \varepsilon, y \rangle \in S(\alpha, v_1)$. So, by Lemma 3.4, $a_{\varepsilon,y}$ is restrained with priority at least β or $\langle \varepsilon, y \rangle \leqslant \langle \varepsilon'', y'' \rangle$ (and so $\langle \varepsilon, y \rangle \leqslant \langle \varepsilon', y' \rangle$) or $\varepsilon \supset \beta\hat{\ }\langle \varepsilon'', y'' \rangle$. In each of these cases our dumping action for γ cannot directly put such a number $a_{\varepsilon,y}$ into A by the definition of this action (case (3) of $R_{e,y}$).

Thus to show that $r(\alpha, v)$ is never injured after t' it suffices to prove that if only numbers greater than $r(\alpha, v)$ are put into A directly by any requirement at points v after v_0 then the cascade they initiate also puts only numbers greater than $r(\alpha, v)$ into A.

The crucial claim here is that if any $a_{\varepsilon',y'}$ is less than $r(\alpha, v)$ and $b_{\varepsilon',y'}$ was put into $B_{\varepsilon'}$ by γ (assigned to $R_{e',y'}$) at substage v' of stage s' before v then $\phi_{e'}(A; y)[s'] \leqslant r(\alpha, t)$ for every point t of the construction that is after v. This claim clearly suffices for our purposes by the definition of the cascade procedure. It is certainly true at v_0 when we define $r(\alpha, v)$ as it is set to be a new large number. The only worry is that for some $a_{\varepsilon',y'} \leqslant r(\alpha, v)$, some γ may put a $b_{\varepsilon',y'}$ into $B_{\varepsilon'}$ at some substage v' of a stage t after v with $\phi_{e'}(A; y')[t]$ larger than $r(\alpha, v')$. As $\langle \varepsilon', y' \rangle \in S(\alpha, v) \subseteq S(\alpha, v')$ (it has not been initialized by induction) and no higher priority η can have its restraint increased by assumption, α must be the highest priority node with $\langle \varepsilon', y' \rangle \in S(\alpha, v')$ and so we set $r(\alpha, v')$ to be a new large number (and so bigger than $\phi_{e'}(A; y')[t]$) at v' by construction. Thus $r(\alpha, v)$ is never injured after t'.

Now $r(\alpha, v)$ changes only when some $b_{\varepsilon',y'}$ enters $B_{\varepsilon'}$ for some $\langle \varepsilon', y' \rangle \in S(\alpha, v)$. Once $b_{\varepsilon',y'}$ enters $B_{\varepsilon'}$ at substage u of stage t, $\phi_{\varepsilon'}(A; y')[t] \leqslant r(\alpha, u)$. As this restraint is never injured, $a_{\varepsilon',y'}$ is never put into A and so $b_{\varepsilon',y'}$ never becomes undefined. (The only other way for $b_{\varepsilon',y'}$ to become undefined is for ε' to be initialized. However, $\langle \varepsilon', y' \rangle \in S(\alpha, v)$ and so $\varepsilon' < \alpha$ by Lemma 3.3 and, by our assumption, no node of higher priority than α is ever initialized again.) Thus, for each $\langle \varepsilon', y' \rangle \in S(\alpha, t)$, $b_{\varepsilon',y'}$ can enter $B_{\varepsilon'}$ at most once. As $r(\alpha, v)$ changes only when such a $b_{\varepsilon',y'}$ enters $B_{\varepsilon'}$, to prove that $r(\alpha, v)$ eventually stabilizes, it clearly suffices then to show that $S(\alpha, v)$ is eventually constant.

When first defined $S(\alpha, v)$ consists of a finite set. It expands at a later point t by our putting in those $\langle \varepsilon'', y'' \rangle$ for which $a_{\varepsilon'',y''}$ is defined only when some $b_{\varepsilon',y'}$ is put into $B_{\varepsilon'}$ by some γ where $\langle \varepsilon', y' \rangle \in S(\alpha, t)$. Remember that, by Lemma 3.3, this implies that $\varepsilon' < \alpha$. Now, by construction, before we put $b_{\varepsilon',y'}$ into $B_{\varepsilon'}$ we put into A every $a_{\varepsilon'',y''}$ with $\varepsilon'' \not\supseteq \alpha\hat{}\langle \eta, z \rangle$ for any $\langle \eta, z \rangle$ such that $\langle \varepsilon'', y'' \rangle > \langle \varepsilon', y' \rangle$ and $a_{\varepsilon'',y''}$ is not restrained by requirements of priority at least α and so make such $a_{\varepsilon'',y''}$ undefined. The markers with $\varepsilon'' \supseteq \alpha\hat{}\langle \eta, z \rangle$ for some $\langle \eta, z \rangle$ are then initialized by the construction and so do not make any contribution to $S(\alpha, v)$. Of course, the restraints of strictly higher priority than α have already stabilized by our choice of v_0 and so $\langle \varepsilon'', y'' \rangle$ is already in $S(\alpha, t)$ for all $a_{\varepsilon'',y''}$ ever restrained by any requirement with strictly higher priority than α. Thus every new $\langle \varepsilon'', y'' \rangle$ put into $S(\alpha, v)$ is strictly smaller than $\langle \varepsilon', y' \rangle$ in the lexicographic ordering of such pairs. As there are only finitely many $\varepsilon' < \alpha$, there are only finitely many such pairs that can ever be put into $S(\alpha, v)$. Thus this process of putting a new $b_{\varepsilon'',y''}$ into $B_{\varepsilon''}$ and new pairs into $S(\alpha, v)$ must eventually stop and so $S(\alpha, v)$ is eventually constant as required. □

We would now like to argue that the requirements are satisfied along the true path. However, because of the possibility of jumping to the left there may be gaps in the set of leftmost nodes visited infinitely often. We consider instead the classes TN of true nodes and STN of semitrue nodes rather than the true path. We must prove that such exist and that every requirement is assigned to some true node.

Definition 3.10. The set of *true nodes*, TN, is defined as follows:

$$TN = \{\alpha \mid \alpha \text{ is accessible infinitely often but no } \beta <_L \alpha \text{ has this property}\}.$$

The set of *semitrue nodes*, STN, is defined as follows:

$$STN = \{\alpha \mid \text{infinitely often some } \gamma \supseteq \alpha \text{ is accessible but no } \beta <_L \alpha \text{ has this property}\}.$$

Lemma 3.11. *$TN \subseteq STN$ which is an infinite path in T.*

Proof. It is clear from the definitions that $TN \subseteq STN$ which is linearly ordered by $\subseteq$. Suppose $\alpha \in STN$. We must show that some immediate successor of α is in STN. If α has only finitely many immediate successors, this is immediate from the definition of STN. If α is assigned to a requirement of the form P_i or R_i then it has only two possible immediate successors. If it is assigned to some N_i or $R_{i,y}$ then it has only finitely many immediate successors by Lemma 3.9. □

Lemma 3.12. *If $\delta\hat{}\langle v, z \rangle \in STN$, then there is an $\varepsilon \subseteq v \subset \delta$ and a $\beta\hat{}\langle \varepsilon, y \rangle \supseteq \delta\hat{}\langle v, z \rangle$ such that $\varepsilon, \beta\hat{}\langle \varepsilon, y \rangle \in TN$.*

Proof. First, note that, by Lemma 3.3, $v \subset \delta$. At the first substage u of any stage s at which some node $\rho \supseteq \delta\hat{}\langle v,z\rangle$ in STN becomes accessible, we must jump to ρ from a node γ to the right of $\delta\hat{}\langle v,z\rangle$. Thus ρ must be of the form $\beta\hat{}\langle\varepsilon,y\rangle$ and γ must be associated with $\langle\varepsilon,y\rangle$. By Lemma 3.4, $\varepsilon \subseteq v$. (We cannot have $\varepsilon \supseteq \delta\hat{}\langle v,z\rangle$ because γ is associated with $\langle\varepsilon,y\rangle$ (which implies that $\varepsilon \subset \gamma$) and is to the right of $\delta\hat{}\langle v,z\rangle$.) If there is a single such node ρ that is accessible infinitely often, it supplies the desired witness for the lemma. Otherwise, there must be an infinite sequence of distinct such $\beta_i\hat{}\langle\varepsilon_i,y_i\rangle \in STN$ and we must jump to each of them from some γ_i to the right of $\delta\hat{}\langle v,z\rangle$. As STN is linearly ordered by extension, we may assume that $\beta_i\hat{}\langle\varepsilon_i,y_i\rangle \subset \beta_{i+1}\hat{}\langle\varepsilon_{i+1},y_{i+1}\rangle$. As above, each $\varepsilon_i \subset v$. As there are only finitely many nodes contained in v, there is a node ε which is the value of ε_i for infinitely many i. By Lemma 3.4, we would then have an infinite nonascending sequence $\langle\varepsilon,y_j\rangle$ in the lexicographic ordering. This can only happen if the y_j are eventually constant. This would mean that there are $\beta_i \subset \beta_j$ such that $\langle\varepsilon_i,y_i\rangle = \langle\varepsilon_j,y_j\rangle = \langle\varepsilon,y\rangle$. This cannot happen for we can never jump to $\beta_j\hat{}\langle\varepsilon,y\rangle$ as the instructions of the construction would always send us to $\beta_i\hat{}\langle\varepsilon,y\rangle$ instead. □

Lemma 3.13. *Every requirement of the form P_i, N_i or R_i is assigned to some node $\alpha \in TN$. Every requirement of the form $R_{e,y}$ is assigned to a node $\alpha \in TN$; or some node of the form $\beta\hat{}\langle\varepsilon,y\rangle$ is in TN; or $\varepsilon\hat{}0$ with ε assigned to R_e is in TN. Moreover, for each requirement Q assigned to a node $\alpha \in TN$ there is a last node $\alpha \in TN$ which is assigned to Q.*

Proof. We define two sequences of nodes $\varepsilon_i, \beta_i\hat{}\langle\varepsilon_i,y_i\rangle$ in TN as follows: $\varepsilon_0 = \emptyset$; if ε_i is defined and in TN, we let ε_{i+1} be the first node in TN extending $\beta_i\hat{}\langle\varepsilon_i,y_i\rangle$ such that some $\beta_{i+1}\hat{}\langle\varepsilon_{i+1},y'\rangle \in TN$ and let y_{i+1} be the least such y'. (We consider $\beta_0\hat{}\langle\varepsilon_0,y_0\rangle$ to be $\emptyset$ for technical convenience.) If there is no such node, the sequence terminates. Suppose the nodes ε_i are assigned to the requirements R_{e_i}.

We claim that the nodes in the interval $(\beta_i, \varepsilon_{i+1}]$ are all in TN and are assigned requirements from $<_{\beta_i\hat{}\langle\varepsilon_i,y_i\rangle}$ in order except that if a node v is assigned to R_e and $v\hat{}0$ is in the interval, then all requirements $R_{e,y}$ are left out. Moreover, $<_{\varepsilon_{i+1}}$ (if it exists) is just $<_{\beta_i\hat{}\langle\varepsilon_i,y_i\rangle}$ starting immediately after $R_{e_{i+1}}$ with all such $R_{e,y}$ omitted. (Of course, $<_{\beta_i\hat{}\langle\varepsilon_i,y_i\rangle}$ itself was just $<_{\varepsilon_i}$ starting after R_{e_i} with all $R_{e_i,y}$ omitted.) If ε_{i+1} does not exist, we simply claim that, after β_i, $STN = TN$ and we just keep assigning requirements from $<_{\beta_i\hat{}\langle\varepsilon,_i y_i\rangle}$ in this way.

We proceed by induction through the nodes in the interval (which are all in STN by definition if ε_{i+1} exists). We start with $\beta_i\hat{}\langle\varepsilon_i,y_i\rangle$. It is in TN and assigned the first element of $<_{\beta_i\hat{}\langle\varepsilon_i,y_i\rangle}$ by definition. Suppose we have reached $\gamma \in TN$ but not yet ε_{i+1}. If γ is assigned some requirement of the form P_i or R_i then the immediate successor $\gamma\hat{}w$ in STN is accessible infinitely often and so in TN. Unless γ is assigned to R_i and $w = 0$, the priority ordering $<_{\gamma\hat{}w}$ is just that $<_\gamma$ with the first element removed and we continue the induction. Otherwise, $<_{\gamma\hat{}w}$ is $<_\gamma$ with R_i and all $R_{i,y}$ removed and we also continue the induction. Suppose then that γ is assigned to a requirement of the form N_i or $R_{i,z}$. Let w be such that $\gamma\hat{}w \in STN$. If $w = 0, 1$, $\gamma\hat{}w \in TN$ and we are in the same situation as for P_i. If w is of the form $\langle v, z\rangle$ then, by Lemma 3.12, there are $\varepsilon \subset \gamma$ and $\beta\hat{}\langle\varepsilon, y\rangle \supseteq \gamma\hat{}w$ such that $\varepsilon, \beta\hat{}\langle\varepsilon, y\rangle \in TN$. If $\varepsilon \supseteq \beta_i\hat{}\langle\varepsilon_i, y_i\rangle$, then ε_{i+1} exists by definition and $\varepsilon \supseteq \varepsilon_{i+1}$ by minimality of ε_{i+1}. In this case, we have finished the induction argument by arriving at ε_{i+1}. Otherwise, we argue for a contradiction. By Lemma 3.4, $\varepsilon \subseteq \varepsilon_i$ and $\langle\varepsilon, y\rangle \leqslant \langle\varepsilon_i, y_i\rangle$. Let $j < i$ be least such that $\varepsilon_j \subset \varepsilon \subseteq \varepsilon_{j+1}$. By minimality of ε_{j+1}, $\varepsilon = \varepsilon_{j+1}$. By Lemma 3.4 again, $\langle\varepsilon, y\rangle \leqslant \langle\varepsilon_{j+1}, y_{j+1}\rangle$ but as $\varepsilon = \varepsilon_{j+1}$ and y_{j+1} was chosen least, $y = y_{j+1}$. In this case $\beta\hat{}\langle\varepsilon, y\rangle$ could never be accessible (we would always jump to $\beta_{j+1}\hat{}\langle\varepsilon_{j+1}, y_{j+1}\rangle$) for the desired contradiction.

Thus, as we proceed through the nodes in the intervals $(\beta_i, \varepsilon_{i+1}]$ in TN (if there is a last node ε_i we understand the interval $(\beta_i, \varepsilon_{i+1}]$ to be all of $STN = TN$ after ε_i), we assign all requirements in order except those $R_{e,y}$ such that $e = e_i$ for some i or such that one of these nodes γ is assigned to R_e and $\gamma\hat{}0$ is also one of these nodes and so in TN. Moreover, it is clear that, once we have assigned a requirement Q to a node α in some interval $(\beta_i, \varepsilon_{i+1}]$, we never again assign Q to any other node $\beta \in TN$. □

We are now ready to prove that the requirements are satisfied and the construction succeeds.

Lemma 3.14. *Each requirement P_e and N_e is satisfied, i.e. $A \neq \Phi_e$ and $\Phi_e^A \neq C$.*

Proof. Consider a node $\alpha \in TN$ assigned to P_e. By Lemma 3.9, the restraints $r(\gamma, s)$ for $\gamma < \alpha$ are eventually constant. Thus it is immediate from the instructions for α that there is some x such that $\Phi_e(x) \neq A(x)$ by α's actions if not by some other means. Next, consider an $\alpha \in TN$ assigned to N_e. Suppose x is a follower of α after all action for higher priority nodes has ceased and so appointed after α is initialized in that way for the last time. If $\Phi_e(A; x)[s] = 0$ at any later stage, we put x into C and preserve the computation. It is never injured by Lemma 3.9 and so $\Phi_e(A; x) = 0 \neq C(x)$. If $\Phi_e(A; x)[s] \neq 0$ for any later stage s, $\Phi_e(A; x) \neq 0 = C(x)$. Thus in either case, $\Phi_e^A \neq C$. □

Lemma 3.15. *If the hypotheses of R_e hold, i.e. $W_e^A = \Psi_e^K$, and $\varepsilon \in TN$ is assigned to R_e then neither $\varepsilon\hat{}0$ nor any node $\beta\hat{}\langle\varepsilon, y\rangle$ is in TN.*

Proof. Suppose $W_e^A = \Psi_e^A$ and $\varepsilon \in TN$ is assigned to R_e. We first show that for any x there is an ε-stage s such that $\ell(\varepsilon, s) > x$. By our assumptions, there is an ε-stage r such that, for every $t \geqslant r$ and $y \leqslant x$, $\Psi_e(K; y)[t] = \Psi_e(K; y)$ with use $\psi_e(y)$ and, if $y \in W_e^A$ (i. e. $\Phi_e(A; y)\downarrow$), then $\Phi_e(A; y)[t] = \Phi_e(A; y)$ with use $\phi_e(y)$. Thus for $y < x, y \in W_e^A$, $\Psi_\varepsilon(K; y)[t] = \Psi_e(K; y) = \Phi_\varepsilon(A; y)[t] = \Phi_e(A; y) = 1$ for every ε-stage $t > r$. Let u be the first A-true stage after r. (Recall that u is an A-true stage if no number less than that enumerated in A at u is ever enumerated in A after u.) Let s be the first ε-stage greater than or equal to u. It is now immediate from the definitions that $W_\varepsilon^A(y)[s] = \Psi_\varepsilon(K;y)[s]$ for every $y \leqslant x$ and so $\ell(\alpha,s) > x$ as required. (The only possible concern is for $y \notin W_e^A$. Of course, $\Psi_\varepsilon(K;y)[s] = \Psi_e(K;y) = 0$ by assumption. If, however, $\Phi_\varepsilon(A;y)[s]\downarrow$, then the use of this computation at s is the same as at the previous ε-stage t. As $r \leqslant t \leqslant u \leqslant s$ and u is an A-true stage, this could happen only if the computation at s is actually A-correct, contradicting our assumption that $y \notin W_e^A$. Thus $\varepsilon\hat{}\infty$ is accessible infinitely often as required.)

Next, suppose that some $\beta\hat{}\langle\varepsilon,y\rangle \in TN$ for the sake of a contradiction. $\beta\hat{}\langle\varepsilon,y\rangle$ can be accessible only when there is a node $\gamma \supseteq \varepsilon\hat{}\infty$ associated with $\langle\varepsilon,y\rangle$ which is accessible at a stage s such that $\Phi_e(A;y)\downarrow$ and $a_{\varepsilon,y}$ is defined and so $\Psi_e(K;y) = 1$ (as $a_{\varepsilon,y}$ is defined and so $\ell(\varepsilon,s) > y$). In order for $\beta\hat{}\langle\varepsilon,y\rangle$ to become accessible our action at γ must kill the computation $\Phi_e(A;x)$. As $\beta\hat{}\langle\varepsilon,y\rangle \in TN$, this happens infinitely often and so $x \notin W_e^A$ but there are infinitely many stages at which $\Psi_e(K;x) = 1$ for the desired contradiction. □

We can now conclude the proof of the theorem with the following lemma.

Lemma 3.16. *If $W_e^A = \Psi_e^K$ and ε is the last node on TN assigned to R_e then $W_e^A \leqslant_T B_\varepsilon \oplus A$ and $B_\varepsilon \leqslant_T W_e^A \oplus A$.*

Proof. First, by Lemma 3.13, there is a last node $\varepsilon \in TN$ assigned to R_e. By Lemmas 3.13 and 3.15 there is, for each y, a node $\alpha \in TN$ associated with $\langle\varepsilon,y\rangle$. Moreover, for each y we eventually define markers $b_{\varepsilon,y}$, $a_{\varepsilon,y}$ and, if they ever become undefined, we redefine them at the next $\varepsilon\hat{}\infty$-stage. Let α be the node on TN below ε assigned to $R_{e,y}$ and s_α the stage s_2 proved to exist Lemma 3.9. Now if $b_{\varepsilon,y}$ is ever in B_ε at an α-stage s after s_α, Lemma 3.9 states that the restraint now imposed (by α if not by some requirement of higher priority) that protects the computation associated with $\Phi_e(A;y)$ because of which we put $b_{\varepsilon,y}$ into B_ε is never injured and so $a_{\varepsilon,y} \notin A$ and $y \in W_e^A$. On the other hand, it is obvious from the construction that if $y \in W_e^A$ then we must eventually have $b_{\varepsilon,y} \in B_\varepsilon$ at an α-stage $s > s_\alpha$.

Now, in general, $b_{\varepsilon,y}$ may enter B_ε at some stage $s > s_\alpha$ which is not an α-stage [it can be put in by some γ assigned to $R_{e,y}$ which is to the right of the true path]. However, this can only happen if $\Phi_e(A;y)[s]$ is convergent and $\Psi_e(K;y) = 1[s]$. If A is not correct on the use $\phi_e(A;y)[s]$ then, when A changes, we put $a_{\varepsilon,y}$ into A by construction. If this happens infinitely often then $y \notin W_e^A$ but $\Psi_e(K;y) = 1[t]$ for infinitely many t and so $W_e^A \neq \Psi_e^K$ for a contradiction. Thus $y \in W_e^A$ if and only if we eventually have a pair of markers such that $b_{\varepsilon,y} \in B_\varepsilon$ and $a_{\varepsilon,y} \notin A$ while $y \notin W_e^A$ if and only if there is eventually a marker $b_{\varepsilon,y} \notin B_\varepsilon$. This shows that $W_e^A \leqslant_T B_\varepsilon \oplus A$.

Finally, we prove that $B_\varepsilon \leqslant_T W_e^A \oplus A$. To decide if some b is in B_ε we wait until stage b to see if b has been appointed as a marker $b_{\varepsilon,y}$ for some y. If not then $b \notin B_\varepsilon$. If so then we see if $y \in W_e^A$. If so then the construction guarantees that $b_{\varepsilon,y} \in B_\varepsilon$. If not then $b_{\varepsilon,y}$ can enter B_ε at $t > b$ only if $a_{\varepsilon,y}$ is later put into A (at a stage when the computation $\Phi_e(A;y)[t]$ is seen to be incorrect). Thus we ask if $a_{\varepsilon,y} \in A$. If so then

$b = b_{\varepsilon,y}$ is in B_ε if and only if it has entered by the stage at which $a_{\varepsilon,y}$ is put into A. If $a_{\varepsilon,y} \notin A$ then $b = b_{\varepsilon,y} \notin B_\varepsilon$. □

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