Stress energy minimization as a tool in the material layout design of shallow shells

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A B S T R A C T

The present research deals with the compliance minimization problem of an elastic thin shallow shell subjected to simultaneous in-plane and bending loads. In this context, our goal is to lay out a given amount of material in the volume of a shell assuming that the distribution in the direction transversal to its middle surface $S$ is homogeneous. The discussion hence reduces to the question of finding the optimal material arrangement on $S$. Similar problems were solved in the framework of two dimensional elasticity or Kirchhoff plate theory and the present research attempts to generalize these results. Following the pattern emerging from the above mentioned considerations, our research starts from the minimum compliance problem of a structure made of two elastic materials whose volumetric fractions are fixed. The existence of a solution to thus posed optimization task is guaranteed if the fine-scale microstructural composites are admitted in the analysis. Their constitutive tensors can be obtained by certain averaging ensuing from the theory of homogenization for periodic media. Additionally, by the Castigliano Theorem, the compliance minimization problem is equivalent to the one for structural stress energy. In turn, the lower estimation of the energy is achieved in two steps: (i) its modification by a certain energy-like functional, and (ii) utilizing the quasiconvexity property of thus obtained expression. As a result, formulae describing the effective stress energy of one-material shallow shell and the material distribution function are explicitly derived.

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1. Introduction

In the late 1960s, the following optimization problem was set: “For given $\Omega$, denoting the design area, consider all material distribution functions $\chi : \Omega \rightarrow \{0, 1\}$ constrained by $\int_\Omega \chi(x)dx < \text{vol}(\Omega)$. Among them, find $\hat{\chi}$ such that the compliance $J = \int_\Omega f(x)\chi(x)dx$ of a structure subjected to a given load achieves its minimal value $\hat{J} = \int_\Omega f(\hat{\chi}(x))dx$”.

It became clear already at the early stage of research, see e.g. Kozlowski and Mróz (1969) and Rozvany et al. (1982), that the above-mentioned task was badly posed, because sequences $\{\chi_n\}$ appeared to be, in general, non-convergent in the standard norm of $L^\infty(\Omega, \{0, 1\})$ hence $\hat{\chi}$ determining the optimal layout of materials could not be computed. Due to this phenomenon, often referred to as “non-existence of classical solutions”, a need for regularizing the optimal design problem appeared.

Regularization techniques considered in the literature can be roughly divided into two groups. The first consists of the methods restricting the set of characteristic functions $\chi$ and in this way ensuring the existence of solutions in some subset of $L^\infty(\Omega, \{0, 1\})$, see Ambrosio and Butazzo (1993), Niordson (1983), Peterson and Sigmund (1998) and Petersson (1999). Thus obtained designs are dependent on the choice of the restriction method and they are usually suboptimal.

Alternatively, one may extend the space of classical designs. More precisely, the space of characteristic functions can be supplemented with the weak-$*$ limits of sequences $\{\chi_n\}$ belonging to $L^\infty(\Omega, \{0, 1\})$ in this way enlarging this set to $L^\infty(\Omega, [0, 1])$, where the latter stands for the space of generalized designs whose main property is that the functions corresponding to material distributions can take any value between 0 and 1. The extension of this type is called “relaxation” and, from the mechanical point of view, it results in allowing the microstructural composites of basic materials in the analysis of the problem. The mathematical foundation of the relaxation method, known as the homogenization theory, is being developed simultaneously with its mechanical applications from the 1970s. The detailed exposition of homogenization lies beyond the scope of this paper, hence we refer the reader to Allaire (2002), Cherkaev (2000), Lewinski and Telega (2000), Milton (2002) and Tartar (2000) for further references.

The relaxed problems of minimum compliance in the Kirchhoff thin plate as well as plane stress theory settings were solved by Gibianski and Cherkaev (1984, 1987). Due to infinite variety of admissible microstructural designs, in both cases the common goal was to eliminate them from the analysis by estimating the effective potential representing the stress energy of a homogenized microstructure and, in the second step, to prove the attainability
of the estimation on a certain microscopic material layout. It is worth pointing out that in both cases the second rank orthogonal sequential laminate turned out to be the microstructure on which the minimal value of the stress energy was attained.

Loosely speaking, the Cherkaev–Gibianski idea of energy estimation was to modify the constitutive tensors of basic materials by a certain fourth-rank tensor thus introducing a free parameter that was optimally adjusted on the further stage of calculations. This approach, usually referred to as “the translation method”, see Milton (1990), was developed in the early 1980s by several groups of mathematicians, see e.g. Cherkaev (2000, Chapter 8) for further explanations. In the present paper we make use of the translation method in discussing the broader issue of minimal compliance posed for the class of structures subjected to simultaneous in-plane and bending loadings.

Many recently published articles and the wide spectrum of subjects concerning various aspects of shell optimization show the importance and interest of the researchers in this field of structural mechanics. Equations of the relaxed optimization problem for a thin shell were obtained by Lewiński and Telega (2000) with the use of the homogenization formulae found by the same authors in Telega and Lewiński (1998). Microstructures known from the in-plane and thin plate solutions and their application in the optimization of shells are dealt with in e.g. Ansolà et al. (2002a,b, 2004), Tenek and Hagiwara (1993a,b, 1994).

The objective of our research is to implement the translation method in the analysis of the minimum compliance problem in the framework set by the theory of shallow shells. More precisely, we determine the lower bound on the stress energy functional of an effective shell and this estimation is valid for any microstructure regardless of its complexity. The idea is justified by the above-mentioned successful application of Gibianski–Cherkaev approach in separately treated cases of plane stress and Kirchhoff’s plate. The question of attainability of the estimation, even though at present far from being answered in the context of shell theory, is not brought up in this paper.

Obtained results show that the originally separated fields of couple and stress resultants are linked through the optimal solution, hence the optimization task cannot be replaced by two independent problems. Partial results of the research were announced in Dzierzanowski and Lewiński (2003, 2005) and Dzierzanowski (2011).

The paper is organized as follows: the background for the research including the notation used throughout the paper and the statement of the optimization problem is set in Section 2. Section 3 is devoted to the description of the method used for obtaining the lower estimation of the stress energy of a homogenized shell made of two isotropic materials. Thus established pattern is next used in Section 4 for the material-void case. It results in deriving the explicit formulae describing the lower bound on the effective stress energy accumulated in a particle of a homogenized shell. Analytical considerations tackled in the paper are illustrated in Section 5. Some technical details of the calculations are gathered in Appendix A.

2. Problem statement

2.1. Notation

Throughout the paper, Greek indices take values 1 or 2 while Latin ones range from 1 to 3, unless otherwise stated, and the usual summation convention applies.

Set Ω for a given subset of R² with a Cartesian basis {e₁, e₂}. Certain geometrical analogy allows treating second-order plane symmetric tensors \( X = X_{ijkl}e_i \otimes e_j \otimes e_k \otimes e_l \) and fourth-order tensors \( X = X_{ijkl}e_i \otimes e_j \otimes e_k \otimes e_l \) endowed with Hooke’s symmetry, i.e. such that \( X_{ijkl} = X_{jikl} = X_{jilk} \), respectively as vectors and second-order tensors in \( R^3 \). Indeed, if one adopts a basis \( E_1 = \frac{1}{\sqrt{2}}(e_1 \otimes e_1 + e_2 \otimes e_2) \),
\( E_2 = \frac{1}{\sqrt{2}}(e_1 \otimes e_1 - e_2 \otimes e_2) \),
\( E_3 = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 + e_2 \otimes e_1) \),
then \( x = x_k E_k \), where
\[ [x_k] = \frac{1}{\sqrt{2}}[x_{11} + x_{22}, x_{11} - x_{22}, 2x_{12}]^T. \]

Consequently, the trace of \( x \) and norm of its deviator are given by
\[ \text{tr} x = \sqrt{2}x_1, \quad ||(\text{dev} x_k)||^2 = x_1^2 + x_2^2. \]

Tensor \( X = X_{ij}E_i \otimes E_j \) is represented by the symmetric matrix \( [X_0] \) such that
\[ X_{11} = \frac{1}{2}(X_{1111} + X_{2222} + 2X_{1212}), \quad X_{12} = \frac{1}{2}(X_{1111} - X_{2222}), \]
\[ X_{22} = \frac{1}{2}(X_{1111} + X_{2222} - 2X_{1212}), \quad X_{13} = X_{1112} + X_{2221}, \]
\[ X_{33} = 2X_{1212}, \quad X_{12} = X_{1112} - X_{2221}. \]

For brevity of further derivation define the following operations
\[ \mathbf{a} \cdot \mathbf{y} = x_1 y_1, \quad \mathbf{a} \times \mathbf{y} = (x_2 y_1 - x_1 y_2)E_i, \]
and extend them to vectors and matrices whose components are respectively given by vectors and second-order tensors in \( R^3 \). Namely for
\[ \mathbf{a} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \]
one may conclude that
\[ \mathbf{a} \cdot \mathbf{b} = x_1 y_1, \quad \mathbf{a} \times \mathbf{b} = \begin{bmatrix} (X_{13} y_2 - X_{23} y_1)E_1 \\ (X_{23} y_1 - X_{31} y_2)E_2 \end{bmatrix}. \]

2.2. Equilibrium equation of a shallow shell

Assume that \( S \) represents the middle surface of a shell of constant thickness \( t \). Next, parameterize this surface by \( \xi \equiv (\xi_1, \xi_2) \in \Omega \) mapped onto \( S \) and introduce standard definitions of a local basis \( \{ g_i, g_j \} \) on \( S \) complemented by the unit normal vector \( g_z \); the co-basis \( \{ g^i, g^j, g^z \} \); the metric tensor \( g = g_{ij}g^{i} \otimes g^{j} \); the curvature tensor \( b = b_{ijk}g^i \otimes g^j \); and the covariant derivative \( \nabla |x| \). The area of an elementary segment at \( S \) is defined by \( dS = \sqrt{g}d\Omega \), where \( g = \det[g_{ij}], \quad d\Omega = d\xi_1 \cdot d\xi_2 \).

Let \( V \) denote the space of kinematically admissible fields \( u(\xi) \equiv u_1(\xi) g_1 + u_2(\xi) g_2 \) and \( w(\xi) \equiv w_0(\xi) g_z \) respectively representing the displacements tangent and normal to \( S \). Deformation fields in the Mush–Donnell–Vlasov (MDV for short) theory of shallow shells are fixed in the form
\[ \varepsilon_{xy}(u, w) = \frac{1}{2}(u_{xy} + u_{yx} - b_{xy}w), \quad \chi_{xy}(w) = -w_{xy}, \]
with the second formula being a rough approximation of the changes of curvature tensor known from the general Kirchhoff–Love shell model and suitable for applications in the shallow shell theories. Recall that a thin shell can be assumed “shallow” if \( t/R_{\text{min}} \leq 1/30 \), where \( R_{\text{min}} \) denotes the smallest radius of curvature, see Vlasov (1949, Chapter 7.1). If this is the case, basis \( \{ g_i \} \), and co-basis \( \{ g^i \} \) vectors are equivalent to the fixed basis \( \{ e_1, e_2, e_3 \} \) in \( \Omega \), where
\( \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 \), hence the difference between contra- and covariant quantities vanishes. In the sequel the basis \( \{ \mathbf{e}_i \} \) is assumed orthogonal. Consequently, metric tensor \( \mathbf{g} \) on \( \Omega \) is sufficiently well approximated by the unit tensor \( I_2 = \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \) from which we conclude that \( \mathbf{g} \approx 1, \ D \approx d \Omega \).

Let \( \mathbf{p}(\xi) = p_\alpha(\xi) \mathbf{e}_\alpha \), and \( q(\xi) \) stand for the loading intensities respectively tangential and normal to \( \Omega \) and let \( \mathbf{N}(\xi) = N_{\alpha\beta}(\xi) \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \) and \( \mathbf{N}(\xi) = M_{\alpha\beta}(\xi) \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \) represent the stress and couple resultant fields. Next, set

\[
K_{\alpha\beta} = \frac{t}{\sqrt{12}} X_{\alpha\beta}, \quad R_{\alpha\beta} = \frac{\sqrt{12}}{t} M_{\alpha\beta},
\]

thus matching the measurement units of deformation components as well as the stress and couple resultants.

Assume that two isotropic materials are distributed within the volume of a shell uniformly with respect to its thickness but arbitrarily in \( \Omega \) and we set \( \Omega = \Omega_1 + \Omega_2 \), where \( \Omega_2 \) denotes the subdomain occupied by material \( \alpha \). Compliance tensors of both materials are represented in the basis \( \{ \mathbf{e}_i \} \) by

\[
\mathbf{C}_\alpha = \text{diag} \left[ \frac{1}{2} K_{x}, \quad \frac{1}{2} L_{x}, \quad \frac{1}{2} L_{y} \right],
\]

where

\[
K_x = \frac{2(1 - v_\alpha)}{E_x}, \quad L_x = \frac{2(1 + v_\alpha)}{E_x},
\]

and we set \( K_1 < K_2, L_1 < L_2 \). Symbols \( E_\alpha, v_\alpha \), respectively stand for Young's modulus and Poisson's coefficient of the \( \alpha \)th material.

By introducing the characteristic function

\[
\chi(\xi) = \begin{cases} 1 & \xi \in \Omega_1, \\ 0 & \xi \in \Omega_2, \end{cases}
\]

one may define the formula \( \mathbf{C} = \chi \mathbf{C}_1 + (1 - \chi) \mathbf{C}_2 \) describing the constitutive properties of a shell. Stress and couple resultant tensors \( \mathbf{N} \) are thus linked with the deformation tensors \( \mathbf{e}, \mathbf{k} \) by the following compact notation

\[
\mathbf{y} = \mathbf{S} \mathbf{y},
\]

where

\[
\mathbf{y} = [\mathbf{e}, \mathbf{k}]^T, \quad \mathbf{S} = [\mathbf{N}, \mathbf{R}]^T, \quad \mathbf{S} = \text{diag} [\mathbf{C}, \mathbf{C}_2].
\]

The components of \( \Sigma \) are assumed statically admissible which is denoted by \( \Sigma \in \Phi \) in the sequel, i.e. \( \Sigma \) satisfies the variational equilibrium equation of the MDV shell given by

\[
\int_\Omega \boldsymbol{\Sigma} \cdot \boldsymbol{\eta} \, d\Omega = \int_\Omega (p_x u_x + q v) \, d\Omega,
\]

for all \( (\mathbf{v}, \mathbf{w}) \in \mathbf{V} \).

### 2.3. Compliance minimization of a homogenized shallow shell

In the compliance minimization problem the task is to find the optimal distribution of materials in the volume of a shell, i.e. to determine a certain function \( \tilde{\chi} \) such that \( J = J(\chi) \) is given by

\[
J = \min_{\chi \in \Phi} \int_\Omega (p_x u_x + q v) \, d\Omega.
\]

Displacements \( u_x = u_x(\chi) \), \( w = w(\chi) \) are statically admissible and \( \tilde{\chi} \) is chosen among all characteristic functions \( \{ \chi \} \) satisfying the isoperimetric constraint

\[
\frac{1}{|\Omega|} \int_\Omega \chi \, d\Omega \leq m,
\]

where \( m \in \{ 0, 1 \} \) determines the amount of material 1 in \( \Omega \). The fraction of material 2 is thus equal to \( 1 - m \).

The optimal control problem in the above mentioned form is badly-posed and requires regularization. More precisely, the solution of (16) does not exist in the set of characteristic functions. It may be achieved in its extension, i.e. the set comprised of functions \( \Theta : \Omega \rightarrow [0, 1] \) representing the fraction of phase 1 in the microstructural layout of constituent materials in \( \Omega \). One may thus replace the original minimal compliance problem (16) with its relaxed counterpart written in the form

\[
J_{\text{hom}} = \min_{\chi \in \Phi} \int_\Omega (p_x u_x + q v) \, d\Omega,
\]

where \( J_{\text{hom}} = J(\tilde{\chi}) \). The generalized material distribution functions satisfy the condition

\[
\int_\Omega \chi \, d\Omega \leq m.
\]

For full justification of the passage from the badly-posed compliance minimization problem to its relaxed, well-posed form we refer the reader to Lewiński and Telega (2000, Chapter 28). Here we only mention that these subtle mathematical considerations are based on the homogenization theory which incorporates certain averaging of material properties locally at \( \xi \in \Omega \). They result in the derivation of effective constitutive relations for thin shells given by formula (13) with \( \mathbf{C}, \mathbf{S} \) replaced with their homogenized counterparts \( \mathbf{C}_{\text{hom}}, \mathbf{S}_{\text{hom}} \).

By introducing the Castigliano Theorem the relaxed compliance minimization problem of a shallow shell in (18) can be rewritten as

\[
J_{\text{hom}} = \min_{\chi \in \Phi} \inf_{\mathbf{S} \in \Sigma} \int_\Omega (2W_{\text{hom}}(\Sigma, \chi) \Theta d\Omega),
\]

where \( \ell \in \mathbb{R} \) denotes the Lagrange multiplier for the isoperimetric condition (19). Effective stress energy density of a homogenized shell \( W_{\text{hom}} \) is determined at given \( \xi \in \Omega \) due to the local character of homogenization, see (Tartar, 2000), by

\[
2W_{\text{hom}}(\Sigma, \chi),
\]

where \( \Sigma \) denotes the set of all homogenized constitutive tensors obtained by mixing basic materials \( \mathbf{C}_1 \) and \( \mathbf{C}_2 \) in proportions locally fixed by \( \theta \) and \( 1 - \theta \).

The problem of determining \( \mathbf{S}_{\text{hom}} \) in (21) is crucial for the solution of the optimization task posed in (20). It is equivalent to finding the microstructural layout of materials realizing the minimum for given \( (\Sigma, \chi) \) and it remains unsolved in the framework of shell theory. One may examine any heuristically chosen class of microstructures thus obtaining approximate solutions to the optimization problem at hand. In this context, however, the following question arises: Is it possible to remove the dependence on microscopic layout of materials from the analysis in (21) thus enveloping all suboptimal solutions to (20)? The affirmative answer follows from the idea of introducing on the r.h.s. of (21) an energy-like functional dependent on several parameters. Optimal adjustment of these parameters allows for obtaining the lower estimation of \( W_{\text{hom}}(\Sigma, \chi) \) which is valid for all possible microstructures.

### 3. Lower estimation of the effective stress energy

#### 3.1. Outline of the estimation method

By the theory of G-convergence applied to the case under study and due to the Dal Maso–Kohn–Raitum theorem, see Lewiński and Telega (2000, Chapter 26), without loss of generality we can restrict our considerations to the Y-periodic distributions of materials in \( \Omega \), where \( Y \) denotes a basic periodicity cell parameterized by \( y \in \mathbb{Y} \). Consequently, we introduce the characteristic function describing the layout of materials in \( Y = Y_1 \cup Y_2 \), such that
\[ \chi'(y) = \begin{cases} 1 & \text{if } y \in Y_1, \\ 0 & \text{if } y \in Y_2. \end{cases} \quad \langle \chi' \rangle = \theta, \quad \langle \cdot \rangle = \int_\Omega \cdot \, dy. \]  

(22)

The value of \( \theta \in [0, 1] \) is fixed in the course of the estimation procedure hence in the remainder of this section \( \theta \) is dropped from the notation and we write

\[ U_{\text{hom}}(\Sigma) = W_{\text{hom}}(\Sigma, \theta) \]  

(23)

for the locally homogenized stress energy functional.

Conversely to the macroscale of \( \Omega \), both constituent materials are disjoint in the microstructural layout of \( Y \) and the distribution of their physical properties is defined by \( S' = \chi' S + (1 - \chi') S_2 \), where \( S_2 = \text{diag}(\mathcal{C}_1, \mathcal{C}_2) \).

Let \( \sigma \in \mathfrak{a} \) stand for the vector of stress and couple resultant fields statically admissible in \( Y \), i.e. set

\[ \sigma = [n, \mathbf{r}]^T, \quad \langle \sigma \rangle = \Sigma, \]  

(24)

where, similar to (9), \( \mathbf{r} \) stands for the scaled couple resultant field. With this notation, rewrite (21) in the form

\[ 2U_{\text{hom}}(\Sigma) = \min_{\chi'Y \setminus \{0, 1\}} \inf_{\mathfrak{a} \in \mathfrak{a}_Y} \langle \sigma - \Sigma \rangle \cdot (S' \sigma) \]  

(25)

The key issue of the estimation method is to determine the quasiconvex envelope of the operand on the r.h.s. of (25). For this purpose, introduce the matrix \( T_0 \), represented in the basis (1) by

\[ T_0 = \text{diag}[-1, 1, 1] \]  

(26)

and the bilinear function

\[ F(x, y) = \langle x \cdot (T_0 y) \rangle. \]  

(27)

According to Cherkaev (2000, Chapter 8) \( F(x, y) \) is quasiconvex, i.e.

\[ F(x, y) \geq \langle x \cdot (T_0 y) \rangle \]  

(28)

provided \( x, y \) are periodic and the differential constraints set on \( \sigma \) by the requirement of statical admissibility in \( Y \) are satisfied.

Next, define the matrix

\[ T = \frac{1}{2} \begin{bmatrix} T_0 & y T_0 \\ y T_0 & \beta T_0 \end{bmatrix}. \]  

(29)

In what follows \( T \) and \( x, \beta, \gamma \) are referred to as “the translation matrix” and “the translation parameters”, respectively. The range of the latter is determined in Section 3.2.

Adding and subtracting (29) in (25) gives

\[ \langle \sigma \cdot (S' \sigma) \rangle = \langle \sigma \cdot [(S' - T) \sigma] \rangle + \langle \sigma \cdot (T \sigma) \rangle. \]  

(30)

Scalar quantities on the r.h.s. of (30) are estimated by

\[ \langle \sigma \cdot [(S' - T) \sigma] \rangle \geq \Sigma \cdot \left\{ \left( [S' - T]^{-1} \right)^{-1} \right\} \]  

(31)

and

\[ \langle \sigma \cdot (T \sigma) \rangle \geq \Sigma \cdot (T \Sigma). \]  

(32)

The former inequality represents the classical harmonic bound obtained by neglecting the differential constraints imposed on \( \sigma \) while the explanation of the latter one is dealt with in Appendix A. Consequently, one may write

\[ \langle \sigma \cdot (S' \sigma) \rangle \geq \Sigma \cdot (S_{\text{low}} \Sigma), \]  

(33)

where

\[ S_{\text{low}} = \left( [S' - T]^{-1} \right)^{-1} + T. \]  

(34)

In this way, the dependence on the microstructural layout of materials in \( Y \) is bypassed and the lower (translation) estimation of the homogenized stress energy accumulated in a particle of a composite shell is given by

\[ 2U_{\text{low}}(\Sigma) = \max_{(x, \beta, \gamma) \in \mathcal{C}} \Sigma \cdot [S_{\text{low}}(x, \beta, \gamma) \Sigma] \]  

(35)

see Section 3.2 for the description of the set \( \mathcal{C} \). From (25) it follows that

\[ U_{\text{hom}}(\Sigma) \geq U_{\text{low}}(\Sigma). \]  

(36)

The solution to the maximization problem in (35) involves coupling the fields \( \mathbf{N} \) and \( \mathbf{R} \) in the effective potential \( U_{\text{low}}(\Sigma) \). Indeed, one may check by inspection of (34) that

\[ S_{\text{low}} = \begin{bmatrix} S_{11} & 0 & 0 & 0 \\ 0 & 0 & S_{25} & 0 \\ 0 & S_{33} & 0 & 0 \\ m & S_{55} & 0 & S_{66} \end{bmatrix}, \]  

(37)

where \( S_{33} = S_{22}, S_{36} = S_{23}, S_{66} = S_{55} \), and

\[ S_{11} = \frac{1}{2} \]  

\[ S_{14} = \frac{1}{2} \]  

\[ S_{12} = \frac{2}{2} \]  

\[ S_{22} = \frac{2}{2} \]  

\[ S_{25} = \frac{2}{2} \]  

\[ S_{33} = \frac{2}{2} \]  

\[ S_{34} = \frac{2}{2} \]  

\[ S_{55} = \frac{2}{2} \]  

(38)

with \( (F)_{a} = 0 \) for \( F_{1}(1 - \theta)F_{2}, F_{a} = (1 - \theta)F_{1} + \theta F_{2}, A F = |F_{1} - F_{2}| \).

Thus we obtain

\[ U_{\text{low}}(\Sigma) = \max_{(x, \beta, \gamma) \in \mathcal{C}} U(\Sigma, x, \beta, \gamma), \]  

(39)

where

\[ 4U(\Sigma, x, \beta, \gamma) = S_{11} tr(N)^2 + 2S_{22} \| dev(N) \|^2 + S_{34} (tr(R))^2 \]  

\[ + 2S_{55} \| dev(R) \|^2 + 2S_{44} tr(N) tr(R) \]  

\[ + 4S_{25} \| dev(N) \cdot dev(R) \|. \]  

(40)

It can be seen that \( U_{\text{low}}(\Sigma) \) is isotropic as \( U(\Sigma, x, \beta, \gamma) \) depends on the invariants of \( \mathbf{N} \) and \( \mathbf{R} \) only.

Consequently, the compliance minimization problem set in (20) takes the form

\[ J_{\text{low}} = \min_{x, \beta, \gamma \in \mathcal{C}} \inf_{\theta \in \mathcal{C}} \int_{\Omega} (2W_{\text{low}}(\Sigma, \theta) + t \theta) d\Omega \]  

(41)

and \( J_{\text{low}} \geq J_{\text{low}} \) by (36).

3.2. Range of the translation parameters

The range of translation parameters \( x, \beta, \gamma \) is determined in two steps. Considerations in Appendix A give

\[ x \in \mathbb{R}, \quad \beta \geq 0, \quad \gamma \in \mathbb{R} \]  

(42)

but these results are narrowed by the requirement of semi-positive definiteness of matrix \( S' - T \) on the i.h.s. of (31). We require that for \( i = 1, 2 \) the matrices
are semi-positively defined. By reshaping (43) in the form

\[ S_{\gamma} - T = \frac{1}{2} \begin{bmatrix} K_{2} + \alpha & 0 & 0 & \gamma & 0 & 0 \\ 0 & L_{2} - \alpha & 0 & 0 & -\gamma & 0 \\ 0 & 0 & L_{2} - \alpha & 0 & 0 & -\gamma \\ \gamma & 0 & 0 & K_{2} + \beta & 0 & 0 \\ 0 & -\gamma & 0 & 0 & L_{2} - \beta & 0 \\ 0 & 0 & -\gamma & 0 & 0 & L_{2} - \beta \end{bmatrix} \] (43)

are semi-positively defined. By reshaping (43) in the form

\[ S_{\gamma} - T = \frac{1}{2} \text{diag}[H_{1}, H_{2}, H_{2}], \] (44)

where

\[ H_{1} = \begin{bmatrix} K_{2} + \alpha & \gamma \\ \gamma & K_{2} + \beta \end{bmatrix}, \quad H_{2} = \begin{bmatrix} L_{2} - \alpha & -\gamma \\ -\gamma & L_{2} - \beta \end{bmatrix}. \] (45)

one obtains the following conditions:

\[ K_{2} + \alpha \geq 0, \quad (K_{2} + \alpha)(K_{2} + \beta) - \gamma^{2} \geq 0, \]

\[ L_{2} - \alpha \geq 0, \quad (L_{2} - \alpha)(L_{2} - \beta) - \gamma^{2} \geq 0. \] (46)

Combining (42) with (46) leads to determination of the set

\[ \mathcal{Z} = \left\{ (\alpha, \beta, \gamma) : -K_{1} \leq \alpha \leq L_{1}, \quad 0 \leq \beta \leq L_{1}, \quad \gamma^{2} \leq (\gamma_{1})^{2}, \quad \gamma^{2} \leq (\gamma_{2})^{2} \right\}. \] (47)

comprising the admissible values of translation parameters, where

\[ \gamma_{1}(\alpha, \beta) = \sqrt{(K_{2} + \alpha)(K_{2} + \beta)}, \quad \gamma_{2}(\alpha, \beta) = \sqrt{(L_{2} - \alpha)(L_{2} - \beta)}. \] (48)

It is a simple matter to check that \( \mathcal{Z} \) is convex, see Fig. 1.

Comparison of the restrictions imposed on \( \gamma \) in (47) yields

\[ \gamma \in \left\{ -\gamma_{1}(\alpha, \beta), \gamma_{1}(\alpha, \beta) \right\} \quad \text{if} \quad (\alpha, \beta) \in D_{1}, \]

\[ -\gamma_{2}(\alpha, \beta), \gamma_{2}(\alpha, \beta) \quad \text{if} \quad (\alpha, \beta) \in D_{2}, \] (49)

where \( D_{1} \cup D_{2} = D \), see Fig. 2, and

\[ D_{1} = \{(\alpha, \beta) : \mathcal{Z}, \alpha + \beta \leq (L_{1} - K_{1}) \}, \]

\[ D_{2} = \{(\alpha, \beta) : \mathcal{Z}, \alpha + \beta \geq (L_{1} - K_{1}) \}. \] (50)

4. Explicit energy estimation in a material-void case

4.1. A brief guide to the translation parameters calculation

Based on considerations in previous chapters is the problem, in which the more compliant (weaker) material becomes void. Maximization formula in (39) remains valid also in the limiting case \( k_{2} \to +\infty, k_{1} \to +\infty \) with (40) taking the form

\[ 8U_{\mathbf{\Sigma}}(\Sigma, \alpha, \beta, \gamma) = \frac{K + \alpha(1 - \theta)}{\theta} \left\| \text{devN} \right\|^{2} + 2L - \alpha(1 - \theta) \left\| \text{devR} \right\|^{2} + 2\left( K + \beta(1 - \theta) \right) \left\| \text{devR} \right\|^{2} + 2\frac{\gamma^{2}(1 - \theta)}{\theta} \left( \text{trN} \text{trR} - 2\text{devN} \cdot \text{devR} \right), \] (51)

where \( K \equiv K_{1} \) and \( L \equiv L_{1} \). Introduce

\[ 2\left\| \text{devy} \right\|^{2} = 2\text{trx}^{2} - (\text{trx})^{2}, \]

\[ 2\text{det} \mathbf{x} = (\text{trx})^{2} - (\text{trx})^{2}, \]

\[ \text{trxtr} \mathbf{y} - 2\text{devx} \cdot \text{devy} = 2(\text{trx} \text{trx} \mathbf{y} - \text{tr}(\mathbf{x} \mathbf{y})). \]

Consequently, formula (51) can be re-written as

\[ U(\Sigma, \alpha, \beta, \gamma) = U_{0}(\Sigma) + \frac{1}{\theta} \left[ U_{0}(\Sigma) + H(\Sigma, \alpha, \beta, \gamma) \right], \] (52)

where \( U_{0} \) given by

\[ 8U_{0}(\Sigma) = (K - L) \left[ (\text{trN})^{2} + (\text{trR})^{2} \right] + 2L \left( \text{trN}^{2} + \text{trR}^{2} \right), \] (53)

\[ 2H(\Sigma, \alpha, \beta, \gamma) = \alpha \text{det} \mathbf{N} + \beta \text{det} \mathbf{R} + \gamma(\text{trN} \text{trR} - \text{tr}(\mathbf{NR})). \] (54)

The set

\[ I_{\mathcal{N}} = \{ \text{trN}, \text{trR}, \text{trN}^{2}, \text{trR}^{2}, \text{tr}(\mathbf{NR}) \} \] (55)

consists of five linearly independent invariants of \( \mathbf{N} \) and \( \mathbf{R} \).

Linear dependence of \( H \) on \( \gamma \) allows for setting the extremal value of \( \gamma \) given by (48) followed by reformulating (55) in the form

\[ 4H(\Sigma, \alpha, \beta, \gamma) = H_{1}(\alpha, \beta) \text{trN} \text{trR} - \text{tr}(\mathbf{NR}), \] (56)

where

\[ H_{1}(\alpha, \beta) = \begin{cases} 2z_{\mathbf{N}} + 2\sqrt{(K + \alpha)(K + \beta)} & (\alpha, \beta) \in D_{1}, \\ 2z_{\mathbf{N}} + 2\sqrt{(L - \alpha)(L - \beta)} & (\alpha, \beta) \in D_{2} \end{cases} \] (57)

and

\[ z_{\mathbf{N}} = \frac{2\text{det} \mathbf{N}}{\text{trN} \text{trR} - \text{tr}(\mathbf{NR})}, \quad z_{\mathbf{R}} = \frac{2\text{det} \mathbf{R}}{\text{trN} \text{trR} - \text{tr}(\mathbf{NR})}. \] (58)

It follows that the signs of \( z_{\mathbf{N}} \) and \( z_{\mathbf{R}} \) depend on the sign of their numerators only. If one sets

\[ 4H_{\text{U}o}(\Sigma) = \max_{(\alpha, \beta)} H_{1}(\alpha, \beta) \text{trN} \text{trR} - \text{tr}(\mathbf{NR}), \] (59)

then the lower estimation \( U_{\text{U}o}(\Sigma) \) of the stress energy \( U_{\text{hom}}(\Sigma) \), see (36), accumulated in the particle of a homogenized shell made of one material follows from (60) and reads
The search for the maximizing pair of translation parameters \((x, \beta)\) in \(D\) is based on the property of concavity and continuity of \(H_1\) in \(D\) and uses the directional derivative

\[
\nabla_h H_1(x, \beta) = \frac{\partial H_1}{\partial x}(x, \beta) h_x + \frac{\partial H_1}{\partial \beta}(x, \beta) h_\beta
\]

where, for \((x, \beta) \in D_1\),

\[
\frac{\partial H_1}{\partial x}(x, \beta) = z_N + \frac{K + \beta}{K + x}
\]

and for \((x, \beta) \in D_2\)

\[
\frac{\partial H_1}{\partial x}(x, \beta) = z_N - \sqrt{L - \beta}.
\]

As any local maximum of a concave and continuous function is also the global one, the starting point of the search is arbitrary. In what follows these points are determined from the necessary conditions of optimality related to (63) and (65) for different assumptions on \(z_N\) and \(z_k\).

As a result of the calculations in the sequel, the plane \((z_N, z_k)\) is divided into five regions (regimes), see Fig. 3. Each of these regions correspond to certain translation parameters \((x, \beta)\) determining \(U_{opt}\) in (61). Certain pair \((x, \beta)\) which is optimal in given region can be uniquely localized in one of the segments: \(\overline{AB}, \overline{BC}, \overline{CD}\) or points: \(B, D\) in the domain \(D\), see Fig. 2. Translation parameters related to the lines which separate the regimes may not be uniquely determined. In this way, the relation between the plane \((z_N, z_k)\) and set \(D\) is established. Table 1 provides a brief guide to Sections 4.2–4.4 where the calculations of translation parameters are discussed.

### 4.2. An instance of \(z_N < 0\) and arbitrary \(z_k\)

One of the necessary conditions for local maximum of \(H_1\) in \(D_1\) can be derived from (63). By this one may define the line

\[
\beta = \frac{z_N}{K} x + K (z_k^0 - 1).
\]

parallel to the vector \(h_1 = [1, z_k^0]^T\). Calculating the derivative \(\nabla_h H_1(x, \beta)\) along (67) gives

\[
\nabla_h H_1(z_N, z_k) = \frac{z_N (z_N z_k - 1)}{\sqrt{1 + z_N^2}}
\]

For assumed \(z_N < 0\) it follows that:

- (A1) if \(z_N z_k > 1\) then \(\nabla_h H_1(z_N, z_k) < 0\) and \(H_1\) is monotonically decreasing along (67).
- (B1) if \(z_N z_k = 1\) then \(\nabla_h H_1(z_N, z_k) = 0\) and \(H_1\) is constant and attains its maximum along whole line (67).
- (C1) if \(z_N z_k < 1\) then \(\nabla_h H_1(z_N, z_k) > 0\) and \(H_1\) is monotonically increasing along (67).

4.2.1. Case of \(z_N z_k \geq 1\) (A1 and B1)

Cases A1 and B1 may occur only if \(z_k < 0\). Function \(H_1\) decreases in case A1 hence the search for optimal translation parameters is reduced to the boundary segment \(\overline{AB}\), see Fig. 2. By taking the direction parallel to \(\overline{AB}\), that is \(h_2 = [1, 0]^T\), in (62) and by considering \(\nabla_h H_1(x, \beta) = 0\) for \((x, \beta) \in \overline{AB}\) one obtains

\[
x_{opt} = -K \left(1 - \frac{1}{z_N^0}\right), \quad \beta_{opt} = 0.
\]

restricted by \(x_{opt} \leq L - K\) corresponding to \(z_N < -\sqrt{K/L}\). For \(-\sqrt{K/L} \leq z_N < 0\) optimal translation parameters are localized at point \(B\). In case B1, the maximum of \(H_1\) is achieved for each pair \((x, \beta) \in D_1\) satisfying (67) which means that the maximizers are not determined uniquely.

4.2.2. Case of \(z_N z_k < 1\) (C1)

In case C1, the function \(H_1\) is increasing hence optimal pair \((x, \beta)\) lies within the segment \(\overline{BC}\) of the straight line given by

\[
b = -x + L - K.
\]

By taking the vector \(h_3 = [1, 1]^T\) parallel to \(\overline{BC}\) one may calculate

\[
\nabla_h H_1(x, \beta) = \frac{1}{\sqrt{2}} \left(\frac{z_N - z_k + (L - z_k) - (K + \beta)}{(L - L)/(K + \beta)}\right)
\]

From \(\nabla_h H_1(x, \beta) = 0\) and (70) it follows that

\[
x_{opt} = \frac{1}{2} \left(\frac{(z_N - z_k)(K + L)}{(L - z_k)^2} + (L - K)\right)
\]

and this result is restricted by the condition \((x, \beta) \in \overline{BC}\) to

\[
z_N - z_k < \sqrt{L - K} - \frac{K}{L - L}.
\]

Proof of this fact is straightforward therefore it is omitted here.

If (73) is not satisfied then \((x_{opt}, \beta_{opt})\) are localized at point \(B\).

### Table 1

A brief guide to the results of optimal translation parameters calculations.

<table>
<thead>
<tr>
<th>(z_N)</th>
<th>Value of (z_N z_k)</th>
<th>((x, \beta)) in Fig. 2</th>
<th>((z_N, z_k)) in Fig. 3</th>
<th>Section number</th>
</tr>
</thead>
<tbody>
<tr>
<td>(z_N &lt; 0)</td>
<td>(z_N z_k &gt; 1)</td>
<td>(\overline{AB}) or (\overline{B})</td>
<td>(R_3) or (R_4^a)</td>
<td>4.2.1</td>
</tr>
<tr>
<td>(z_N &lt; 0)</td>
<td>(z_N z_k = 1)</td>
<td>Not unique</td>
<td>(\overline{BC}) or (\overline{B})</td>
<td>4.2.2</td>
</tr>
<tr>
<td>(z_N &lt; 0)</td>
<td>(z_N z_k &lt; 1)</td>
<td>(\overline{D})</td>
<td>(R_1)</td>
<td>4.2.3</td>
</tr>
</tbody>
</table>

* a Due to certain restriction imposed on \(z_N\).
* b Due to certain restriction imposed on \(z_N - z_k\).
* c Case of \(z_N > 0\). \(z_N z_k < 1\) splits into two subcases.
4.2.3. Remarks on the continuity of $U_{\text{low}}$ for $z_N < 0$

Whole region $\mathcal{R}_3$ and the lines separating it from $\mathcal{R}_2$ and $\mathcal{R}_5$ in Fig. 3 correspond to the point $B$ in Fig. 2. Hence the values of translation parameters continuously change across both boundaries. Consequently, $H_1$ in (58) and $U_{\text{low}}$ in (61) are also continuous.

Any point at the curve $z_N z_R = 1$ dividing $\mathcal{R}_2$ and $\mathcal{R}_3$ is related by (67) to a straight line which links $EB$ and $AB$ in Fig. 2. Consequently, infinitely many pairs $(x, \beta)$ correspond to this point. The values of translation parameters morph across the line bounding $\mathcal{R}_1$. Function $H_1$ is thus continuous along whole line.

It follows that $U_{\text{low}}$ is continuous on the half-plane $z_N < 0$.

4.4. Summary of the results

Considerations in Sections 4.2 and 4.3 result in the division of the plane $(z_N, z_R)$ into five regions (regimes) $\mathcal{R}_i$, $i = 1, \ldots, 5$. Their range and the values of optimal translation parameters are given by formulae below, see also Fig. 3.

Regime $\mathcal{R}_1$ (corresponding to $D$ in Fig. 2)

\[
(z_N, z_R) : \{z_N > 0, z_R > 0, z_N z_R > 1 \}.
\]

\[
\begin{align*}
x_{\text{opt}} &= L, \\
\beta_{\text{opt}} &= L, \\
\gamma_{\text{opt}} &= 0.
\end{align*}
\]

Regime $\mathcal{R}_2$ (corresponding to a certain point belonging to $EB$ in Fig. 2)

\[
(z_N, z_R) : \left\{ z_N z_R < \sqrt{\frac{K}{L}} z_R z_N z_R < 1 \right\}.
\]

\[
\begin{align*}
x_{\text{opt}} &= \frac{1}{2} \left[ (z_N - z_R)(K + L) \right] \sqrt{(z_N - z_R)^2 + 4} + (L - K), \\
\beta_{\text{opt}} &= -\frac{1}{2} \frac{2 (z_N - z_R)(K + L)}{\sqrt{(z_N - z_R)^2 + 4}}, \\
\gamma_{\text{opt}} &= \frac{K + L}{\sqrt{(z_N - z_R)^2 + 4}}, \\
H_{\text{opt}}(\Sigma) &= \frac{1}{8} \left[ (L - K)(z_N + z_R) + (L + K) \sqrt{(z_N - z_R)^2 + 4} \right].
\end{align*}
\]

Fig. 3. Five regions on the plane $(z_N, z_R)$ related to different pairs $(x, \beta)$.
Regime $\mathcal{R}_3$ (corresponding to a certain point belonging to $\overline{BC}$ in Fig. 2)

$$z_N > \sqrt{\frac{L}{K}} z_N z_K < 1,$$

$$z_N = L \left(1 - \frac{1}{z_N^2}\right), \quad \beta_{opt} = 0, \quad \gamma_{opt} = \frac{L}{z_N},$$

$$\begin{aligned}
\frac{H_{opt}(\Sigma)}{trNtrR - tr(NR)} &= \frac{1}{4} \left(1 + z_N^2\right).
\end{aligned}$$

Regime $\mathcal{R}_4$ (corresponding to $B$ in Fig. 2)

$$z_N \in \left(\frac{L}{K}, \sqrt{\frac{L}{K}}\right), \quad z_N - z_K > \sqrt{\frac{L}{K}} - \sqrt{\frac{K}{L}}.$$ 

$$z_N = L - K, \quad \beta_{opt} = 0, \quad \gamma_{opt} = \sqrt{KL},$$

$$\begin{aligned}
\frac{H_{opt}(\Sigma)}{trNtrR - tr(NR)} &= \frac{1}{4} \left(L - K\right) z_N + \sqrt{LK}.
\end{aligned}$$

Regime $\mathcal{R}_5$ (corresponding to a certain point belonging to $\overline{AB}$ in Fig. 2)

$$z_N < -\sqrt{\frac{K}{L}} z_N < 0, \quad z_N > 1,$$

$$z_N = -K \left(1 - \frac{1}{z_N^2}\right), \quad \beta_{opt} = 0, \quad \gamma_{opt} = -\frac{K}{z_N},$$

$$\begin{aligned}
\frac{H_{opt}(\Sigma)}{trNtrR - tr(NR)} &= -\frac{1}{4} \left(1 + z_N^2\right).
\end{aligned}$$

4.5. Inverse constitutive equations and material distribution function related to the estimated energy of a shell

Inverse constitutive equations related to $U_{low}(\Sigma)$ are given by

$$\mathbf{e} = \frac{\partial U_{low}}{\partial \mathbf{N}}, \quad \mathbf{K} = \frac{\partial U_{low}}{\partial \mathbf{R}}.$$ 

The general formulae for shell deformations read

$$\mathbf{e} = \frac{\partial U_{low}}{\partial \mathbf{N}} \partial \mathbf{N} + \frac{\partial U_{low}}{\partial \mathbf{R}} \partial \mathbf{R} + \frac{\partial U_{low}}{\partial tr\mathbf{R}} \partial tr\mathbf{R},$$

$$\mathbf{K} = \frac{\partial U_{low}}{\partial \mathbf{N}} \partial \mathbf{R} + \frac{\partial U_{low}}{\partial \mathbf{R}} \partial \mathbf{N} + \frac{\partial U_{low}}{\partial tr\mathbf{R}} \partial tr\mathbf{R} + \frac{\partial U_{low}}{\partial tr\mathbf{R}} \partial tr\mathbf{R},$$

hence the coupled constitutive equations are given by

$$\begin{aligned}
\mathbf{e} &= \frac{\partial U_{low}}{\partial \mathbf{N}} \mathbf{I}_2 + 2 \frac{\partial U_{low}}{\partial \mathbf{N}} \mathbf{N} + \frac{\partial U_{low}}{\partial \mathbf{R}} \mathbf{R}, \\
\mathbf{K} &= \frac{\partial U_{low}}{\partial \mathbf{R}} \mathbf{I}_2 + 2 \frac{\partial U_{low}}{\partial \mathbf{R}} \mathbf{R} + \frac{\partial U_{low}}{\partial \mathbf{R}} \mathbf{N},
\end{aligned}$$

where $\mathbf{I}_2 = \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$.

Material distribution function related to the obtained energy estimation can be determined locally. Indeed, by the Rockafellar Theorem, see (Rockafellar, 1976) one may exchange the minimization over $\theta$ with the integration in (41) thus obtaining the condition

$$\frac{d}{d\theta} \left\{ \frac{1}{2} \beta_{opt}^2 + H_{opt}(\Sigma) + \theta \right\} = 0.$$ 

From this it follows that

$$\tilde{\theta}_z = \min \left\{ \frac{1}{2} \frac{U_{opt}(\Sigma)}{\beta_{opt}^2} \right\}. $$

By making use of the results obtained in previous sections, all derivatives in (84) and the function $\beta_{opt}$ can be calculated explicitly in each region $\mathcal{R}_i, i = 1, \ldots, 5$, but for the reasons of space they are not reported here. It has to be stressed that the distribution given by (86) is optimal for given Lagrange multiplier $\ell$ and $U_{low}(\Sigma)$ determined by the method proposed in this paper. However, there is no proof that this estimation is attainable on some microstructure.

4.6. Limiting cases of $\mathbf{R} = \mathbf{0}$ and $\mathbf{N} = \mathbf{0}$

In case of $\mathbf{R} = \mathbf{0}$ the translation method provides exact lower estimation for the stress energy $W_{low}$, see Gibianski and Cherkaev (1987), Cherkaev (2000) and Allaire (2002). The lower bound on this functional is given by

$$8W_{low}(\mathbf{N}) = K(N_i + N_j)^2 + L(N_i - N_j)^2 + \frac{1}{\theta_{opt}} (K + L)(|N_i| + |N_j|)^2,$$

where $N_i, N_j$ stand for the principal values of tensor $\mathbf{N}$. Exactness of this estimate is proved by pointing out that the energy stored in certain microstructures realize (87). Formula for the optimal material distribution function $\theta_{opt}$ reads

$$\theta_{opt} = \min \left\{ \frac{1}{2} \frac{K + L}{\ell} (|N_i| + |N_j|) \right\}.$$ 

These results cannot be obtained by taking $\mathbf{R} = \mathbf{0}$ in (77)-(81). Variables $\xi_1, \xi_2$ do not make sense in the limiting case as the denominator of the rationals in (59) tends to infinity. Hence all subsequent calculations are invalid. In order to link the formulae established on the grounds of shallow shell theory with those related to plane elasticity on needs to start from starting $\mathbf{R} = \mathbf{0}$ in (54) and (55). In this way (87) and (88) follow. Similarly, by assuming $\mathbf{N} = \mathbf{0}$ in the expressions for $U_{low}$ and $H$ the problem reduces to the one of optimal material distribution in a flat Kirchhoff plate. It was solved by Gibianski and Cherkaev (1984) and the specification of results in the solid-void case can be found in (Lewiński and Telega, 2000). The compliance $f_{low}$ is estimated by making use of the formula for stress energy

$$8W_{low}(\mathbf{R}) = K(R_i + R_j)^2 + L(R_i - R_j)^2 + \frac{1}{\theta_{opt}} (K + L)(|R_i| + |R_j|)^2,$$

The estimation is exact as it is attainable on certain microstructures. Optimal material distribution function $\theta_{opt}$ is given by

$$\theta_{opt} = \min \left\{ \{ \rho_1 \right\},$$

where

$$\rho_1 = \left\{ \begin{array}{ll}
\frac{1}{\sqrt{8\pi}} |R_i| + |R_j| \quad \text{if} \ R_i R_j > 0,
\frac{1}{\sqrt{8\pi}} |K(R_i + R_j)^2 + L(R_i - R_j)^2 \quad \text{if} \ R_i R_j < 0.
\end{array} \right.$$ 

5. Illustrative examples

5.1. General remarks on numerical implementation

Numerical results presented in this section are obtained by assuming that the middle surface of a shallow shell is occupied by the isotropic material microscopically mixed with void. The total amount of the solid phase of the mixture is restricted by the isoperimetric condition (19) with $m = 0.5$.

Two types of structures are considered. In each case, the plot of $\tilde{\theta}_z$, see (86), corresponds to a shell whose compliance $f_{low}$ corre-
sponds through (41) to the lower bound of the stress energy $W_{low}$ calculated by the translation method.

The values $\theta_2(x)$, $x \in \Omega$, are set between 0 and 1. The lower/greater density of the solid phase is reflected in the lighter/darker shades of grey in the corresponding figures.

### 5.2. Twisted cantilever

In this section we consider the twisted cantilever shell formed on a rectangular plane $\Omega$. The structure is subjected to the load concentrated at the middle point of the free edge and inclined to it at different angles as shown in Figs. 4, 6 and Fig. 8. Change in the material distribution function $\theta_2$ with respect to the inclination angle of the loading is shown in Figs. 5, 7 and 9.

Material distribution in Fig. 10 is related to the flat cantilever with pure in-plane loading.

---

**Fig. 4.** Twisted cantilever clamped along $A - B$ with the load inclined at equal angles to both $A - B$ and $C - D$.

**Fig. 5.** Optimal material distribution for the twisted cantilever in Fig. 4 with $f/a = 1/10$.

**Fig. 6.** Twisted cantilever clamped along $A - B$ with the load directed along this edge.

**Fig. 7.** Optimal material distribution for the twisted cantilever in Fig. 6 with $f/a = 1/10$.

**Fig. 8.** Twisted cantilever clamped along $A - B$ with the load directed along $C - D$.

**Fig. 9.** Optimal material distribution for the twisted cantilever in Fig. 8 with $f/a = 1/10$.

**Fig. 10.** Material distribution for a flat cantilever.

---

### 5.3. Twisted square plate clamped along the boundary

In the next example we generalize the problem of optimal material distribution for minimal compliance of the Kirchhoff plate. The structure is subjected to the uniform load $q(x,y) = \text{const.}$, in the direction parallel to the axis $z$ as shown in Fig. 11.

Material layouts shown in Figs. 12 and 13 reflect the change in the $f/a$ ratio, where $f$ denotes the maximal rise or depression of the shell and $a$ is related to the plane dimension of $\Omega$. Material distribution in Fig. 14 corresponds to a thin plate in pure bending ($f = 0$).
6. Conclusions

The main result of the paper consists in regularization of a two-material distribution problem for compliance minimization of shallow shells or thin plates subjected to simultaneous in-plane and bending loadings. This objective is achieved by allowing composite materials with microstructure as possible solution. Consequently, the question of choosing the appropriate microstructural material layout and studying the problem of its homogenized properties naturally appears in the optimization task. Due to the infinite variety of possible micro-designs the corresponding calculations are not easy to perform. This difficulty is removed from the analysis by using the Gibianski–Cherkaev and Murat–Tartar translation method in establishing the lower estimation of stress energy accumulated in a particle of a homogenized shell. In this way, the compliance minimization problem is reduced to local determination of optimal translation parameters which are independent of any microstructure. In the present research, the translator was assumed in the form (27) compatible with the one proposed in Gibianski and Cherkaev (1984) and Gibianski and Cherkaev (1987).

As a result, the functional representing stress energy of an effective shell turns out to be isotropic and nonlinearly dependent on the stress and couple resultants. Hence, practical applications of obtained results are realized by an iterative procedure whose typical loop consists in: (i) solving the equilibrium problem of a shell for fixed material distribution, and (ii) redistributing the material in an optimal way. These iterations run until the sequence of compliances converges with assumed accuracy.

In case of a material-void optimization problem, the lower estimation \( U_{\text{low}}(\Sigma) \) of the stress energy \( U_{\text{hom}}(\Sigma) \) accumulated in the particle of the homogenized shell is calculated explicitly by optimal adjustment of the translation parameters. This result seems to be the novel generalization of the solutions previously obtained for separately treated cases of the in-plane and bending deformation.

For \( N = 0 \), formulae (53)–(55) describe the lower bound of a stress energy for the shape design of a Kirchhoff plate reported in Lewiński and Telega (2000, Chapter 26.7). Similarly, if \( R = 0 \) then the analysis degenerates to the optimal material-void distribution in two-dimensional elasticity, see Allaire (2002, Chapter 4.2). It is a matter of straightforward calculations to prove these facts, hence the corresponding transformations are omitted here.
Acknowledgements

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Appendix A

The study of inequality (32) can be reduced to determining the range of \( x, \beta, \gamma \in \mathbb{R} \) satisfying

\[
\alpha \{ \mathbf{F}(\mathbf{n}, \mathbf{n}) - (\mathbf{n} \cdot \mathbf{T}_0(\mathbf{n})) \} \geq 0,
\]

\[
\beta \{ \mathbf{F}(\mathbf{r}, \mathbf{r}) - (\mathbf{r} \cdot \mathbf{T}_0(\mathbf{r})) \} \geq 0,
\]

\[
\gamma \{ \mathbf{F}(\mathbf{n}, \mathbf{r}) - (\mathbf{n} \cdot \mathbf{T}_0(\mathbf{r})) \} \geq 0
\]

see (26) and (27). Recall that \( \mathbf{n} = \mathbf{N}, (\mathbf{r}) = \mathbf{R} \). In what follows, we make use of the Fourier representation of periodic functions. For given \( f \) we set

\[
f(y) = \sum_{s_1, s_2 = -\infty}^{\infty} \sum_{s_1, s_2 = -\infty}^{\infty} f(s_1, s_2) e^{i(s_1 \varphi_1 + i s_2 \varphi_2)},
\]

where \( k_1 = (2\pi s_1)/l_1; s_1, s_2 \) are real numbers and \( (\varphi_1, \varphi_2) \in Y = (0, l_1) \times (0, l_2) \), where \( Y \) denotes a periodicity cell. Moreover,

\[
f(s_1, s_2) = \langle f(y) e^{-i(s_1 \varphi_1 + i s_2 \varphi_2)} \rangle,
\]

\[
\frac{\partial f}{\partial y_j} = \sum_{s_1, s_2 = -\infty}^{\infty} \sum_{s_1, s_2 = -\infty}^{\infty} \frac{\partial f}{\partial y_j} \langle (s_1, s_2) e^{i(s_1 \varphi_1 + i s_2 \varphi_2)} \rangle,
\]

\[
\frac{\partial^2 f}{\partial y_j \partial \mu} = \sum_{s_1, s_2 = -\infty}^{\infty} \sum_{s_1, s_2 = -\infty}^{\infty} \frac{\partial^2 f}{\partial y_j \partial \mu} (s_1, s_2) e^{i(s_1 \varphi_1 + i s_2 \varphi_2)}
\]

where

\[
\frac{\partial f}{\partial y} = k_1 f_j, \quad \frac{\partial^2 f}{\partial y_j \partial \mu} = -k_2 k_1 f_j.
\]

If \( f_1, f_2 \) take real values then

\[
\langle f_1 f_2 \rangle = \langle f_1 \rangle \langle f_2 \rangle + \frac{1}{2} \sum_{s_1, s_2 = 0}^{\infty} \left( \langle f_1 \rangle \langle f_2 \rangle + \langle f_1 f_2 \rangle \right),
\]

where \( f_j = f_j(s_1, s_2) \) and \( (\bar{\cdot}) \) denotes the complex conjugate.

Case of \( \mathbf{F}(\mathbf{n}, \mathbf{r}) \)

Representing second-order tensors \( \mathbf{n}, \mathbf{r} \) in the basis (1) gives

\[
\mathbf{F}(\mathbf{n}, \mathbf{r}) = -\langle n_{11} r_{22} - n_{22} r_{11} + 2n_{12} r_{12} \rangle.
\]

Next, applying (A.7) leads to

\[
\mathbf{F}(\mathbf{n}, \mathbf{r}) = -\langle n_{11} \rangle \langle r_{22} \rangle - \langle n_{22} \rangle \langle r_{11} \rangle + 2\langle n_{12} \rangle \langle r_{12} \rangle + Q_1,
\]

or simply

\[
\mathbf{F}(\mathbf{n}, \mathbf{r}) - \mathbf{N} \cdot \mathbf{T}_0(\mathbf{R}) = Q_1,
\]

with \( Q_1 \) given by

\[
Q_1 = \frac{1}{2} \sum_{s_1, s_2 = 0}^{\infty} \left( -\tilde{n}_{11} \tilde{q}_{22} - \tilde{n}_{22} \tilde{q}_{11} - \tilde{n}_{22} \tilde{q}_{11} + 2\tilde{n}_{12} \tilde{q}_{12} + 2\tilde{n}_{12} \tilde{q}_{12} \right).
\]

The equilibrium equations

\[
\frac{\partial n_{11}}{\partial y_1} = 0, \quad \frac{\partial^2 T_{11}}{\partial y_1 \partial \mu} = 0
\]

are imposed by the static admissibility of \( \mathbf{n}, \mathbf{r} \). They can be transformed, by applying (A.6), to the form

\[
\tilde{n}_{11} + \frac{k_2}{k_1} \tilde{n}_{12} = 0, \quad \frac{k_1}{k_2} \tilde{n}_{12} + \tilde{n}_{22} = 0,
\]

\[
\tilde{r}_{12} = -\frac{1}{2} \left( \frac{k_1}{k_2} \tilde{r}_{11} + \frac{k_2}{k_1} \tilde{r}_{22} \right).
\]

Moreover, by replacing (A.13), (A.14) in (A.11) and by recalling that \( k_1/k_2 = k_1/k_2 \) we finally obtain

\[
Q_1 = 0
\]

hence (A.1) is satisfied for any \( \gamma \in \mathbb{R} \).

Case of \( \mathbf{F}(\mathbf{n}, \mathbf{n}) \)

Similarly, we may write

\[
\mathbf{F}(\mathbf{n}, \mathbf{n}) = -2\langle n_{11} \rangle \langle n_{12} \rangle + 2\langle n_{12} \rangle^2 + Q_2,
\]

or simply

\[
\mathbf{F}(\mathbf{n}, \mathbf{n}) - \mathbf{N} \cdot \mathbf{T}_0(\mathbf{N}) = Q_2,
\]

with \( Q_2 \) given by

\[
Q_2 = \sum_{s_1, s_2 = 0}^{\infty} \left( -\tilde{n}_{11} \tilde{n}_{22} - \tilde{n}_{11} \tilde{n}_{22} + 2\tilde{n}_{12} \tilde{n}_{12} \right).
\]

Applying (A.13) in (A.18) yields

\[
Q_2 = 0
\]

thus satisfying (A.1) for any \( \alpha \in \mathbb{R} \).

Case of \( \mathbf{F}(\mathbf{r}, \mathbf{r}) \)

The function

\[
\mathbf{F}(\mathbf{r}, \mathbf{r}) = -2\langle r_{11} \rangle \langle r_{22} \rangle + 2\langle r_{12} \rangle^2 + Q_3,
\]

or simply

\[
\mathbf{F}(\mathbf{r}, \mathbf{r}) - \mathbf{R} \cdot \mathbf{T}_0(\mathbf{R}) = Q_3,
\]

with \( Q_3 \) given by

\[
Q_3 = \sum_{s_1, s_2 = 0}^{\infty} \left( -\tilde{r}_{11} \tilde{q}_{22} - \tilde{q}_{11} \tilde{r}_{22} + 2\tilde{r}_{12} \tilde{q}_{12} \right)
\]

satisfies (A.1) provided \( \beta > 0 \). The latter is proved by applying (A.14) in (A.22) which leads to

\[
Q_3 = \left[ \frac{1}{2} \sum_{s_1, s_2 = 0}^{\infty} \left( \frac{k_1}{k_2} \tilde{r}_{11} - \frac{k_2}{k_1} \tilde{r}_{22} \right)^2 \right.
\]

and \( Q_3 \geq 0 \).

References


