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A technique to prove parameter-uniform convergence for a singularly perturbed convection–diffusion equation

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Abstract

A priori parameter explicit bounds on the solution of singularly perturbed elliptic problems of convection–diffusion type are established. Regular exponential boundary layers can appear in the solution. These bounds on the solutions and its derivatives are obtained using a suitable decomposition of the solution into regular and layer components. By introducing extensions of the coefficients to a larger domain, artificial compatibility conditions are not imposed in the derivation of these decompositions.

Keywords: Decomposition; Regular boundary layers; Elliptic equation

1. Introduction

In this paper we discuss the derivation of a priori parameter explicit bounds on the derivatives of the solutions to linear singularly perturbed partial differential equations. These bounds are derived by decomposing the solution into a sum of components—a regular component (whose first and second partial derivatives are bounded independently of the singular perturbation parameter) and several layer components. The layer components are characterized as being the solution of the associated homogeneous differential equation and being small (relative to the perturbation parameter) in the domain except in the neighbourhood of one particular edge or corner of the domain. These decompositions were given first in [13].

After the appearance of [13], initial efforts at understanding this decomposition confined the discussion to the original domain, which led to technical difficulties with compatibility conditions [10,2] or to modifications to the decomposition [9] (where a remainder term is involved in the decomposition). In this paper, the above difficulties are avoided by including the additional idea of extending the problem domain, which was a central component in the original decomposition given in [13]. Note that here we identify sufficient compatibility for the decomposition to be valid and we do not address the question of necessary and sufficient compatibility conditions.

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The decomposition of the solution into a regular component and several layer components is related to the ideas of an asymptotic expansion [6,12], but it is not an asymptotic expansion [2]. There are no remainder terms. The nature of the layer components are identified implicitly (rather than explicitly as in an asymptotic expansion). In order to derive first order asymptotic error bounds for any proposed numerical method, all that is required in a standard stability and consistency argument are adequate parameter-explicit upper bounds on the second and third order partial derivatives of each of the components.

Both [11,7] (albeit [7] studies the case of no compatibility and the possibility of layers being generated due to discontinuities in the data at the corners) deal with constant coefficients. In the decomposition discussed here, variable coefficient problems are considered and bounds on the derivatives of the components are derived based on classical bounds on the derivatives given in [8].

The analysis here is close in character to the analysis presented in [9]. However, there are some marked differences. In [9] a decomposition of the solution to $Lu = f$ of the form

$$u = S + \varepsilon^2 R + (E_1 + E_2 + E_{12})$$

is established. The terms E_1, E_2, E_{12} corresponding to the layer components satisfy bounds of the form

$$|LE_1(x, y)| \leqslant C\varepsilon \mathrm{e}^{-\alpha_1(1-x)/\varepsilon}.$$

However, $LE_1 \neq 0$ and consequently $L(S + \varepsilon^2 R) \neq f$. In the decomposition presented here, where

$$u = v + w,$$

the layer components w satisfy the homogeneous problem $Lw = 0$ and the regular component v satisfies the differential equation $Lv = f$. In our opinion, this is a neater splitting of the components and the discrete error analysis in Section 6 is simplified with the aid of this decomposition.

Note that throughout this paper C denotes a constant independent of ε and $f^* : \Omega^* \to \mathbb{R}$ denotes an extension of the function $f : \Omega \to \mathbb{R}$ from the domain Ω to Ω^* where $\Omega \subset \Omega^*$.

2. Extensions and compatibility issues

Let D be an open set containing the closed unit square $\overline{\Omega} = [0, 1] \times [0, 1]$. Let D_1, D_2 be open sets so that $D \subset D_1 \subset D_2$. For any $f \in C^n(D)$ there exists an extension f^* of f such that

$$f^*(x, y) = f(x, y), \quad (x, y) \in \overline{\Omega},$$
$$f^*(x, y) = C, \quad (x, y) \in D_2 \backslash D_1, \quad f^* \in C^n(D_2).$$

In the subsequent sections, the data are extended in such a way that in the process of decomposing the solutions into regular and layer components the imposition of artificial compatibility conditions is minimized.

For non-negative integers k, we define the following semi-norms on $C^k(D)$, $D \subset \mathbb{R}^2$:

$$|v|_{k,D} = \sum_{i+j=k} \sup_{(x,y)\in D} \left| \frac{\partial^{i+j} v}{\partial x^i \partial y^j} \right|$$

and the related norms

$$\|v\|_{k,D} = \sum_{0 \leqslant j \leqslant k} |v|_{j,D}.$$

When the domain D is obvious we drop D from this notation and if a norm is not subscripted then it is the norm with $k = 0$. The space $C^\gamma(D)$ is the set of all functions that are Hölder continuous of degree γ with respect to the Euclidean norm $\|\cdot\|_e$. That is $f \in C^\gamma(D)$ if

$$\lceil f \rceil_{0,\gamma,D} = \sup_{\mathbf{u}\neq\mathbf{v},\ \mathbf{u},\mathbf{v}\in D} \frac{|f(\mathbf{u}) - f(\mathbf{v})|}{\|\mathbf{u} - \mathbf{v}\|_{\mathbf{e}}^{\gamma}}$$

is finite. The space $C^{n,\gamma}(D)$ is the set of all functions in $C^n(D)$ whose derivatives of order n are Hölder continuous of degree γ. That is,

$$C^{n,\gamma}(D)=\left\{z:\frac{\partial^{i+j}z}{\partial x^i\partial y^j}\in C^{\gamma}(D),\,0\leqslant i+j\leqslant n\right\}.$$

Also $\|\cdot\|_{n,\gamma}$ are the associated norms and $\lceil\cdot\rceil_{n,\gamma}$ is the associated Hölder semi-norm defined by

$$\lceil v\rceil_{n,\gamma}=\sum_{i+j=n}\left\lceil\frac{\partial^n v}{\partial x^i\partial y^j}\right\rceil_{0,\gamma},\quad \|v\|_{n,\gamma}=\sum_{0\leqslant k\leqslant n}|v|_k+\lceil v\rceil_{n,\gamma}.$$

Consider the first order problem

$$a_1r_x+a_2r_y=f,\quad (x,y)\in D=(0,L)\times(0,L), \tag{1a}$$

$$r(x,0)=g_s(x),\quad r(0,y)=g_w(y), \tag{1b}$$

$$a_1,a_2,\,f_0\in C^{n,\gamma}(\bar D),\quad a_1>0,\ a_2>0. \tag{1c}$$

Bobisud [1] gives explicit compatibility and regularity conditions so that $r\in C^2(\bar D)$ and indicates how to derive necessary and sufficient conditions so that $r\in C^n(\bar D)$. Linß and Stynes [9] extend this result to identify necessary and sufficient conditions for $r\in C^{n,\gamma}(\bar D)$. For example, if $a_1,a_2\in C^{\infty}(\bar D)$, $g_s,g_w\in C^{1,\gamma}([0,1])$, $f\in C^{1,\gamma}(\bar D)$ then $r\in C^{1,\gamma}(\bar D)$ if and only if

$$g_s(0)=g_w(0)\quad\text{and}\quad a_1(0,0)g_s'(0)+a_2(0,0)g_w'(0)=f(0,0).$$

Consider the elliptic problem

$$\varepsilon\triangle z+a_1z_x+a_2z_y=f,\quad (x,y)\in\Omega=(0,1)\times(0,1), \tag{2a}$$

$$z=0,\quad (x,y)\in\partial\Omega, \tag{2b}$$

$$a_1\geqslant\alpha_1>0,\quad a_2\geqslant\alpha_2>0,\quad (x,y)\in\overline{\Omega}. \tag{2c}$$

From Vulkov [14] and Ladyzhenskaya and Ural'tseva [8] (see also [4, Theorem 3.2]) if a_1,a_2 are smooth and $f\in C^{1,\gamma}(\overline{\Omega})$, $f(0,0)=f(1,0)=f(0,1)=f(1,1)=0$ then $z\in C^{3,\gamma}(\overline{\Omega})$. It also should be noted that for a variable coefficient problem local compatibility conditions to ensure that $z\in C^{n,\gamma}(\overline{\Omega})$ for $n>3$ are not available in general [4]. If a_1,a_2 are constant in a neighbourhood of each of the four corners, then compatibility conditions exist so that $z\in C^{n,\gamma}(\overline{\Omega})$.

Using the stretching transformations $\eta=x/\varepsilon$, $\zeta=y/\varepsilon$ the differential equation (2a) transforms to

$$\tilde z_{\eta\eta}+\tilde z_{\zeta\zeta}+\tilde a_1\tilde z_\eta+\tilde a_2\tilde z_\zeta=\varepsilon\tilde f_2,\quad (\eta,\zeta)\in\Omega_\varepsilon=(0,1/\varepsilon)\times(0,1/\varepsilon),$$

where $\tilde z(\eta,\zeta)=z(x,y)$. From Ladyzhenskaya and Ural'tseva [8, p. 110, (1.11)], we deduce that (see [9, Theorem 3.2]) the following global a priori bounds on the solution of (2)

$$|z|_0\leqslant C|f|_0, \tag{3a}$$

$$|z|_1+\varepsilon^{\gamma}\lceil z\rceil_{1,\gamma}\leqslant C\varepsilon^{-1}\{\varepsilon|f|_0+\varepsilon^{1+\gamma}\lceil f\rceil_{0,\gamma}+|z|_0\}, \tag{3b}$$

$$|z|_2+\varepsilon^{\gamma}\lceil z\rceil_{2,\gamma}\leqslant C\varepsilon^{-2}\{\varepsilon|f|_0+\varepsilon^{1+\gamma}\lceil f\rceil_{0,\gamma}+|z|_0\}, \tag{3c}$$

$$|z|_3+\varepsilon^{\gamma}\lceil z\rceil_{3,\gamma}\leqslant C\varepsilon^{-3}\{\varepsilon|f|_0+\varepsilon^2|f|_1+\varepsilon^{2+\gamma}\lceil f\rceil_{1,\gamma}+|z|_0\}. \tag{3d}$$

Using these bounds we derive the basic parameter-explicit derivative estimates on the solutions of (2)

$$\left\|\frac{\partial^{i+j}z}{\partial x^i\partial y^j}\right\|\leqslant C\varepsilon^{-i-j},\quad 0\leqslant i+j\leqslant 3, \tag{4}$$

when $z \in C^{3,\gamma}(\overline{\Omega})$. In the case of non-zero boundary conditions, where $u = g \neq 0$, $(x, y) \in \partial\Omega$ then the above bounds on the solutions of (2) have the additional terms (see [9, Theorem 3.2]) on the right-hand side of the respective bounds in (3)

$$|z|_0 \leqslant C(|f|_0 + |g|_0),$$

$$|z|_1 + \varepsilon^\gamma \lceil z \rceil_{1,\gamma} \leqslant C\varepsilon^{-1} \left(\varepsilon(|f|_0 + \varepsilon^\gamma \lceil f \rceil_{0,\gamma}) + |z|_0 + \sum_{i=0}^{2} \varepsilon^i |g|_i + \varepsilon^{2+\gamma} |g|_{2,\gamma} \right), \tag{5a}$$

$$|z|_2 + \varepsilon^\gamma \lceil z \rceil_{2,\gamma} \leqslant C\varepsilon^{-2} \left(\varepsilon|f|_0 + \varepsilon^{1+\gamma} \lceil f \rceil_{0,\gamma} + |z|_0 + \sum_{i=0}^{2} \varepsilon^i |g|_i + \varepsilon^{2+\gamma} |g|_{2,\gamma} \right), \tag{5b}$$

$$|z|_3 + \varepsilon^\gamma \lceil z \rceil_{3,\gamma} \leqslant C\varepsilon^{-3} \left(\varepsilon|f|_0 + \varepsilon^2 |f|_1 + \varepsilon^{2+\gamma} \lceil f \rceil_{1,\gamma} + |z|_0 + \sum_{i=0}^{3} \varepsilon^i |g|_i + \varepsilon^{3+\gamma} |g|_{3,\gamma} \right). \tag{5c}$$

3. Problem class with no parabolic boundary layers

In this section, we discuss the decomposition ideas in relation to the singularly perturbed elliptic problem

$$Lu \equiv \varepsilon \triangle u + a_1 u_x + a_2 u_y = f, \quad (x, y) \in \Omega, \tag{6a}$$

$$u = g, \quad (x, y) \in \partial\Omega, \tag{6b}$$

$$a_1 > \alpha_1 > 0, \quad a_2 > \alpha_2 > 0, \quad a_1, a_2, f \in C^{5,\gamma}(D), \quad \overline{\Omega} \subset D, \tag{6c}$$

where no parabolic boundary layers occur in the solution. The data a_1, a_2, f, g are assumed to be sufficiently regular and sufficiently compatible at the four corners, so that only exponential boundary layers appear near the outflow edges $x = 0$, $y = 0$ and a simple corner layer appears in the vicinity of $(0, 0)$. This corner layer is induced not by any lack of sufficient compatibility, but by the presence of the singular perturbation parameter. In this case, there is no loss in generality in dealing with homogeneous boundary data. That is

$$g \equiv 0. \tag{6d}$$

Below we require the following compatibility:

$$f(1, 0) = f(0, 1) = f(0, 0) = f(1, 1) = 0, \tag{6e}$$

$$\frac{\partial^k f}{\partial x^i \partial y^j}(1, 1) = 0, \quad 0 \leqslant k \leqslant 4. \tag{6f}$$

From the previous section, we note that $u \in C^{3,\gamma}(\overline{\Omega})$ and

$$\left\| \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \right\| \leqslant C\varepsilon^{-i-j}, \quad 0 \leqslant i + j \leqslant 3. \tag{7}$$

More informative bounds on how the parameter ε effects the solution u locally can be derived by decomposing the solution of (6) into the sum

$$u = v + w_{\mathrm{L}} + w_{\mathrm{B}} + w_{\mathrm{BL}}.$$

Here v is the regular component, w_{L} is a regular boundary layer function associated with the left edge $x = 0$, w_{B} is a regular boundary layer function associated with the bottom edge $y = 0$ and w_{BL} is a corner layer function associated with the corner $(0, 0)$. The decomposition into regular and layer components will be defined so that each of the layer functions satisfy the homogeneous differential equation $Lw = 0$.

In the case of the ordinary differential equation,

$$\varepsilon y'' + ay' = f(x), \quad x \in (0, 1), \quad a > 0,$$

many publications (e.g., [10,3]) begin the standard decomposition by defining the regular component $s(x)$ to be

$$s(x) = s_0(x) + \varepsilon s_1(x) + \varepsilon^2 s_2(x),$$

where

$$as_0' = f, \quad s_0(1) = y(1),$$
$$as_1' = -s_0'', \quad s_1(1) = 0,$$
$$\varepsilon s_2'' + as_2' = -s_1'', \quad s_2(0) = s_2(1) = 0.$$

Hence

$$\varepsilon s'' + as' = f, \quad s(0) = s_0(0) + \varepsilon s_1(0), \quad s(1) = y(1)$$

and $|s|_i \leqslant C(1 + \varepsilon^{2-i}),\ i \leqslant 3$. In this paper, we will construct the regular component in a different way. The functions a, f are smoothly extended to be functions a^*, f^* defined on the interval $[-d, 1],\ d > 0$. The regular component is first defined on the extended domain $[-d, 1]$ as follows. Let

$$v^* = v_0^* + \varepsilon v_1^* + \varepsilon^2 v_2^*,$$

where

$$a^*(v_0^*)' = f^*, \quad v_0^*(1) = u(1),$$
$$a^*(v_1^*)' = -(v_0^*)_0'', \quad v_1^*(1) = 0,$$
$$\varepsilon(v_2^*)'' + a(v_2^*)' = -(v_1^*)'', \quad v_2^*(-d) = v_2^*(1) = 0.$$

The regular component is taken to be the restriction of v^* to the original domain [0, 1]. Hence

$$\varepsilon v'' + av' = f, \quad v(0) = v^*(0), \quad v(1) = u(1).$$

Note that $v^*(0) = s_0(0) + \varepsilon s_1(0) + \mathcal{O}(\varepsilon^2)$ and so in the case of the ordinary differential equation the difference between v and s is marginal. However, in the construction of the regular component v there is freedom in the choice of the parameter d and in the choice of the extensions. In the following sections, we see that this additional freedom is useful when dealing with singularly perturbed partial differential equations.

4. Regular component

Define the extended domain $\Omega^* = (-d_1, 1) \times (-d_2, 1),\ d_1, d_2 > 0$. Since $u = 0$ on the boundary $\partial\Omega$, the reduced solution v_0^* is the solution of the first order problem

$$a_1^* \frac{\partial v_0^*}{\partial x} + a_2^* \frac{\partial v_0^*}{\partial y} = f^*, \quad (x, y) \in [-d_1, 1) \times [-d_2, 1),$$
$$v_0^*(1, y) = 0, \quad v_0^*(x, 1) = 0.$$

A first order correction to this reduced solution is the solution of the first order problem

$$a_1^* \frac{\partial v_1^*}{\partial x} + a_2^* \frac{\partial v_1^*}{\partial y} = -\triangle v_0^*, \quad (x, y) \in [-d_1, 1) \times [-d_2, 1),$$
$$v_1^*(1, y) = v_1^*(x, 1) = 0$$

and a second order correction is the solution of the elliptic problem

$$\varepsilon \triangle v_2^* + a_1^*(v_2^*)_x + a_2^*(v_2^*)_y = -\triangle v_1^*, \quad (x, y) \in \Omega^*,$$
$$v_2^* = 0, \quad (x, y) \in \partial\Omega^*.$$

The regular component v^* on the extended domain Ω^* is taken to be

$$v^* = v_0^* + \varepsilon v_1^* + \varepsilon^2 v_2^*.$$

The extensions are constructed so that $f^* = 0$ and $a_1^* = a_2^* = \gamma > 0$ in $\Omega^* \backslash \{(-0.5d_1, 1] \times (-0.5d_2, 1]\}$. The extension can then be organized (e.g., let d_1 and d_2 be such that $d_2 + 0.5d_1 > 1$) so that

$$\triangle v_1^*(1, -d_2) = \triangle v_1^*(-d_1, 1) = \triangle v_1^*(-d_1, -d_2) = 0.$$

We impose compatibility conditions on f at the inflow corner $(1, 1)$ so that $\triangle v_1^* \in C^{1,\gamma}(\bar{\Omega}^*)$. Given that v_0^*, v_1^* satisfy first order problems, if we impose

$$f \in C^{5,\gamma}(\bar{\Omega}^*) \quad \text{and} \quad \frac{\partial^k f}{\partial x^i \partial y^j}(1, 1) = 0, \quad 0 \leqslant k \leqslant 4,$$

then $v_0^* \in C^{5,\gamma}(\bar{\Omega}^*)$, $v_1^* \in C^{3,\gamma}(\bar{\Omega}^*)$ (see e.g., [9]), $\triangle v_1^*(1, 1) = 0$ and $\triangle v_1^*$ is zero at the other three corners of the domain $\bar{\Omega}^*$. Hence $v_2^* \in C^{3,\gamma}(\bar{\Omega}^*)$. The freedom introduced by the extension has allowed us deal with compatibility at the artificial corners $(-d_1, 1)$, $(-d_2, 1)$, $(-d_1, -d_2)$. Define the regular component v to be the restriction of v^* to the closed domain $\bar{\Omega}$. Hence the regular component $v \in C^{3,\gamma}(\bar{\Omega})$ is the solution of

$$\varepsilon \triangle v + a_1 v_x + a_2 v_y = f, \quad (x, y) \in \Omega, \tag{8a}$$

$$v = v^*, \quad (x, y) \in \partial\Omega. \tag{8b}$$

Moreover v_0, v_1 do not depend on ε and from (3) we have

$$|v_2|_i + \varepsilon^\gamma \lceil v_2 \rceil_{i,\gamma} \leqslant C\varepsilon^{-i}, \quad 1 \leqslant i \leqslant 3.$$

Hence, we have the following bounds on the regular component:

$$|v| \leqslant C, \quad |v|_i + \varepsilon^\gamma \lceil v \rceil_{i,\gamma} \leqslant C(1 + \varepsilon^{2-i}), \quad 1 \leqslant i \leqslant 3. \tag{9}$$

5. Layer components

Let us now define the layer component w_{L} associated with the left edge $x = 0$. Define the extended domain $\Omega^{**} = (0, 1) \times (-d, 1)$, $d > 0$. The extended regular layer component w_{L}^* is defined to be the solution of the problem

$$L^{**} w_{\mathrm{L}}^* = 0, \quad (x, y) \in \Omega^{**}, \tag{10a}$$

$$w_{\mathrm{L}}^*(0, y) = (u - v)^*(0, y), \quad w_{\mathrm{L}}^*(1, y) = w_{\mathrm{L}}^*(x, -d) = w_{\mathrm{L}}^*(x, 1) = 0. \tag{10b}$$

The extension of the boundary value $(u - v)^*(0, y)$ is defined in such a way that $(u - v)^*(0, y) = 0$, $y < -d/2$. Note also that $u \in C^{3,\gamma}(\bar{\Omega})$ and $v \in C^{3,\gamma}(\bar{\Omega})$. Hence $u - v$ and 0 are compatible at the corner $(0, 1)$. Therefore, the extension can be arranged so that $w_{\mathrm{L}}^* \in C^{3,\gamma}(\bar{\Omega}^{**})$. Using a maximum principle we deduce that

$$|w_{\mathrm{L}}^*(x, y)| \leqslant C\mathrm{e}^{-\alpha_1 x/\varepsilon}, \quad (x, y) \in \Omega^{**} \tag{11a}$$

and from (5) and (9) it follows that at all points in the domain $\bar{\Omega}^{**}$

$$|w_{\mathrm{L}}^*|_i + \varepsilon^\gamma \lceil w_{\mathrm{L}}^* \rceil_{i,\gamma} \leqslant C\varepsilon^{-i}, \quad i = 1, 2, 3. \tag{11b}$$

Sharper bounds on the derivatives of w_{L}^* in the direction parallel to the side $x = 0$ will be required in the discrete error analysis. To obtain these bounds, we introduce the following expansion of w_{L}^*:

$$w_{\mathrm{L}}^*(x, y) = (u - v)^*(0, y)\phi(x, y) + \varepsilon z_{\mathrm{L}}(x, y), \tag{12}$$

where for all $y \in [-d, 1]$ the function ϕ is taken to be

$$\phi(x, y) = \frac{\mathrm{e}^{-a_1^*(0,y)x/\varepsilon} - \mathrm{e}^{-a_1^*(0,y)/\varepsilon}}{1 - \mathrm{e}^{-a_1^*(0,y)/\varepsilon}},$$

which is the solution of the boundary value problem

$$\varepsilon\phi_{xx} + a_1^*(0, y)\phi_x = 0, \quad x \in (0, 1), \quad \phi(0, y) = 1, \quad \phi(1, y) = 0.$$

Note that, by using $t^n e^{-t} \leqslant C e^{-t/2}$, $n \geqslant 1$, $t \geqslant 0$, we have for all $(x, y) \in \Omega^{**}$

$$\left|\frac{\partial\phi}{\partial y}(x, y)\right| \leqslant C\left(\frac{1}{\varepsilon} e^{-a_1^*(0,y)/\varepsilon} + \frac{x}{\varepsilon} e^{-a_1^*(0,y)x/\varepsilon}\right) \leqslant C e^{-a_1^*(0,y)x/2\varepsilon},$$

$$|\phi(x, y)| \leqslant e^{-a_1^*(0,y)x/\varepsilon},$$

$$\left|\frac{\partial^2\phi}{\partial y^2}(x, y)\right| \leqslant C e^{-a_1^*(0,y)x/2\varepsilon},$$

$$\left|\frac{\partial\phi}{\partial x}(x, y)\right| \leqslant \frac{C}{\varepsilon} e^{-a_1^*(0,y)x/\varepsilon}.$$

Note that $z_L = 0$ on $\partial\Omega^{**}$ and

$$-\varepsilon L^{**} z_L = w_L^*(0, y)(a_1^*(x, y) - a_1^*(0, y))\frac{\partial\phi}{\partial x} + \left(\varepsilon\frac{\partial^2 w_L^*(0, y)}{\partial y^2} + a_2^*(x, y)\frac{\partial w_L^*(0, y)}{\partial y}\right)\phi + w_L^*(0, y)\left(\varepsilon\frac{\partial^2\phi}{\partial y^2} + a_2^*(x, y)\frac{\partial\phi}{\partial y}\right) + 2\varepsilon\frac{\partial w_L^*(0, y)}{\partial y}\frac{\partial\phi}{\partial y}.$$

Thus, using (9) and (11a)

$$|L^{**} z_L(x, y)| \leqslant \frac{C}{\varepsilon}\left(1 + \frac{x}{\varepsilon}\right) e^{-\alpha_1 x/2\varepsilon} \leqslant \frac{C}{\varepsilon} e^{-\alpha_1 x/4\varepsilon}.$$

From (3a)

$$|z_L(x, y)| \leqslant C e^{-\alpha_1 x/4\varepsilon}, \quad (x, y) \in \Omega^{**}.$$

Then using the bounds (5) and the facts that

$$\varepsilon^\gamma \lceil e^{-\alpha x/\varepsilon} \rceil_{0,\gamma} \leqslant C \quad \text{and} \quad |L^{**} z_L(x, y)|_{1,\gamma} \leqslant C\varepsilon^{-2} e^{-\alpha x/4\varepsilon}$$

(see also [9, Theorem 5.1]) we have from (5) and (9) that

$$|z_L|_k + \varepsilon^\gamma \lceil z_L \rceil_{k,\gamma} \leqslant C\varepsilon^{-k}, \quad k = 1, 2, 3.$$

The regular layer component w_L is the restriction of w_L^* to Ω and is the solution of the homogeneous problem

$$L w_L = 0, \quad (x, y) \in \Omega, \tag{13a}$$

$$w_L(0, y) = (u - v)(0, y), \quad w_L(1, y) = w_L(x, 1) = 0, \tag{13b}$$

$$w_L(x, 0) = w_L^*(x, 0). \tag{13c}$$

Hence, it follows that

$$\left\|\frac{\partial^j w_L}{\partial x^i}\right\| \leqslant C(1 + \varepsilon^{-i}), \quad i \leqslant 3, \qquad \left\|\frac{\partial^j w_L}{\partial y^j}\right\| \leqslant C(1 + \varepsilon^{1-j}), \quad j \leqslant 3. \tag{14}$$

Corresponding bounds hold for w_B, which is the boundary layer function associated with the edge $y = 0$.

Finally, we consider the corner layer function, which is defined on the original domain as follows:

$$L w_{BL} = 0, \quad (x, y) \in \Omega, \tag{15a}$$

$$w_{BL}(x, 0) = -w_L(x, 0), \quad w_{BL}(0, y) = -w_B(0, y), \tag{15b}$$

$$w_{BL}(1, y) = 0, \quad w_{BL}(x, 1) = 0. \tag{15c}$$

Recall that $u-v, w_{\mathrm{L}}, w_{\mathrm{B}} \in C^{3,\gamma}(\bar{\Omega})$ and $L(u-v)=Lw_{\mathrm{L}}=Lw_{\mathrm{B}}=0$. Also note that w_{B} is compatible with $u-v$ at the corner $(0,0)$ and, in turn, $u-v$ is compatible with w_{L} at the corner $(0,0)$. Hence the corner layer function $w_{\mathrm{BL}} \in C^{3,\gamma}(\bar{\Omega})$. From the comparison principle and the bounds on w_{L} and w_{B} established above, we have that

$$|w_{\mathrm{BL}}(x,y)| \leqslant C\mathrm{e}^{-\alpha_1 x/\varepsilon}\mathrm{e}^{-\alpha_2 y/\varepsilon}, \quad (x,y) \in \Omega. \tag{16a}$$

Using the bounds given in (5) and (11b) we deduce that

$$|w_{\mathrm{BL}}|_i \leqslant C\varepsilon^{-i}, \quad i=1,2,3. \tag{16b}$$

Hence for all three layer functions $w_{\mathrm{L}}, w_{\mathrm{B}}, w_{\mathrm{BL}}$ the bounds on the derivatives given in (4) are applicable.

6. Numerical method

The bounds (9), (11), (14), (16) and (7) suffice to establish first order convergence (up to logarithmic factors) for standard upwinding on the standard piecewise-uniform mesh when applied to the elliptic problem (6).

We present error bounds for the approximations generated by using a standard upwind difference operator

$$L^N U = \varepsilon(\delta_x^2 + \delta_y^2)U + a_1 D_x^+ U + a_2 D_y^+ U = f, \quad (x_i, y_j) \in \Omega^N \tag{17a}$$

on a mesh

$$\Omega^N = \omega_x \times \omega_y, \tag{17b}$$

which is a tensor product of two piecewise-uniform one-dimensional meshes ω_x, ω_y. The finite difference operators D_x^+ and δ_x^2 are the standard first order forward difference and the second order centered difference on a non-uniform mesh [3]. Here the mesh ω_x places $N/2$ mesh intervals into both $[0,\sigma_x]$ and $[\sigma_x, 1]$. The mesh points in the y direction are distributed in the same fashion. The transition parameters are taken to be

$$\sigma_x = \min\left\{0.5, \frac{\varepsilon}{\alpha_1}\ln N\right\} \quad \text{and} \quad \sigma_y = \min\left\{0.5, \frac{\varepsilon}{\alpha_2}\ln N\right\}. \tag{17c}$$

The discrete solution is decomposed in an analogous fashion to the continuous solution. That is

$$U = V + W_{\mathrm{L}} + W_{\mathrm{B}} + W_{\mathrm{BL}},$$

where

$$L^N V = f, \quad L^N W = 0, \quad (x_i, y_j) \in \Omega^N \quad \text{and} \quad V = v, \quad W = w, \quad (x_i, y_j) \in \partial\Omega^N.$$

On an arbitrary mesh using the bounds (9) one has the truncation error estimate

$$|L^N(v-V)| \leqslant CN^{-1}\left(\varepsilon\left\|\frac{\partial^3 v}{\partial x^3}\right\| + \varepsilon\left\|\frac{\partial^3 v}{\partial y^3}\right\| + \left\|\frac{\partial^2 v}{\partial x^2}\right\| + \left\|\frac{\partial^2 v}{\partial y^2}\right\|\right) \leqslant CN^{-1}$$

and hence $\|v-V\|_{\overline{\Omega}^N} \leqslant CN^{-1}$. If $\sigma_x = 0.5$ or $\sigma_y = 0.5$ then the same argument coupled with the bounds (7) yields

$$\|u-U\|_{\overline{\Omega}^N} \leqslant CN^{-1}(\ln N)^2 \quad \text{if } \sigma_x = 0.5 \text{ or } \sigma_y = 0.5.$$

In the remainder of this section we assume that

$$\sigma_x = \frac{\varepsilon}{\alpha_1}\ln N \quad \text{and} \quad \sigma_y = \frac{\varepsilon}{\alpha_2}\ln N.$$

Note that from (11a) and by the choice of the transition point σ_x

$$|w_{\mathrm{L}}(x,y)| \leqslant CN^{-1}, \quad x \geqslant \sigma_x.$$

Consider the discrete one-dimensional barrier function $\Psi(x_i)$ which is the solution of the constant coefficient difference equation

$$\varepsilon\delta_x^2\Psi(x_i) + \alpha D_x^+\Psi(x_i) = 0, \quad x_i \in \omega_x, \quad \Psi(0) = 1, \quad \Psi(1) = 0.$$

Note that by (11a)

$$|W_{\mathrm{L}}(x_i,0)| = |w_{\mathrm{L}}(x_i,0)| \leqslant C\mathrm{e}^{-\alpha_1 x_i/\varepsilon} \leqslant C\Psi(x_i) + CN^{-1}.$$

Hence by using a discrete comparison principle and the fact that $L^N\Psi = (a_1-\alpha_1)D^+\Psi \leqslant 0$ we conclude that

$$|W_{\mathrm{L}}(x_i,y_j)| \leqslant C\Psi(x_i) + CN^{-1}, \quad (x_i,y_j) \in \bar{\Omega}^N.$$

Thus, as in [10, Chapter 7]

$$|W_{\mathrm{L}}(x_i,y_j)| \leqslant CN^{-1}, \quad x_i \geqslant \sigma_x.$$

Thus, we have the following bound on the error

$$|(W_{\mathrm{L}}-w_{\mathrm{L}})(x_i,y_j) \leqslant CN^{-1}, \quad x_i \geqslant \sigma_x.$$

In the fine mesh region $(0,\sigma_x)\times(0,1)$, using the bounds (14) we have the truncation error bound

$$\begin{aligned}|L^N(w_{\mathrm{L}}-W_{\mathrm{L}})| &\leqslant Ch\left(\varepsilon\left\|\frac{\partial^3 w_{\mathrm{L}}}{\partial x^3}\right\| + \left\|\frac{\partial^2 w_{\mathrm{L}}}{\partial x^2}\right\|\right) + CN^{-1}\left(\varepsilon\left\|\frac{\partial^3 w_{\mathrm{L}}}{\partial y^3}\right\| + \left\|\frac{\partial^2 w_{\mathrm{L}}}{\partial y^2}\right\|\right)\\ &\leqslant C\,\frac{N^{-1}\ln N}{\varepsilon},\end{aligned}$$

where $Nh = 2\sigma_x$. Use the barrier function

$$C\,\frac{N^{-1}\ln N}{\varepsilon}(\sigma_x - x_i) + CN^{-1}$$

in the region $(0,\sigma_x)\times(0,1)$ to derive the error bound

$$\|W_{\mathrm{L}} - w_{\mathrm{L}}\|_{\Omega^N} \leqslant CN^{-1}(\ln N)^2.$$

Note that

$$|W_{\mathrm{BL}}(x_i,y_j)| \leqslant C\Psi(x_i) + CN^{-1}, \quad (x_i,y_j)\in\bar{\Omega}^N,$$

which implies that

$$|W_{\mathrm{BL}}(x_i,y_j)| \leqslant CN^{-1}, \quad x_i \geqslant \sigma_x.$$

In analogous fashion we have that

$$|W_{\mathrm{BL}}(x_i,y_j)| \leqslant CN^{-1}, \quad y_j \geqslant \sigma_y,$$

which implies that

$$|W_{\mathrm{BL}}(x_i,y_j) - w_{\mathrm{BL}}(x_i,y_j)| \leqslant CN^{-1}, \quad (x_i,y_j)\notin(0,\sigma_x)\times(0,\sigma_y).$$

In the fine mesh corner region $(0,\sigma_x)\times(0,\sigma_y)$, use the bounds given in (16b) to obtain

$$|L^N(w_{\mathrm{BL}} - W_{\mathrm{BL}})| \leqslant C\,\frac{N^{-1}\ln N}{\varepsilon}.$$

Use the barrier function

$$C\,\frac{N^{-1}\ln N}{\varepsilon}(\sigma_x - x_i) + CN^{-1}$$

in the corner region to derive the error bound

$$\|w_{\mathrm{BL}} - W_{\mathrm{BL}}\|_{\Omega^N} \leqslant CN^{-1}(\ln N)^2.$$

Collecting all the error bounds on the components and we have the final error bound:

Theorem 1. *If U is the solution of* (17) *and u is the solution of the elliptic problem* (6) *then*

$$\|U - u\|_{\Omega^N} \leqslant CN^{-1}(\ln N)^2.$$

Numerical results illustrating the performance of this numerical method are given in [3,5].

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