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Voting Fairly: Transitive Maximal Intersecting Families of Sets[1]

Daniel E. Loeb[2]

Daniel H. Wagner, Associates, 40 Lloyd Avenue, Suite 200, Malvern, Pennsylvania 19355
E-mail: loeb@pa.wagner.com
http://dept-info.labri.u-bordeaux.fr/~loeb/index.html

and

Andrew R. Conway[3]

Silicon Genetics, 2601 Spring Street, Redwood City, California 94063

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DEDICATED TO THE MEMORY OF GIAN-CARLO ROTA

There are several applications of maximal intersecting families (MIFs) and different notions of fairness. We survey known results regarding the enumeration of MIFs, and we conclude the enumeration of the 207,650,662,008 maximal families of intersecting subsets of X whose group of symmetries is transitive for $|X| < 13$. © 2000 Academic Press

Key Words: homogeneous games; fair; transitive; regular; coterie; maximal intersecting family; strong simple game; voting scheme; ipsodual element of free distributive lattice; self-dual monotone Boolean function; self-dual anti-chain; critical tripartite hypergraph.

Contents.

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1. INTRODUCTION

> [This paper] attempts to promote better communication and less duplication of mathematical effort by identifying and describing several other theories, formally equivalent ... that are founded in fields ranging from sociology to electrical engineering.
>
> —Dubey and Shapley [7]

How large can a collection of pairwise intersecting subsets of a given n-element set X be? It is easy to see not only that any intersecting family contains at most 2^{n-1} sets, but furthermore that any intersecting family can be extended to a maximal intersecting family containing exactly 2^{n-1} sets [1, Theorem 1.1.1].

The number a_n of maximal intersecting families (MIFs) on X has been found to grow quite quickly as $n=|X|$ increases. (See Table I.) Korshunov

TABLE I

Number a_n of Maximal Intersecting Families on a Set of Cardinality n

n	a_n	Notes
0	0	
1	1	
2	2	
3	4	
4	12	Dedekind 1897 [6]
5	81	Von Neumann 1944 [6]
6	2 646	Gurk and Isbell 1959 [9]
7	1 422 564	Loeb 1992 [16] and Bioch and Ibarki 1994 [3]
8	229 809 982 112	Conway and Loeb (12 min. computer calculation, Nov. 30 (1995) and Bioch and de Boer (6 month computer calculation, Jun–Dec. 1995)

proved the following asymptotic formulas for a_n depending on the parity of n.

$$a_n \sim 2^{\binom{n-1}{\lfloor n/2 \rfloor}} \exp(e_n),$$

where

$$e_n = \begin{cases} \binom{n-1}{(n-1)/2} (2^{-(n-1)/2} + 3n^2 \cdot 2^{-n-4} - n \cdot 2^{-n-2}) \\ \qquad + \binom{n-1}{(n+3)/2} (2^{-(n+3)/2} + n^2 \cdot 2^{-n-6} - n \cdot 2^{-n-5}) & \text{for } n \text{ odd} \\ \text{and} \\ \binom{n-1}{n/2-1} (2^{-n/2-1} + n \cdot 2^{-n-4}) \\ \qquad + \binom{n}{n/2+1} (2^{-n/2-1} + n^2 \cdot 2^{-n-5} - n \cdot 2^{-n-4}) & \text{for } n \text{ even.} \end{cases}$$

The notion of a maximal intersecting family has arisen independently in a surprisingly large number of contexts besides extremal combinatorics.

Interactive decision making. MIFs are known as *strong simple games* [25], and are used to model situations in every coalitions is either "all-powerful" or "ineffectual." A *game* (or *upset* or *filter*) on a set of players X is a set $\mathscr{F}$ of coalitions $A \subseteq X$ closed under inclusion. The coalition A is *winning* (resp. *losing*, *blocking*) if $A \in \mathscr{F}$ (resp. $A \notin \mathscr{F}$, $X \backslash A \notin \mathscr{F}$). The game $\mathscr{F}$ is *simple* (resp. *strong*) if winning implies blocking (resp. blocking implies winning).

Distributed computing. The set $\min(\mathscr{F})$ of minimal elements of an intersecting family $\mathscr{F}$ is called a *coterie* whereas if $\mathscr{F}$ is maximal, then $\min(\mathscr{F})$ is called a *non-dominated coterie* [8]. (Recall that a MIF is determined by its minimal elements.) They are used in mutual exclusion protocols (to limit access to a protected resource) and replication protocols (to manage a distributed memory system).

Logic or linear programming. The characteristic function $\xi(\mathscr{F})$ of a MIF $\mathscr{F}$ is a self-dual monotone boolean function. Conversely, given any self-dual monotone boolean function f, the preimage $f^{(-1)}(\text{True})$ is a MIF. (This is related to Dedekind's problem [6] of enumerating all monotone boolean functions.)

Category theory. The ipsodual elements of the free distributive lattice [24] and the elements of the free median set [21] generated by X correspond to MIF's.

Social science. Arrow's impossibility theorem [2] states that nondictatorial, unanimous social choice functions independent of irrelevant alternatives exist only when the public faces at most two choices. Using a MIF to determine winning coalitions gives an effective voting scheme when there are exactly two choices [10].

Graph theory. A coloring of a hypergraph is an assignment of colors to vertices such that each nontrivial edge contains at least two colors. The minimal sets of a MIF can be regarded as the edges of a *critical tripartite hypergraph H*. That is to say, H is 3-colorable, and if any edge is removed from H then it would be 2-colorable.

Reliability theory. Games are thought of as semi-coherent structure functions [11, 22].

...

Each rediscovery of a theory gives birth to alternate notation and terminology. An attempt has been made here to choose a consistent terminology which makes our results as clear as possible. The above references are useful in adapting our results to other fields of interest.

In Section 2, we define homomorphisms or quotients of MIFs. Voters in the same orbit of the automorphism group of a MIF can be said to play the equivalent roles when the MIF is thought of as a voting system.

This notion of equivalence is used in Section 3 as a measure of fairness. The remainder of the paper is devoted to the enumeration of MIFs whose automorphism groups act transitively on the set of voters, so that all voters play the same role.

In Section 4, we enumerate all 207,650,662,008 transitive MIFs on up to 12 voters. For completeness, we survey previous results on MIFs with up to 7 voters before giving the classification of larger MIFs. Such a list is important in applications, since the "best" transitive MIF can be selected from it, depending upon your personal criteria that define what is best [3, 8].

Finally in Section 5, we explain the search techniques used in our research. We believe that similar techniques can be helpful in the enumeration of other combinatorial objects according to their symmetries.

2. QUOTIENTS AND ISOMORPHISMS

Let $\mathscr{F}$ be a MIF on X, and let σ: $X \to Y$ be some function. It is easy to define the *quotient voting scheme* $\sigma(\mathscr{F}) = \{A \subseteq Y : \sigma^{(-1)}(A) \in \mathscr{F}\}$. Note that $\min(\sigma(\mathscr{F}))$ is equal to $\sigma(\min(\mathscr{F}))$ in the usual sense.

PROPOSITION 1. *Let $\mathscr{F}$ be a MIF on X, and let σ: $X \to Y$ be some function. Then $\sigma(\mathscr{F})$ is a MIF on Y.*

Proof. σ^{-1} is a monotone function from 2^X to 2^Y, thus $\sigma(\mathscr{F})$ is a game.
Suppose $A \in \sigma(\mathscr{F})$. Then $\sigma^{(-1)}(A) \in \mathscr{F}$. Hence, $\sigma^{(-1)}(Y-A) = X - \sigma^{(-1)}(A) \notin \mathscr{F}$. Thus, $Y - A \notin \sigma(\mathscr{F})$, so $\sigma(\mathscr{F})$ is simple.
Similarly, suppose $B \notin \sigma(\mathscr{F})$. Then $\sigma^{(-1)}(B) \notin \mathscr{F}$. Hence, $\sigma^{(-1)}(Y-A) = X - \sigma^{(-1)}(A) \in \mathscr{F}$. Thus, $Y - A \in \sigma(\mathscr{F})$, so $\sigma(\mathscr{F})$ is strong. ▮

X can be thought of as a set of *offices* and Y as a set of *voters*. σ describes which offices are held by which voters. If σ is non-surjective, then certain voters will hold no office, and thus are powerless (*dummies*). If σ is non-injective, then certain voters will combine the functions of several offices. The single vote of each such voter is then taken into account as the vote of each of his offices.

If σ is bijective, then $\mathscr{F}$ and $\sigma(\mathscr{F})$ are said to be *isomorphic*. Furthermore, if $\mathscr{F} = \sigma(\mathscr{F})$, then σ is said to be an *automorphism* of $\mathscr{F}$. An automorphism is a permutation of X taking winning sets into winning sets.

Let $\mathrm{Aut}(\mathscr{F})$ be the set of automorphisms of $\mathscr{F}$.

THEOREM 2. *Let $\mathscr{F}$ be a MIF on X. Then* $\mathrm{Aut}(\mathscr{F})$ *is a permutation group of X.*

Proof. Observe that $\tau(\sigma(\mathscr{F})) = (\tau \circ \sigma)(\mathscr{F})$. Thus, the composition of two automorphisms or the inverse of an automorphism is again an automorphism. ▮

A permutation group containing only one permutation (the identity) is said to be *trivial*. For large n, most MIFs have trivial automorphism groups.

THEOREM 3. *Let b_n be the number of MIFs with trivial automorphism groups on an n-element set. ($b_n/n!$ is the number of isomorphism classes of such MIFs.) Then the fraction of MIFs (resp. isomorphism classes of MIFs) with trivial automorphsim groups tends to* 1 *as n tends to infinity.*

$$\lim_{n \to \infty} \frac{b_n}{a_n} = 1, \qquad \lim_{n \to \infty} \frac{b_n/n!}{\tilde{a}_n} = 1.$$

Sketch of proof. From [15], we recall that all but a vanishingly small fraction of all MIFs have all but a vanishingly small fraction of their minimal sets of cardinality $n/2$ for n even. That is, for all $\varepsilon < 1$ there is an N such that for $n > N$, over εa_n MIFs on an n element set have over $1 - \varepsilon$ of their minimal sets of cardinality $n/2$.

The $\binom{n}{n/2}$ such sets are divided by a non-trivial permutation group G into a number of orbits not exceeding

$$\begin{aligned} c_n &= \binom{n-2}{n/2} + \binom{n-2}{(n/2)-2} + \binom{n-2}{(n/2)-1} \\ &= \binom{n}{n/2} - \binom{n-2}{(n/2)-1} \\ &= \sim \frac{3}{4}\binom{n}{n/2}. \end{aligned} \tag{1}$$

For n even, the orbits form complementary pairs (otherwise there is no MIF with automorphism group G) and we must choose one orbit from each such pair. Thus, the logarithm (base two) of the number of MIFs with automorphism group G is asymptotically bounded by $\frac{3}{8}\binom{n}{n/2}$ whereas $\log_2 a_n \sim \frac{1}{2}\binom{n}{n/2}$.

For n odd, all but a vanishingly small fraction of MIFs have all their sets of cardinality at least $i = (n-1)/2$ and at most $\binom{n}{i} 2^{-n/2}$ sets of cardinality exactly i. In fact these MIFs are uniquely determined by their sets of cardinality i. As above, the $\binom{n}{i}$ such sets are divided by a non-trivial permutation group G into at most about $\frac{3}{4}\binom{n}{i}$ orbits. Hence, $\log_2 b_n / \log_2 a_n$ is asymptotically bounded by

$$\left(\frac{3}{4}\right)^{\binom{n}{i} 2^{-n/2}} \ll 1. \quad \blacksquare$$

It is difficult to find such examples for small n. For $n \leqslant 6$, the only example is the trivial one-voter MIF. Nonetheless, already for $n = 7$, there are 498,960 MIFs with trivial automorphism groups. They can be divided into 99

TABLE II

Weighted Majority Games with Trivial Automorphism Group and up to Seven Voters

v_1	v_2	v_3	v_4	v_5	v_6	v_7	Quota	ℓ
1							1	1
8	6	5	4	3	2	1	15	155
9	7	5	4	3	2	1	16	78
8	7	6	5	4	3	2	18	299

Note. v_1: Number of votes to be cast by player i; Quota: Number of votes needed to win; ℓ: Line number in [13, Tables 2, 3].

automorphism classes and constitute over 35% of the seven-voter MIFs [3]. Three of the 99 classes involve weighted majority games (or threshhold functions or quota games) [13]. (See Table II and Section 4.1).)

Such MIFs are maximally unfair in the sense that no two voters play the same role. That is, each element of X is in a separate orbit under the action of the permutation group $\mathrm{Aut}(\mathscr{F})$.

Conversely, if all of the elements of X are in the same orbit, then $\mathrm{Aut}(\mathscr{F})$ is a *transitive subgroup* of $\mathrm{Sym}(X)$, and we will say that $\mathscr{F}$ is a *transitive MIF* (or *fair game* or *homogeneous game*).

3. FAIRNESS

Depending on the interpretation chosen, different measures of fairness are appropriate.

- In a democratic country, each voter should play the same role in the system of vote adopted.
- In a game, each player should have the same possibilities of winning.
- In a distributed system, load should be equally divided among all of the processors.

One might require:

- (Regularity [5].) All voters belong to the same number of winning coalitions.
- (Equal Banzhaf index [7].) All voters belong to the same number of minimal winning coalitions.
- (Equal Shapley–Shubik index [7].) All voters have an equal probability of being the *pivot* voter given a random *alignment* of the voters in order of their enthusiasm for a proposal under consideration.

However, we will retain the notion of transitivity as a measure of "fairness" since it is stricter than any of the others mentioned above.

The main result of this paper is the enumeration of all transitive MIFs for $n < 13$. Such a list is important in applications, since the "best" transitive MIF can be selected from it, depending upon your personal criteria that define what is best [3, 8].

Since a permutation group on a set X is defined to be *k-transitive* ($k \leqslant n$) if it acts transitively on the set of k-tuples of distinct elements of X, we can go further and discuss k-transitive MIFs. Presumably, a k-transitive MIF is somehow more "fair" than a 1-transitive MIF, since it does not distinguish among k-tuples of players.

However, the natural object of study is not k-tuples of players, but rather *coalitions*, that is, unordered sets of players. We will therefore define a permutation group on X to be *k-homogeneous* ($k \leqslant n$) if it acts transitively on the set $\binom{X}{j} = \{A \subseteq X : |A| = j\}$ of j element subsets of X for each j $(0 \leqslant j \leqslant k)$.

PROPOSITION 4. *Let G be a permutation group on X. ($|X| = n$.)*

1. *If G is k-transitive, then it is also k-homogeneous.*
2. *If G is $\lfloor \frac{n}{2} \rfloor$-homogeneous, then G is n-homogeneous.*

Proof. 1. Let $(x_1, ..., x_k)$ and $(y_1, ..., y_k)$ be k-tuples of distinct elements of X. By hypothesis, there exists $\sigma \in G$ such that $\sigma x_i = y_i$. Thus, $\sigma\{x_1, ..., x_k\} = \sigma\{y_1, ..., y_k\}$.

2. If $\sigma A = B$, then $\sigma(X - A) = X - B$. Thus, if G acts transitively on $\binom{X}{j}$, then it also acts transitively on $\binom{X}{n-j}$. ∎

Several authors have studied the set A of numbers of voters n such that there exists a transitive MIF. One might think that there is no such game having an even number of players; however, see below for explicit examples in the cases $n = 6$ (Section 4.2) and $n = 10$ (Section 4.5).

See Table III for a list of (possible) non-elements of A. The first few values of n whose membership in A is still in doubt are 40, 72, 80, and 88.

TABLE III

Values of n for Which There Are (or Might Be) No Transitive MIF on an n-Element Set. ($n = c2^k$ with c odd.)

c	$2c$	$4c$	$8c$	$16c$	$32c$	$64c$	$\cdots$	$c2^k$
1	2	4	8	16	32	64	$\cdots$	2^k for all $k \geqslant 1$
3		12	24	48	96	192	$\cdots$	$3 \cdot 2^k$ for all $k \geqslant 2$
5			40?	80?	160?	320?	$\cdots$	
7					224?	448?	$\cdots$	
9			72?	144?	288?	576?	$\cdots$	
11			88?	176?	352?	704?	$\cdots$	
13			104?	208?	416?	832?	$\cdots$	
15					480?	960?	$\cdots$	
17			136?	272?	544?	1088?	$\cdots$	
19			152?	304?	608?	1216?	$\cdots$	
21						1344?	$\cdots$	
23				368?	736?	1472?	$\cdots$	
25					800?	1600?	$\cdots$	
27				432?	864?	1728?	$\cdots$	
29			232?	464?	928?	1856?	$\cdots$	
31						15872?	$\cdots$	

THEOREM 5. 1. [14, I: Lemma 1] *$n \in A$ if and only if there is a transitive permutation group of degree n containing no fixed-point free 2-element.*

2. [5] *A is multiplicatively closed.*

3. [5] *A contains all non-multiples of* 8 (*with the exception of* 2, 4, *and* 12)

4. [5] *A has density* 1.

4. EXAMPLES OF TRANSITIVE MIFs

4.1. *Democracy*

One of the simplest ways to define a MIF is to attribute weights to each of the voters $w\colon X \to \mathbf{N}$. A coalition wins if its total weight is greater than the total weight of its complement.

$$A \in S_w \qquad \text{if and only if} \quad \sum_{a \in A} w(a) > \tfrac{1}{2} \sum_{v \in V} w(v).$$

To enforce duality, the total weight can be taken to be odd.

For n odd, we have the true democracy

$$\begin{aligned} Dem_n &= [\underbrace{1\ \ 1\ \ 1\ \ \cdots\ \ 1}_{n}]_{(n+1)/2} \\ &= \{A \subseteq X : |A| > n/2\}. \end{aligned}$$

Note that the weights are not uniquely defined by the MIF. For example, the weights $[1, 1, 1]_2$ and $[2, 2, 1]_3$ both give the democratic voting scheme Dem_3.

Note however the following proposition.

PROPOSITION 6. *Let $w\colon V \to \mathbf{X}$ be a weight function. Then there are alternative weight functions $w'\colon V \to \mathbf{N}$ constant on all orbits of* $\mathrm{Aut}(S_w)$, *and zero on all dummies such that $S_w = S_{w'}$.*

Conversely, if $w(a) = w(b)$, then a and b lie in the same orbit of S_w, and if $w(a) = 0$, then a is a dummy.

Proof. Without loss of generality, the weight of all dummies is 0. Let $\mathrm{Aut}(S) \subseteq \mathrm{Sym}(V)$ be the automorphism group of S. Then $w'(v) = \sum_{g \in \mathrm{Aut}(S)} w(g(v))$ is the required weight function.

The converse is evident. ∎

Clearly, the democracy has the full group of symmetries $\mathrm{Aut}(Dem_n) = S_n$. Moreover, we have the following results.

TABLE IV

Transitive MIFs for $n \leqslant 13$

n	a_n	$\tilde{a}_n$	t_n	$\tilde{t}_n$	$\min(\lvert A\rvert)$
1	1	1	1	1	1
2	2	1	0	0	
3	4	2	1	1	2
4	12	3	0	0	
5	81	7	1	1	3
6	2 646	30	12	1	3
7	1 422 564	716	31	2	3–4
8	229 809 982 112		0	0	
9	$\sim 9 \times 10^{23}$		570 361	24	4–5
10	$\sim 3 \times 10^{40}$		1 441 440	28	4–5
11	$\sim 6 \times 10^{80}$		207 648 650 161	57 259	4?–6
12	$\sim 5 \times 10^{143}$		0	0	
13	$\sim 5 \times 10^{286}$				4–7

Note. a_n: Number of maximal intersecting families on an n-element set; $\tilde{a}_n$: number of isomorphism classes of maximal intersecting families on an n-element set; t_n: number of transitive maximal intersecting families on an n-element set; $\tilde{t}_n$: number of isomorphism classes of transitive maximal intersecting families on an n-element set; $\min(|A|)$: minimal number of elements in a winning coalition of a transitive MIF.

Corollary 7. *Dem_n is the only transitive strong simple weighted MIF.*

Proof. By Proposition 6, we must be able to assign the same weight to all voters. ∎

Proposition 8. *Dem_n is the only MIF whose automorphism group is $(n-1)/2$-homogeneous.*

Proof. Suppose $\operatorname{Aut}(\mathscr{F})$ is $(n-1)/2$-homogeneous. Then by Proposition 4, $\mathscr{F}$ is n-transitive. Thus, all sets of equal cardinality lie in the same orbit. Since there must be at least one winning coalition of $(n+1)/2$ elements, they are all winning. ∎

Proposition 9. *Every strong simple majority game is a quotient of Dem_n for some n.*

Proof. Let $\mathscr{F} = [v_1\, v_2 \cdots v_k]_q$ be a strong simple majority game. Consider the set $X = \{(i, j)\colon 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant v_i\}$ and the function $f\colon (i, j) \mapsto i$. Then $\mathscr{F}$ is the quotient of democracy on X with the function f. ∎

TABLE V

List of Transitive MIFs: Part I ($n \leqslant 9$)

n	ℓ	MWC	Name	t	t'	Aut($\mathscr{F}$)	Generators	#
1	1	a	$Dem_1 = Dict_1$	1	1	S_1		1
3	2	ab	Dem_3	3	3	$3T2 = S_3$	$(abc), (ab)$	1
5	7	abc	Dem_5	5	5	$5T5 = S_5$	$(abcde), (ab)$	1
6	30	abc	$Icos$	2	2	$6T12 = L(2, 5)$	$(abcde), (af)(bd)$	12
7	716	$abcd$	Dem_7	7	7	$7T5 = S_7$	$(abcdefg)(ab)$	1
7	713	abd	$Fano$	2	2	$7T7 = L(3, 2)$	$(abcdefg)(bc)(dg)$	30
Total								31

Note. n: Number of voters. $n = |X|$; ℓ: Line number in [4, Tables 2, 3]; MCW: A list of representatives of the orbits of the set of minimal winning minority coalitions under the action of Aut($\mathscr{F}$); Name: Notation used to denote $\mathscr{F}$; t: Degree of transitivity of Aut($\mathscr{F}$); t' Degree of homogeneity of Aut($\mathscr{F}$); Aut($\mathscr{F}$): Designation of the automorphism group of $\mathscr{F}$ using the notation of [4] and any common name (see Table VI); Generators: A minimal set of generators of the group Aut($\mathscr{F}$); #: The number of MIF on X which are isomorphic to $\mathscr{F}$ (only one MIF is listed for each isomorphism class).

4.2. $n = 6$: *Icosahedral MIF*

As part of the enumeration of six-player games, Gurk and Isbell [9] discovered a transitive MIF which they described by its minimal winning coalitions:

$$Icos = \{abc, acf, aef, ade, abd, bce, cef, bef, cde, bdf\}.$$

Icos is the smallest transitive MIF on an even number of voters. Note that all majority coalitions are winning and all minority coalitions are losing.

TABLE VI

Groups Appearing in Tables V, VII, VIII, and IX

S_n	Symmetric group
$L(2, q)$	Group of rational linear maps on a q-element field
$L(3, q)$	Group of invertible 3×3 matrices over a q-element field
C_n	Cyclic group
D_{2n}	Dihedral group
A_n	Alternating group
M_{11}	Matthieu group
$G \times H$	Direct product of groups G and H
$G \wr H$	Wreath product of G and H

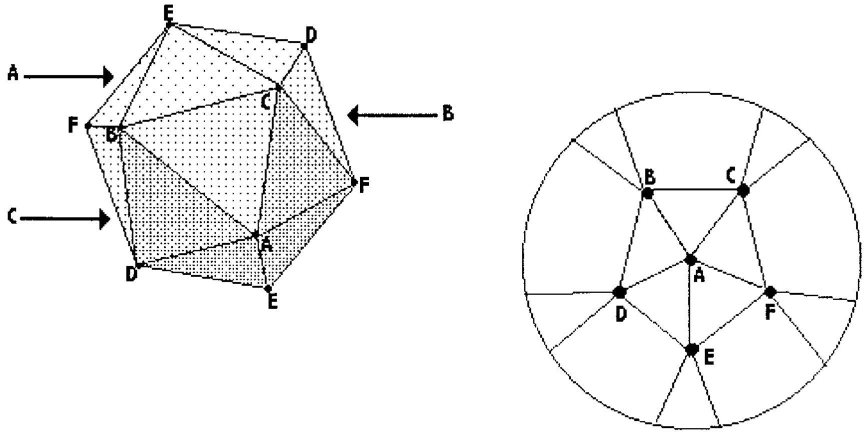

FIG. 1. Transitive MIF: $\mathrm{Aut}(\mathscr{F}) = 6T12$ (icosahedral symmetry).

A more intuitive characterization which highlights the 2-transitivity of the underlying automorphism group was found by Dmitri Zvonkin. Consider an icosahedron and identify opposite vertices (the resulting map is K_6 drawn on the projective plane!). Define the voting scheme *Icos* to be the collection of all sets of vertices which includes a face. It is easy to see that *Icos* is a MIF. Since the icosahedron is a platonic solid, Aut(*Icos*) acts transitively on the edges of the icosahedron; that is, Aut(*Icos*) is 2-transitive.

4.3. $n = 7$: *Fano MIF*

Let $\mathscr{P}_k$ be a projective plane of order k on a set X ($|X| = k^2 + k + 1$). We can define an intersecting family consisting of all collections of points which include a line

$$\mathscr{F}_{\mathscr{P}_k} = \{A \subseteq X : \exists \ell \in \mathscr{P} \text{ such that } \ell \subseteq A\}.$$

This projective plane is not maximal unless $k = 2$ [23, Theorem 1], in which case $\mathscr{F} = \mathscr{P}_2$ is the Fano plane. Any two points in a projective plane determine a line, and all lines are mapped to each other by the automorphism group $L(3, 2)$. Thus, $\mathscr{F}$ is 2-transitive.

For $k > 2$, $\mathscr{F}_{\mathscr{P}}$ can be extended to a transitive MIF in a number of ways (for example, by including all sets with over half the elements whose complement does not contain a line). These transitive MIFs are distinguished by the fact that they have minimal winning coalitions containing as few as a elements where $n = a^2 - a + 1$. More precisely, we have the following result:

PROPOSITION 10 ([5, Theorem 3.a]). *Let $\mathscr{F}$ be a transitive MIF on n voters.*

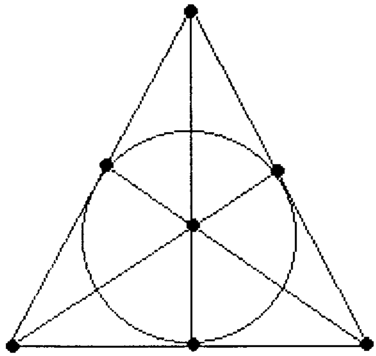

FIG. 2. Transitive MIF: $\mathrm{Aut}(\mathscr{F}) = 7T5$.

$(n > 1)$ *Let* $a = \min_{A \in \mathscr{F}} |A|$. *Then*

$$a \geqslant 1 + \left\lfloor \sqrt{n} \right\rfloor.$$

Proof. Let $A \in \mathscr{F}$. Consider the orbit of A under the action of $\mathrm{Aut}(\mathscr{F})$. What is the average size E of the intersection of two sets in this orbit?

On one hand, E is at least one, $\mathscr{F}$ is an intersecting set, so all pairs of sets intersect. Actually, $E > 1$, since $A \cup A > 1$. On the other hand, E must be exactly $a \times (a/n)$ since each of the a members of A is mapped equally often to each of the n members of X.)

Hence, $a\,\frac{a}{n} > 1$ and $a > \sqrt{n}$. ∎

4.4. $n = 9$

4.4.1. *New results.* All of the transitive MIFs mentioned above had already been known prior to our work. For $n = 8$, there are no transitive MIFs. (Any MIF on 8 voters includes 35 winning 4-element coalitions. To be transitive, each voter would have to be a member of exactly $35 \times 4/8 = 17.5$ of them, which is of course impossible.

Thus, the transitive MIFs for $n = 9$ listed in Table VII represent our first new results. (See Table VII for legend.)

For example, we see that there is a single MIF $\mathscr{F}$ with $\mathrm{Aut}(\mathscr{F}) = 9T8$, where $9T8$ is the 8th transitive permutation group on 9 letters listed in [4]. To determine the winning coalitions in $\mathscr{F}$, we consult the column "MWC." In this column it is indicated that $C = \{a, d, e, h\}$ is a minimal winning coalition. Acting the group $9T8$ on C, we find that the entire orbit

$$[C] = \{adeh, afgi, bdeg, acfg, cefh, befi, acdi, bdgh, adfi, aegi,$$

$$bfhi, bcei, cehi, bcfh, cdfg, cdgi, abch, abeg\}$$

TABLE VII

List of transitive MIFs: Part II ($n=9$)

MWC	Name	t	t'	Aut($\mathscr{F}$)	Generators	#
See Figure 3 for 10 missing lines		1	1	$9T1 = C_9$	$(aehbficdg)$	$10 \times 40{,}320$
aefg		1	1	$9T3 = D_{18}$	$(aehbficdg)$ $(ah)(bg)(ci)(df)$	20,160
dghi		1	1	$9T3 = D_{18}$	$(aehbficdg)$ $(ah)(bg)(ci)(df)$	20,160
See Figure 5 for 5 missing lines		1	1	$9T4 = C_3 \times S_3$	$(adg)(beh)(cfi)$ $(aie)(bgf)(chd)$ $(dg)(eh)(fi)$	$5 \times 20{,}160$
adeh		1	1	$9T8 = S_3^2$	$(adg)(beh)(cfi)$ $(aie)(bgf)(chd)$ $(dg)(eh)(fi)$ $(ag)(bi)(ch)(ef)$	10,080
aefg *dghi*		1	1	$9T13$	$(adg)(beh)(cfi)$ $(aie)(bgf)(chd)$ $(afi)(bdg)(ceh)$ $(bc)(ef)(hi)$	6,720
abfi		1	1	$9T16 = C_3^2 \times D_{18}$	$(abc)(def)(ghi)$ $(bdcg)(fihe)$ $(dg)(eh)(fi)$	5,040
aefg		1	1	$9T18$	$(adg)(beh)(cfi)$ $(aie)(bgf)(chd)$ $(ag)(bi)(ch)(ef)$ $(afi)(bdg)(ceh)$	3,360
bdef		1	1	$9T28$	$(abc), (bc)$ $(abc)(def), (ef)(hi)$ $(ghi), (hi)$ $(adg)(beh)(cfi)$	560
bcde	Dem_3^2	1	1	$9T31 = S_3 \wr S_3$	$(abc), (bc)$ $(adg)(beh)(cfi)$ $(dg)(eh)(fi)$	280
abcde	Dem_9	9	9	$9T34 = S_9$	$(abcdefghi)$ (ab)	1
Total						570,361

of minimal winning coalitions. There are other minimal winning coalitions, namely $\{b, c, d, g, i\}$, $\{a, b, c, f, i\}$, $\{a, b, c, h, i\}$ and their orbits under the action of $9T8$. However, only minimal winning coalitions containing at most half the number of voters are listed. Given the winning minority coalitions, the winning majority coalitions are simply those majority coalitions whose complement is losing.

Due to space constraints, the transitive MIFs with symmetry $9T1$, $9T3$, and $9T4$ are listed in Figs. 3–5.

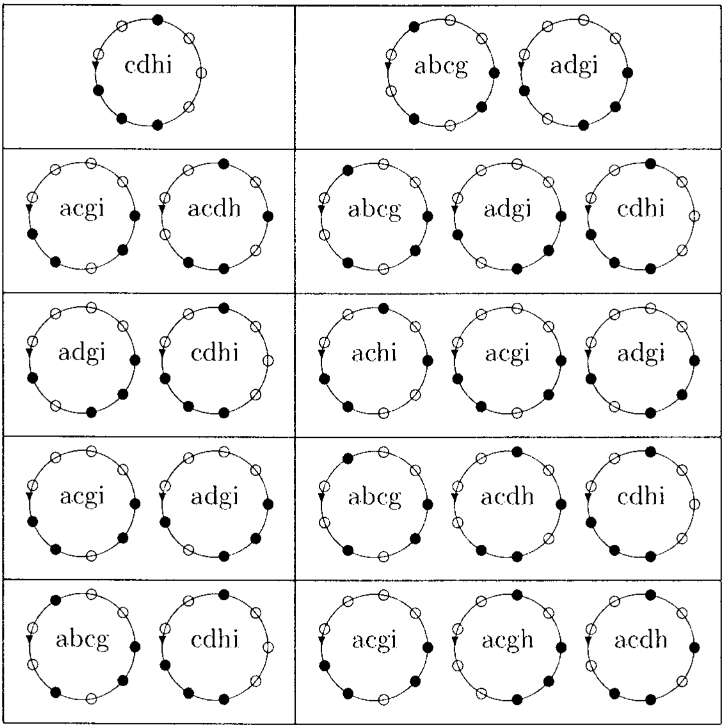

FIG. 3. Transitive MIFs: $\mathrm{Aut}(\mathscr{F}) = 9T1$ (cyclic symmetry).

• In Fig. 3, orbits are represented up to rotation by an oriented cycle (*aehbficdg*). (This oriented cycle is used instead of (*abcdefghi*) in order to conform to the standard notation given in [19].)

For each isomorphism class of maximal intersecting families of sets with the indicated group of symmetries, one MIF is depicted by representing its orbits (under the action of the group) of minimal winning coalitions containing at most half of the elements of X. The complete MIF can then be reconstituted by symmetry, inclusion, and duality.

• In Fig. 4, orbits are represented up to rotation and flip by an unoriented cycle.

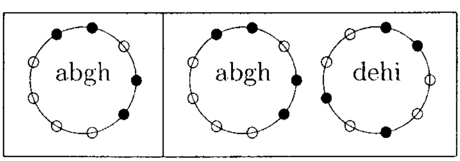

FIG. 4. Transitive MIFs: $\mathrm{Aut}(\mathscr{F}) = 9T3$ (dihedral symmetry).

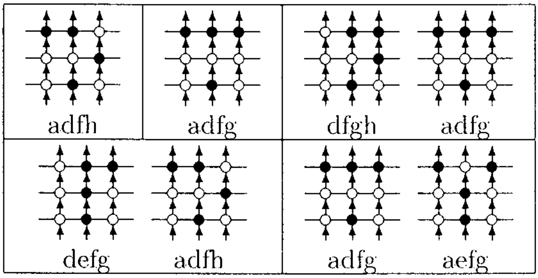

FIG. 5. Transitive MIFs: $\mathrm{Aut}(\mathscr{F}) = 9T4$ (toroidal symmetry, oriented in one direction, but not in the other).

- In Fig. 5, orbits are represented by a "tic-tac-toe" graph drawn on a torus, up to rotation along both axes, and reflection about the vertical axis. (It is understood that edges going off one edge of the diagram reappear on the other side.)
- In Fig. 6, the orbit $[C]$ in the MIF mentioned above is represented by a "tic-tac-toe" graph drawn on a torus, up to rotation and reflection about both axes.
- In Fig. 7, orbits are represented by the complete symmetry group of the "tic-tac-toe" graph. When drawn on the torus, this includes rotation and reflection about both axes and exchange of axes.

4.4.2. *Composition.* Given a MIF $\mathscr{F}$ on n voters, and n MIFs $\mathscr{G}_1, ..., \mathscr{G}_n$ on disjoint sets of voters $X_1, ..., X_n$, respectively. Then one can define the *composition* of $\mathscr{F}$ with $\mathscr{G}_1, ..., \mathscr{G}_n$ to be the set of subsets A of $X = X_1 \cup \cdots \cup X_n$ such that $\{i: A \cap X_i \in \mathscr{G}_i\} \in \mathscr{F}$. In other words, $\mathscr{F}[\mathscr{G}_1, ..., \mathscr{G}_n]$ is the voting scheme in which the voters vote by committee. Each committee votes according to its own rules $\mathscr{G}_i$, and results are combined via the voting scheme $\mathscr{F}$.

If $\mathscr{F}$ is a transitive game on n voters, and $\mathscr{G}$ is a transitive game on m voters, then

$$\mathscr{F}[\underbrace{\mathscr{G}, ..., \mathscr{G}}_{n}]$$

is a transitive game on nm voters. (This is essentially the proof of part 2 of Theorem 5 given by Cameron, Frankl, and Kantor [5].)

PROPOSITION 11 [17]. *Let $S, T_1, ..., T_n$ be strong simple games. If $S[T_1, ..., T_n]$ is transitive, then S is also transitive, and the games $T_1, ..., T_n$ must all be isomorphic transitive strong simple games.*

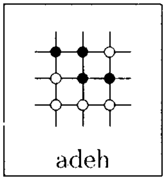

FIG. 6. Transitive MIFs: $\mathrm{Aut}(\mathscr{F}) = 9T8$ (unoriented toroidal symmetry).

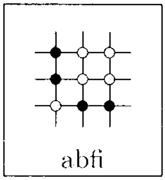

FIG. 7. Transitive MIFs: $\mathrm{Aut}(\mathscr{F}) = 9T16$ (total toroidal symmetry).

TABLE VIII

List of Transitive MIFs: Part III ($n = 10$)

MWC	t	t'	$\mathrm{Aut}(\mathscr{F})$	Generators	#
See Figure 5 16 missing lines	1	1	$10T7 = A_5$	$(bf)(ce)(dg)(ij)$ $(aeg)(bid)(chj)$	$16 \times 60{,}480$
See Figure 9 for 10 missing lines	1	1	$10T8$	$(ab)(cd)$ $(acegi)(bdfhj)$	$10 \times 45{,}360$
dghij, *abefh*	2	2	$10T26 = L(2, 9)$	$(abc)(def)(ghi)$ $(bdcg)(efih)$	10,080
abefh, *abfi*	2	2	$10T26 = L(2, 9)$	$(abc)(def)(ghi)$ $(bdcg)(efih)$	10,080
Total					1,441,440

no minority coalitions	acefg abceg abcgh adehj	abceg adehj defij bdgij
adehj abceg acefg defij	abceg acefg bcfgi defij	abceg abcgh bcfgi bdgij
abceg abcgh adehj bdgij	acefg abceg ahij bcfgi	
one minority coalition	bdhj adehj abceg	abceg ahij bcfgi bdgij
abceg adehj ahij bdgij	abceg abcgh adehj aefg	abcgh adehj aefg dfhij
abceg adehj aefg defij	abceg aefg defij bcfgi	acefg abceg adehj ahij
two minority coalitions	abceg adehj aefg ahij	

FIG. 8. Transitive MIFs: $\mathrm{Aut}(\mathscr{F}) = 10T7$.

no minority coalitions

acdef acdij abehj afhij acdgh adehj | acdef acdij abehj afhij befij adehj
acdef acdij abehj bcdeg acdgh adehj | acdef acdij cdfgi afhij befij adehj
acdef acdij abehj bcdeg befij adehj | acdef acdij cdfgi afhij acdgh adehj
acdef acdij cdfgi bcdeg acdgh adehj | acdef acdij cdfgi bcdeg befij adehj

one minority coalition | acdgh adehj bdef

two minority coalitions | befij adehj bdef

FIG. 9. Transitive MIFs: $\mathrm{Aut}(\mathscr{F}) = 10T8$.

In fact, the symmetries of such a composition $\mathrm{Aut}(S[T, T, ..., T])$ is given by the wreath product $\mathrm{Aut}(S) \wr \mathrm{Aut}(T)$, since each committee can be permuted by $\mathrm{Aut}(T)$ or the committee can be permuted with each other by $\mathrm{Aut}(S)$.

Thus, $Dem_3^2 = Dem_3[Dem_3, Dem_3, Dem_3]$ is a simply transitive MIF on 9 voters with $\mathrm{Aut}(Dem_3^2) = S_3 \wr S_3$.

All MIFs can be expressed as trivial compositions

$$\begin{aligned} \mathscr{F} &= Dem_1[\mathscr{F}] \\ &= \mathscr{F}[Dem_1, ..., Dem_1] \end{aligned}$$

TABLE IX

List of Transitive MIFs: Part IV ($n = 11$)

MWC	Name	t	t'	$\mathrm{Aut}(\mathscr{F})$	Generators	#
57,196 missing lines		1	1	$11T1 = C_{11}$	$(abcdefghijk)$	$57{,}196 \times 3{,}628{,}800$
See Figure 10 for 48 missing lines		1	1	$11T2 = D_{22}$	$(abcdefghijk)$ $(bk)(cj)(di)(eh)(fg)$	$43 \times 1{,}814{,}400$
See Figure 11 for 11 missing lines		1	2	$11T3 = C_5 \times C_{11}$	$(abcdefghijk)$ $(befjd)(cikhg)$	$11 \times 725{,}760$
bcdfg		1	2	$11T4 = C_{10} \times C_{11}$	$(abcdefghijk)$ $(bceifkjhdg)$	$2 \times 362{,}880$
bdefh		1	2	$11T4 = C_{10} \times C_{11}$	$(abcdefghijk)$ $(bceifkjhdg)$	$2 \times 362{,}880$
abefh		2	4	$11T6 = M_{11}$	$(abc)(def)(ghi)$ $(bdcg)(efih)$ $(aj)(dg)(ef)(hi)$ $(dh)(ei)(fg)(jk)$	5040
abcdef	Dem_{11}	11	11	$11T8 = S_{11}$	$(abcdefghijk)$ (ab)	1
Total						207,688,650,161

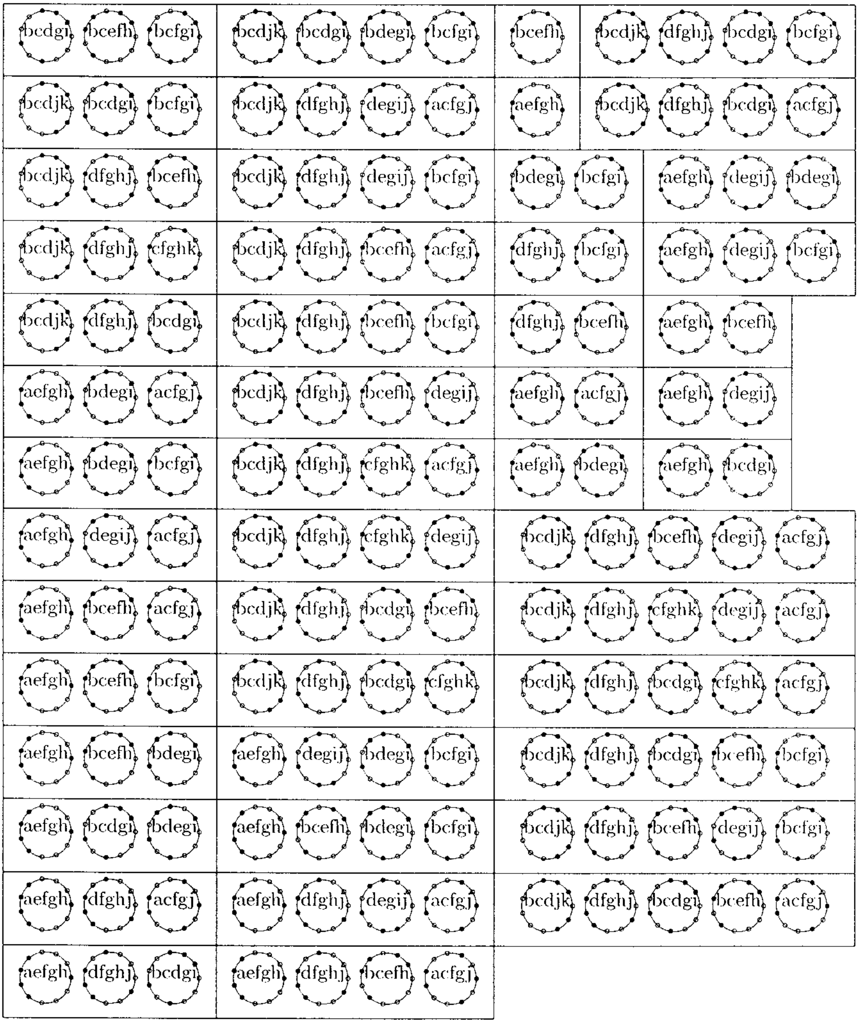

FIG. 10. Transitive MIFs: Aut($\mathscr{F}$) = 11T2.

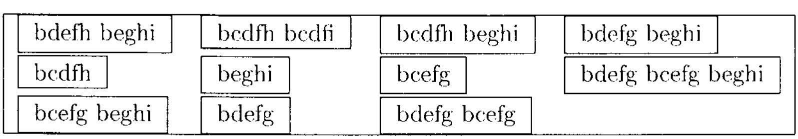

FIG. 11. Transitive MIFs: Aut($\mathscr{F}$) = 11T3.

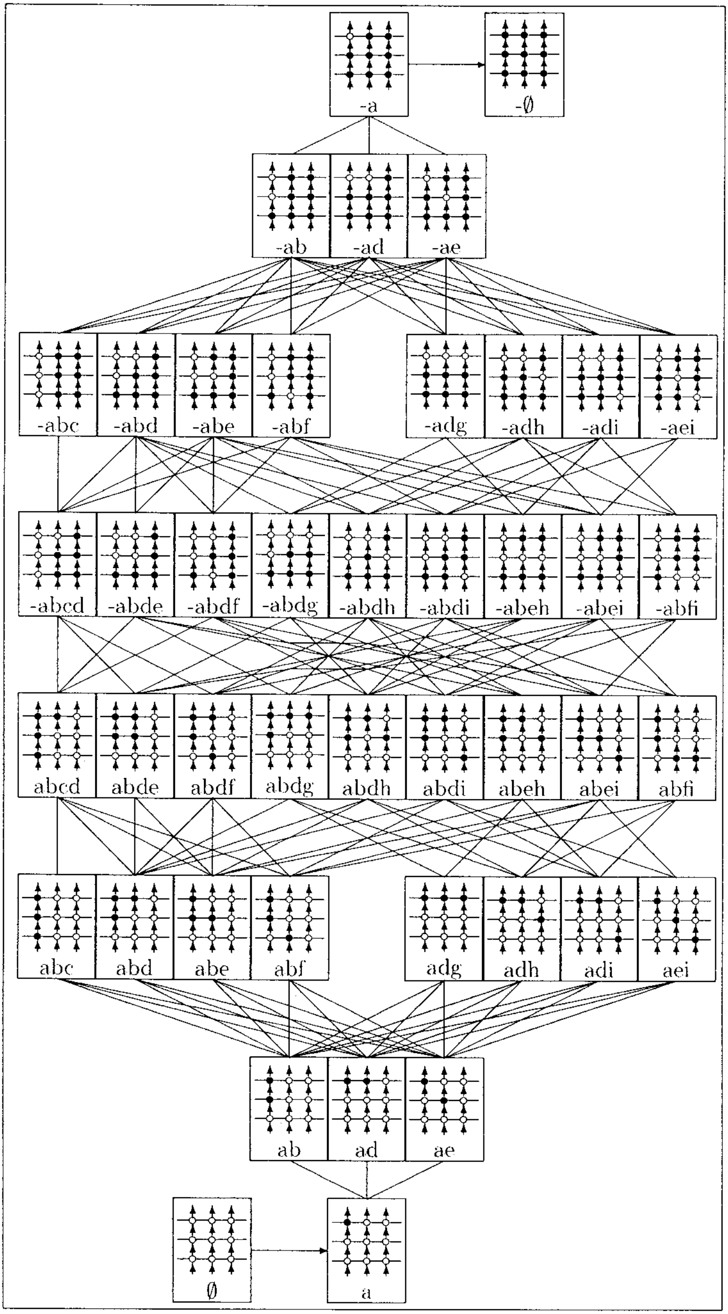

FIG. 12. Lattice of orbits $G = 9T4$.

involving a single "committee" or a large number of one-person "committees." A MIF is said to be *prime* if it cannot be expressed as a composition $\mathscr{F}[\mathscr{G}_1, ..., \mathscr{G}_n]$ in any other way [17].

PROPOSITION 12 [17]. *All 2-transitive strong-simple games are prime strong-simple games.* ∎

4.5. $n = 10$

Transitive MIFs for $n = 10$ are tabulated in Table VIII. (See Table V for legend.) As we saw before in the case of *Icos* (for $n = 6$), an even number of voters is not necessarily a barrier to the existence of a transitive MIF. In fact, there are even 12 classes of transitive MIFs with at least one winning minority coalition.

Due to space constraints, the transitive MIFs with symmetries $10T7$ and $10T8$ are listed in Figs. 8 and 9. They are classified according to how many minority winning coalition they include (4 elements).

4.6. $n = 11$

Transitive MIFs for $n = 11$ are tabulated in Table IX. (See Table V for legend.)

Due to space constraints, the 57,196 isomorphism classes of transitive MIFs with symmetry $11T1$ have been omitted. They each have 11-fold cyclic symmetry.

Transitive MIFs with symmetries $11T2$ and $11T3$ are listed in Figs. 10 and 11, respectively.

In Fig. 12, orbits are represented by an unoriented cycle up to rotation and reflection.

5. SEARCH TECHNIQUES

In this section, we give details concerning our computer search for transitive MIFs. We believe that other symmetrical combinatorial structures can be enumerated by similar techniques. For example, McKay and others [19, 20, 27] have enumerated graphs with vertex transitive automorphism groups and up to 26 vertices.

5.1. *Tree Search*

We first identified and eliminated the transitive groups which were liable to be the automorphism group of some MIF. We began with the catalog of transitive groups of degree up to 11 by Butler and McKay [4]. (For $11 < n \leqslant 23$, one can use the program `Gap`. For $n = 25$ or 26, see A. Hulpke [28].) Groups which contained a fixed-point free 2-element were then identified by using the table of group elements according to cyclic decomposition type.

A program was written in `Caml Special Light (CSL)` [12] that given a group G would generate the set S_G of all MIFs $\mathscr{F}$ with $G \subseteq \mathrm{Aut}(\mathscr{F})$. To do this, the computer calculated the orbits of the action of G on the powerset of X by applying the generators of G repeatedly to the subsets of X. Given an orbit α, define its dual α^* to be the orbit generated by $[X \backslash A]$, where $A \in \alpha$. We never have $\alpha = \alpha^*$ since G is assumed to contain no fixed-point free 2-elements. There is no point in studying such groups as they are not the automorphism groups of any MIF. For each pair (α, α^*) we must decide whether α or α^* will be in our MIF.

We write $\alpha \geqslant \beta$ if there is some $A \in \alpha$ and $B \in \beta$ such that $B \subseteq A$. (See Fig. 12.) If $\alpha \geqslant \alpha^*$, then α is a subset of every MIF in S_G.

Our main algorithm considers an orbit $\alpha = [A]$, where A has minimal cardinality among those orbits still under consideration. If $\alpha \leqslant \alpha^*$, then α must be rejected and α^* must be included as above. Otherwise, we either reject α and accept α^*, or else we accept α and we reject all orbits $\beta \leqslant \alpha^*$. In both cases, a recursive call to the algorithm allows us to determine the possible ways to treat the remaining orbits.

If only the *number of solutions* $|S_G|$ is required, then a dynamic programming algorithm can be used by creating a hash table (remember table) and using it to treat most of the recursive calls. Using a 50-Mb table, and taking advantage of the obvious 8-fold symmetry, we were able to compute all 8-voter MIFs in only 12 min. whereas an analogous algorithm by Bioch and Ibarki [3] required 6 months.

5.2. *Inclusion–Exclusion*

The set S_G includes all MIFs with at least the group G as symmetry. To compute the number of MIFs with exactly the group G as symmetry, inclusion–exclusion techniques (Möbius inversion) are required.

S_n acts on its subgroups by conjugation. Given a permutation group $G \subseteq S_n$, let $Z'(G)$ denote its stabilizer under this action $Z'(G) = \{\sigma \in S_n : \sigma G = G\sigma\}$. $Z'(G)$ includes the center $Z(G)$ (and in fact G itself) but also possibly other elements since we require not that σ commute with each element of G but simply that σ commute with G itself. Thus, there are $n!/|Z'(G)|$ conjugate copies of the group G in S_n.

By restricting this action to a group H $(G \subseteq H \subseteq S_n)$, the stabilizer is $Z'(G) \cap H$. We thus deduce that there are $|H|/|Z'(G) \cap H|$ conjugate copies of the group G in H.

For each pair of groups (G, H) under consideration for which we found at least one nondemocratic MIF, we calculated $|Z'(G) \cap H|$ using a short program in `CSL`. The generators of G were conjugated the generators of H. The result was then compared to a list of elements of G.

Since the $n!/|Z'(H)|$ copies of H contain $|H|/|Z'(G)\cap H|$ copies of G and there are $n!/|Z'(G)|$ copies of G, it then follows that each conjugate copy of G is contained in exactly

$$M_{GH}=\frac{n!}{|Z'(H)|}\times\frac{|H|}{|Z'(G)\cap H|}\bigg/\frac{n!}{|Z'(G)|}=\frac{|H|\times|Z'(G)|}{|Z'(H)|\times|Z'(G)\cap H|}$$

copies of H.

PROPOSITION 13. *Given the unitriangular matrix* $\mathbf{M}$ *above and the vector* $\mathbf{v}$ $(v_G=|S_G|)$, *indexed by certain subgroups of* S_n, *we have the identity*

$$\mathbf{v}=\mathbf{M}\mathbf{w}, \tag{1}$$

where w_G *is the number of MIF* $\mathscr{F}$ *with* $\operatorname{Aut}(\mathscr{F})=G$.

Proof.

$$\begin{aligned} v_G &= \sum_{\mathscr{F}\in S_G} 1 \\ &= \sum_{G\subseteq \operatorname{Aut}(\mathscr{F})} 1 \\ &= \sum_{G\subseteq H}\ \sum_{H=\operatorname{Aut}(\mathscr{F})} 1 \\ &= M_{GH}w_H. \quad \blacksquare \end{aligned}$$

Using the `Maple linalg` package, we solved (1) for $\mathbf{w}$. For example, in the case $n=11$ (see Table IX), we solved

$$\begin{bmatrix} 572227 \\ 243 \\ 27 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{w}$$

to find $\mathbf{w}=[571960, 240, 22, 2, 1, 1]^{\mathrm{T}}$.

Among the v_G MIFs we found with symmetry at least G, there are w_G with symmetry exactly G. However, some of these may be isomorphic.

PROPOSITION 14. *The MIFs with automorphism group* G *are divided into* $w_G\,|G|/|Z'(G)|$ *isomorphism classes each of size* $|Z'(G)|/|G|$.

Proof. Let $\mathrm{Aut}(\mathscr{F}) = G$. The $n!/|G|$ isomorphic images $\sigma\mathscr{F}$ of $\mathscr{F}$ have automorphism groups $\mathrm{Aut}(\sigma\mathscr{F}) = \sigma G \sigma^{-1}$ which are conjugates of G. There are $n!/|Z'(G)|$ conjugates of G which are all identical up to permutations of X. Thus, there are $|Z'(G)|/|G|$ isomorphic images of $\mathscr{F}$ with automorphism group exactly equal to G. ∎

Dividing by the appropriate quantities we know, for example, that the group $11T1$ is the automorphism group of 57,196 collections of 10 isomorphic MIFs.

5.3. *Statistics*

To identify the various isomorphism classes, we used a collection of `Maple` routines to apply various statistics (invariant under permutation of X) to the MIFs generated. Typical statistics included:

- The distribution of winning coalitions according to coalition size.
- The distribution of minimal winning coalitions according to coalition size.
- The set of numbers c_j ($t' < j \leqslant n$), where c_j is the number of winning coalitions containing 1, 2, 3, ..., t' and j, and t' is the homogeneity of the group.

In a handful of cases, no easily computed statistic could distinguish certain isomorphism classes. In those cases, coset representatives were found for the quotient $Z'(G)/G$ using the `cosets` function in the `Maple` `group` package. By applying these permutations to these MIFs using the procedure `subs`, we were able to compute the isomorphism classes as they are simply the orbits of the action of $Z'(G)$ on these MIFs.

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