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Characterisation and Properties of r-Toeplitz Matrices

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An r-Toeplitz (rT) matrix can be regarded as a block Toeplitz matrix with order some multiple of r from which a bottom border and a right-hand border have been removed. It is shown that some known properties of Toeplitz matrices extend to this new class. In particular, a necessary and sufficient condition is given, in terms of a linear matrix equation, for a nonsingular matrix to have rT form, from which it follows that the elements of the inverse are completely determined by those in its first and last r rows and columns. In addition, some results are presented for cases when the inverse of an rT matrix has certain banded forms.

1. INTRODUCTION

In two previous papers [1, 6] we introduced a generalisation of Toeplitz matrices, called *conjugate-Toeplitz* (CT) matrices, and showed that certain properties of Toeplitz matrices could be extended to these new matrices. Algorithms for inverting CT matrices and solving CT systems of equations, using $O(n^2)$ flops for matrices of order n, were given in [5]. A further generalisation, called *r-Toeplitz* (rT) matrices, was defined in [7], complete with an inversion procedure. This was used to obtain an algorithm (based on the well-known Levinson–Trench algorithm [12, 14]) for inverting Toeplitz matrices which are not strongly nonsingular, in $O(n^2)$ flops. This case is not covered by most existing algorithms, although recently another scheme which permits weakening of the strong nonsingularity restriction has been presented in [11]. A system of linear equations whose coefficient matrix is rT has arisen in an engineering problem involving sound propagation [3], and an algorithm for solving such a system can also be found in [7].

To introduce the rT class of matrices, we begin by defining a block Toeplitz matrix.

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DEFINITION 1.1. A matrix B of order $n=mr$ whose elements are matrices B_{ij} of order r, $i, j=1, 2,..., m$, is a *block Toeplitz* matrix if

$$B_{i+1, j+1}=B_{ij}, \qquad i, j=1, 2,..., m-1. \tag{1.1}$$

If we examine the individual elements of B, denoted by b_{ij}, $i, j=1, 2,..., mr$, it is easy to see that they satisfy

$$b_{i+r, j+r}=b_{ij}, \qquad i, j=1, 2,...,(m-1) r. \tag{1.2}$$

Thus it is natural to consider the following class of matrix.

DEFINITION 1.2. A matrix A of order n is an *r-Toeplitz* (rT) matrix if

$$a_{i+r, j+r}=a_{ij}, \qquad i, j=1, 2,..., n-r. \tag{1.3}$$

Remark 1.1. An rT matrix is completely defined by its first r rows and columns.

Remark 1.2. A Toeplitz matrix is an rT matrix with $r=1$ and a CT matrix is an example of a 2T matrix.

Although the elements of A and B satisfy the same relationship, the crucial difference is that the order of A is arbitrary, whereas the order of B in Definition 1.1 must be a multiple of r.

The purpose of this paper is to show that some known properties of Toeplitz and CT matrices can now be extended to rT matrices. In the next section we characterise a nonsingular rT matrix A by showing (Theorem 2.1) that it is necessary and sufficient for A to satisfy a simple linear matrix equation; the latter involves a generalisation of the companion matrix. This leads to the important result (Theorem 2.2) that the elements of the inverse of an rT matrix are completely determined by the elements in the first and last r rows and columns of the inverse.

Several authors [2, 10] have considered the particular case when a banded matrix has a Toeplitz inverse, and these matrices occur in the study of stationary time series [13] and in moving weighted average smoothing [9]. In Section 3 two types of banding are considered for the more general rT matrix, and in each instance (Theorems 3.1 and 3.2) it is shown that, apart from certain submatrices in the top left and bottom right-hand corners, the inverse also has an rT pattern of elements.

It is interesting that useful structural properties can still be derived for matrices whose pattern of elements is much more general than the simple Toeplitz or block Toeplitz forms. In a subsequent paper [8] further properties of rT matrices will be developed.

2. Characterisation of an rT Matrix and a Property of Its Inverse

In [1] we showed that a Toeplitz matrix could be characterised by a matrix equation involving the companion matrix. In order to produce a similar equation for an rT matrix we need to define the following matrix.

Definition 2.1. A matrix C of order n is an *r-companion* matrix if

$$C=\begin{array}{cc} \begin{array}{cc} r & \quad n-r \end{array} & \\ \left[\begin{array}{cc} 0 & I \\ \multicolumn{2}{c}{C_1} \end{array}\right] & \begin{array}{l} n-r \\ r \end{array} \end{array} \tag{2.1}$$

where C_1 is an arbitrary $r \times n$ matrix, and I denotes a unit matrix.

Remark 2.1. If X is an ordinary companion matrix, then X^r $(r<n)$ is an r-companion matrix. However not every r-companion matrix can be expressed in the form X^r.

We now have the following characterisation theorem.

Theorem 2.1. *A nonsingular matrix A is rT if and only if there exist r-companion matrices C and D such that*

$$CA = AJD^TJ, \tag{2.2}$$

where J is the reverse unit matrix, i.e., $[J]_{ij}=\delta_{i,n-j+1}$ and δ_{ij} is the Kronecker delta.

Proof. If (2.2) is satisfied, then comparing elements on each side shows that (1.3) holds. We therefore have to show that if A is nonsingular and rT, then there exist r-companion matrices C and D such that (2.2) is satisfied.
Using (2.1) for both C and D, (2.2) can be written as

$$\left[\begin{array}{cc} 0 & I_{n-r} \\ \multicolumn{2}{c}{C_1} \end{array}\right] A = A \left[\begin{array}{cc} JD_1^TJ & \begin{array}{c} I_{n-r} \\ 0 \end{array} \end{array}\right]. \tag{2.3}$$

When A is rT, the ijth element on each side of (2.3) is the same for $i=1, 2,..., n-r$, $j=r+1, r+2,..., n$. If we take $i=1,..., n-r$, $j=1,...,r$ then

$$[0 \quad I_{n-r}]\, A_1 = A_2 JD_1^T J, \tag{2.4}$$

where A_1 consists of the first r columns of A and A_2 consists of the first $(n-r)$ rows of A. Since A is nonsingular, A_2 and hence

$[A_2, [0 \quad I_{n-r}] A_1]$ both have rank $n-r$ and so (2.4) can be solved for D_1, although not uniquely. Finally for $i = n-r+1,..., n$, $j = 1,..., n$, we obtain

$$C_1 A = A_3 D,$$

where A_3 consists of the last r rows of A. Since A is nonsingular and D has already been obtained, there is a solution for C, as required. ∎

Remark 2.2. When $r = 1$ we have a Toeplitz matrix which is persymmetric, i.e., $JA^{\mathrm{T}}J = A$. In this case, as shown in [1], C and D in (2.2) are ordinary companion matrices and $C = D$.

A consequence of Theorem 2.1 is that we can now determine a relationship between the elements of the inverse of an rT matrix.

THEOREM 2.2. *If $Y = [y_{ij}]$ is the inverse of an rT matrix A such that the first and last principal minors of Y of order r, $|Y_{1r}|$ and $|Y_{nr}|$, respectively, are nonzero then for $i, j = 1, 2,..., n-r$,*

$$\begin{aligned} y_{i+r,j+r} = y_{ij} &+ (y_{i+r,1},..., y_{i+r,r})\, Y_{1r}^{-1}(y_{1,j+r},..., y_{r,j+r})^{\mathrm{T}} \\ &- (y_{i,n-r+1},..., y_{in})\, Y_{nr}^{-1}(y_{n-r+1,j},..., y_{nj})^{\mathrm{T}}. \end{aligned} \tag{2.5}$$

Proof. Since A satisfies (2.2) and $Y = A^{-1}$ then

$$YC = JD^{\mathrm{T}}JY \tag{2.6}$$

and, with suitable partitioning, this can be written as

$$\begin{array}{c c} & \begin{array}{ccc} r & n-2r & r \end{array} \\ \begin{array}{c} r \\ n-2r \\ r \end{array} & \begin{bmatrix} Y_{1r} & Y_1 & Y_2 \\ Y_3 & Y_4 & Y_5 \\ Y_6 & Y_7 & Y_{nr} \end{bmatrix} \end{array} \begin{bmatrix} 0 & I_{n-r} \\ C_r & C_{n-r} \end{bmatrix} = \begin{bmatrix} JD_{n-r}^{\mathrm{T}}J & I_{n-r} \\ JD_r^{\mathrm{T}}J & 0 \end{bmatrix} \begin{bmatrix} Y_{1r} & Y_1 & Y_3 \\ Y_3 & Y_4 & Y_5 \\ Y_6 & Y_7 & Y_{nr} \end{bmatrix}. \tag{2.7}$$

Since Y_{1r} and Y_{nr} are nonsingular, the bottom right $r \times (n-r)$ submatrix and the top left $(n-r) \times r$ submatrix in (2.7) give

$$C_{n-r} = Y_{nr}^{-1}\{JD_r^{\mathrm{T}}J[Y_1 \quad Y_2] - [Y_6 \quad Y_7]\} \tag{2.8}$$

and

$$JD_{n-r}^{\mathrm{T}}J = \left\{\begin{bmatrix} Y_2 \\ Y_5 \end{bmatrix} C_r - \begin{bmatrix} Y_3 \\ Y_6 \end{bmatrix}\right\} Y_{1r}^{-1}. \tag{2.9}$$

If we now put (2.8) and (2.9) in the top right $(n-r)\times(n-r)$ submatrix of (2, 7) we obtain

$$\begin{bmatrix} Y_{1r} & Y_1 \\ Y_3 & Y_4 \end{bmatrix} + \begin{bmatrix} Y_2 \\ Y_5 \end{bmatrix} Y_{nr}^{-1}\{JD_r^{\mathrm{T}}J[Y_1 \quad Y_2] - [Y_6 \quad Y_7]\}$$

$$= \left\{ \begin{bmatrix} Y_2 \\ Y_5 \end{bmatrix} C_r - \begin{bmatrix} Y_3 \\ Y_6 \end{bmatrix} \right\} Y_{1r}^{-1}[Y_1 \quad Y_2] + \begin{bmatrix} Y_4 & Y_5 \\ Y_7 & Y_{nr} \end{bmatrix}. \tag{2.10}$$

Last, from the bottom left $r\times r$ submatrix in (2.7) we see that $JD_r^{\mathrm{T}}J = Y_{nr}C_rY_{1r}^{-1}$ and so (2.10) becomes

$$\begin{bmatrix} Y_4 & Y_5 \\ Y_7 & Y_{nr} \end{bmatrix} = \begin{bmatrix} Y_{1r} & Y_1 \\ Y_3 & Y_4 \end{bmatrix} + \begin{bmatrix} Y_3 \\ Y_6 \end{bmatrix} Y_{1r}^{-1}[Y_1 \quad Y_2] - \begin{bmatrix} Y_2 \\ Y_5 \end{bmatrix} Y_{nr}^{-1}[Y_6 \quad Y_7]. \tag{2.11}$$

Taking the ijth term of (2.11) gives (2.5). ▮

Remark 2.3. Equation (2.5) shows that the elements of Y, the inverse of an rT matrix, are completely determined by the elements in the first and last r rows and columns of Y.

Remark 2.4. When $r=1$, Y is the inverse of a Toeplitz matrix which is persymmetric. Thus $y_{in} = y_{1,n-i+1}$, $y_{nj} = y_{n-j+1,1}$, and $y_{nn} = y_{11}$ and so (2.5) reduces to

$$y_{i+1,j+1} = y_{ij} + \frac{1}{y_{11}}(y_{i+1,1}y_{1,j+1} - y_{1,n-i+1}y_{n-j+1,1}).$$

This equation is well known and is given, for example, in [1, 10].

3. rT MATRICES WITH BANDED INVERSES

In this section two types of banding are considered. The first is elementwise banding, i.e., there are triangles of zeros in the top right and/or the bottom-left-hand corners of the inverse. The second form is called incomplete block banding, which is explained in Definition 3.2. This reduces to block banding (to be defined below) when the order of the matrix is a multiple of the order of each block.

In fact, if certain elements in the first r rows and columns of the inverse of an rT matrix are zero then we show, using Theorem 2.2, that one or other of the above types of banding occurs in the inverse. We also show that the inverse has what can be called quasi-rT form, i.e., the elements satisfy (1.3) except for submatrices in its top left and bottom right hand corners.

DEFINITION 3.1. A matrix $A=[a_{ij}]$ of order n is (element-wise) banded if for some positive integers r and s, less than $n+1$,

$$a_{ij}=0, \qquad j-i\geqslant r,\ i-j\geqslant s. \tag{3.1}$$

We now have the following result.

THEOREM 3.1. *If $Y=[y_{ij}]$ is the inverse of an rT matrix of order n satisfying the conditions of Theorem 2.2 and, in addition, for positive integers M and N, $M, N\leqslant n$*

$$y_{ij}=0, \qquad \begin{cases} j-i\geqslant M & \text{and} \quad i\leqslant r \\ i-j\geqslant N & \text{and} \quad j\leqslant r. \end{cases} \tag{3.2}$$

Then Y is banded and is rT, except for the $(N-1)\times(M-1)$ submatrix in the top-left-hand corner and the $(M-1)\times(N-1)$ submatrix in the bottom-right-hand corner.

Proof. To prove that Y is banded we only consider the top-right-hand corner since the proof for the bottom-left-hand corner is similar. First, if $M\geqslant n-r$, we already have a triangle of zeros in the top-right-hand corner and so there is nothing to prove. Suppose therefore that $M<n-r$. From (2.5) the elements in the $(r+1)$th row of Y satisfy

$$\begin{aligned} y_{r+1,j+r}={}&y_{1j}+(y_{r+1,1},\ldots,y_{r+1,r})\,Y_{1r}^{-1}(y_{1,j+r},\ldots,y_{r,j+r})^{\mathrm{T}} \\ &-(y_{1,n-r+1},\ldots,y_{1n})\,Y_{nr}^{-1}(y_{n-r+1,j},\ldots,y_{nj})^{\mathrm{T}}. \end{aligned} \tag{3.3}$$

Condition (3.2) then shows that $y_{i,j+r}=0$, $i=1,2,\ldots,r$, when $j\geqslant M$, $y_{1,k}=0$, $k=n-r+1,\ldots,n$, since $n-r+1>M+1$ and $y_{1j}=0$ when $j\geqslant M+1$. Thus (3.3) gives

$$y_{r+1,j+r}=0, \qquad j=M+1,\ldots,n.$$

If this analysis is continued similarly for the $(r+2)$th row and subsequent rows, it follows that the trapezium of zeros given in (3.2) implies a triangle of zeros in the top-right-hand corner of Y, which satisfies (3.1).

To prove the second part of the theorem we note that (3.2) is now satisfied without the restrictions $i\leqslant r$ and $j\leqslant r$ so that from (2.5)

$$y_{i+r,j+r}=y_{ij} \tag{3.4}$$

and

$$\text{either } y_{i+r,r}=y_{n-r+1,j}=0 \text{ or } y_{r,j+r}=y_{i,n-r+1}=0 \tag{3.5}$$

which hold when

$$\text{either} \quad i \geqslant N, j \leqslant n-N-r+1 \quad \text{or} \quad j \geqslant M, i \leqslant n-M-r+1. \quad (3.6)$$

Thus the rT part of Y consists of the elements y_{ij} which satisfy

$$\text{either} \quad i \geqslant N, j \leqslant n-N+1 \quad \text{or} \quad j \geqslant M, i \leqslant n-M+1$$

and the result then follows. ∎

COROLLARY. *If A is a nonsingular upper (lower) triangular rT matrix then so is its inverse.*

Proof. If A is upper triangular, so is its inverse and so $N=1$ in (3.2). The excluded submatrices in Theorem 3.1 therefore disappear. When A is lower triangular, $M=1$ and the result follows similarly. ∎

If A is a block Toeplitz matrix then it is natural to investigate the case when its inverse Y is block banded.

DEFINITION 3.2. If $A=[A_{ij}]$ is a block matrix of order nr where A_{ij}, $i, j=1, 2, \ldots, n$, is a matrix of order r, then A is *block banded* if for some positive integers r and s, less than $n+1$,

$$A_{ij}=0, \qquad j-i \geqslant r, \; i-j \geqslant s.$$

A similar form of banding is now considered for rT matrices.

DEFINITION 3.3. A matrix L of order $n=mr+p$, $0 \leqslant p < r$ is an *incomplete block matrix* if it is partitioned from the top-left-hand corner using $r \times r$ submatrices as far as possible. Thus

$$V=\begin{bmatrix} V_{11} & \cdots & V_{1m} & V_{1,m+1} \\ \vdots & & \vdots & \vdots \\ V_{m1} & \cdots & V_{mm} & V_{m,m+1} \\ V_{m+1,1} & \cdots & V_{m+1,m} & V_{m+1,m+1} \end{bmatrix}, \tag{3.7}$$

where V_{ij}, $i, j=1, 2, \ldots, m$, is an $r \times r$ matrix, $V_{i,m+1}$ and $V^{\mathrm{T}}_{m+1,j}$, $i, j=1, 2, \ldots, m$, are $r \times p$ matrices and $V_{m+1,m+1}$ is a $p \times p$ matrix.

The following definition is now a natural extension of Definition 3.2.

DEFINITION 3.4. V has *incomplete r-block banding* if for some positive integers M and N ($M, N \leqslant m+1$, $p>0$ or $M, N \leqslant m$, $p=0$),

$$V_{ij}=0 \quad \begin{cases} j-i \geqslant M \\ i-j \geqslant N. \end{cases} \tag{3.8}$$

In particular V is a *lower incomplete block triangular* matrix if $M=1$. If $N=1$, then V is an *upper incomplete block triangular* matrix.

If Y is now partitioned in the manner of (3.7), then the following result can be proved in a similar manner to Theorem 3.2.

Details of this proof can be found in [4].

THEOREM 3.2. *If $Y=[Y_{ij}]$ is the inverse of an rT matrix of order $n=mr+p$, satisfying the conditions of Theorem 2.2 and, in addition, for positive integers M and N ($M, N \leqslant m+1$, $p>0$ or $M, N \leqslant m$, $p=0$)*

$$Y_{1j}=0,\ j \geqslant M+1 \qquad \text{and} \qquad Y_{i1}=0,\ i \geqslant N+1$$

then Y has incomplete r-block banding, and is rT except for an $(N-1)\,r \times (M-1)\,r$ submatrix of elements y_{ij} in the top-left-hand corner and an $[(M-1)\,r+p] \times [(N-1)\,r+p]$ submatrix of elements y_{ij} in the bottom-right-hand corner.

Remark 3.1. If $n=mr$ then Y is the inverse of a block Toeplitz matrix. If it also satisfies the conditions of Theorem 3.2 then Y is also a block Toeplitz matrix except for an $(N-1)\times(M-1)$ submatrix of matrix elements Y_{ij} in the top-left-hand corner and an $(M-1)\times(N-1)$ submatrix of matrix elements Y_{ij} in the bottom-right-hand corner.

Remark 3.2. When $r=1$, both Theorems 3.1 and 3.2 reduce to the Toeplitz case. The second part of the Toeplitz theorem was proved in a different way in [10].

REFERENCES

1. S. BARNETT AND M. J. C. GOVER, Some extensions of Hankel and Toeplitz matrices, *Linear Multilinear Algebra* **14** (1983), 45–65.
2. W. W. BARRETT, Toeplitz matrices with banded inverses, *Linear Algebra Appl.* **57** (1984), 131–145.
3. S. N. CHANDLER-WILDE AND D. HOTHERSALL, Private communication, University of Bradford.
4. M. J. C. GOVER AND S. BARNETT, "An Extension of Block Toeplitz Matrices, Part I, Characterisation and Displacement rank," Mathematical Sciences Report, TAM 83–13, University of Bradford, 1983.
5. M. J. C. GOVER AND S. BARNETT, Inversion of certain extensions of Toeplitz matrices, *J. Math. Anal. Appl.* **100** (1984), 339–353.
6. M. J. C. GOVER AND S. BARNETT, Generating polynomials for matrices with Toeplitz or conjugate-Toeplitz inverses, *Linear Algebra Appl.* **61** (1984), 253–275.
7. M. J. C. GOVER AND S. BARNETT, Inversion of Toeplitz matrices which are not strongly nonsingular, *IMA J. Numer. Anal.* **5** (1985), 101–110.
8. M. J. C. GOVER AND S. BARNETT, Displacement rank and quasi-triangular decomposition for r-Toeplitz matrices, *Linear Algebra Appl.*, to appear.

9. T. N. E. GREVILLE, "Moving-Weighted Average Smoothing Extended to the Extremities of the Data," MRC Tech. Summary Report, No. 1786, Res. Center., University of Wisconsin–Madison, 1977.
10. T. N. E. GREVILLE AND W. F. TRENCH, Band matrices with Toeplitz inverses, *Linear Algebra Appl.* **27** (1979), 199–209.
11. G. HEINIG AND K. ROST, "Algebraic Methods for Toeplitz-Like Matrices and Operators," Akademic-Verlag, Berlin, 1984.
12. W. F. TRENCH, An algorithm for the inversion of finite Toeplitz matrices, *J. SIAM* **12** (1964), 515–522.
13. W. F. TRENCH, Weighting coefficients for the prediction of stationary time series from the finite past, *SIAM J. Appl. Math.* **15** (1967) 1502–1510.
14. S. ZOHAR, Toeplitz matrix inversion: The algorithm of W. F. Trench, *J. Assoc. Comput. Mach.* **16** (1969), 592–601.