Voting Fairly: Transitive Maximal Intersecting Families of Sets

Daniel E. Loeb

Daniel H. Wagner, Associates, 40 Lloyd Avenue, Suite 200, Malvern, Pennsylvania 19355
E-mail: loeb@pa.wagner.com
http://dept-info.labri.u-bordeaux.fr/~loeb/index.html

and

Andrew R. Conway

Silicon Genetics, 2601 Spring Street, Redwood City, California 94063
E-mail: arc@sigenetics.com
http://www.sequence.Stanford.edu/~arc

Communicated by the Managing Editors
Received February 16, 2000

DEDICATED TO THE MEMORY OF GIAN-CARLO ROTA

There are several applications of maximal intersecting families (MIFs) and different notions of fairness. We survey known results regarding the enumeration of MIFs, and we conclude the enumeration of the 207,650,662,008 maximal families of intersecting subsets of \( \mathcal{X} \) whose group of symmetries is transitive for \(|\mathcal{X}| < 13\).

2000 Academic Press
Key Words: homogeneous games; fair; transitive; regular; coterie; maximal intersecting family; strong simple game; voting scheme; ipsodual element of free distributive lattice; self-dual monotone Boolean function; self-dual anti-chain; critical tripartite hypergraph.

Contents
1. Introduction
2. Quotients and Isomorphisms
3. Fairness

1 Work was conducted in 1995–1996 while both authors were at the Laboratoire Bordelais de Recherche en Informatique at the Université de Bordeaux I in Talence, France. We thank the people of LaBRI for their hospitality and the stimulating research environment.

2 Partially supported by URA CNRS 1304, EC Grant CHRX-CT93-0400, the PRC “Mathématiques et Informatique,” and NATO CRG 930554.

3 Partially supported by CHES.
4. Examples of Transitive MIFs

4.1. Democracy
4.2. \( n = 6 \): Icosahedral MIF
4.3. \( n = 7 \): Fano MIF
4.4. \( n = 9 \)
   4.4.1. New Results
   4.4.2. Composition
4.5. \( n = 10 \)
4.6. \( n = 11 \)

5. Search Techniques
   5.1. Tree Search
   5.2. Inclusion-Exclusion
   5.3. Statistics

1. INTRODUCTION

[This paper] attempts to promote better communication and less duplication of mathematical effort by identifying and describing several other theories, formally equivalent ... that are founded in fields ranging from sociology to electrical engineering.

—Dubey and Shapley [7]

How large can a collection of pairwise intersecting subsets of a given \( n \)-element set \( X \) be? It is easy to see not only that any intersecting family contains at most \( 2^{n-1} \) sets, but furthermore that any intersecting family can be extended to a maximal intersecting family containing exactly \( 2^{n-1} \) sets [1, Theorem 1.1.1].

The number \( a_n \) of maximal intersecting families (MIFs) on \( X \) has been found to grow quite quickly as \( n = |X| \) increases. (See Table I.) Korshunov

<table>
<thead>
<tr>
<th>( n )</th>
<th>( a_n )</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>Dedekind 1897 [6]</td>
</tr>
<tr>
<td>5</td>
<td>81</td>
<td>Von Neumann 1944 [6]</td>
</tr>
<tr>
<td>6</td>
<td>2,646</td>
<td>Gurk and Ishell 1959 [9]</td>
</tr>
<tr>
<td>7</td>
<td>1,422,564</td>
<td>Loeb 1992 [16] and Bioch and Ibarki 1994 [3]</td>
</tr>
<tr>
<td>8</td>
<td>229,809,982,112</td>
<td>Conway and Loeb (12 min. computer calculation, Nov. 30 (1995) and Bioch and de Boer (6 month computer calculation, Jun-Dec. 1995)</td>
</tr>
</tbody>
</table>
proved the following asymptotic formulas for $a_n$ depending on the parity of $n$.

$$a_n \sim 2^{\binom{n-1}{n/2}} \exp(e_n),$$

where

$$e_n = \begin{cases}
    \left( \frac{n-1}{(n-1)/2} \right) \left( 2^{-(n-1)/2} + 3n^2 \cdot 2^{-n-4} - n \cdot 2^{-n-2} \right) \\
    + \left( \frac{n-1}{(n+3)/2} \right) \left( 2^{-(n+3)/2} + n^2 \cdot 2^{-n-6} - n \cdot 2^{-n-5} \right) & \text{for } n \text{ odd}
\end{cases}$$

and

$$\begin{cases}
    \left( \frac{n}{n/2-1} \right) \left( 2^{-n/2-1} + n \cdot 2^{-n-4} \right) \\
    + \left( \frac{n}{n/2+1} \right) \left( 2^{-n/2-1} + n^2 \cdot 2^{-n-5} - n \cdot 2^{-n-4} \right) & \text{for } n \text{ even}.
\end{cases}$$

The notion of a maximal intersecting family has arisen independently in a surprisingly large number of contexts besides extremal combinatorics.

**Interactive decision making.** MIFs are known as strong simple games [25], and are used to model situations in every coalitions is either “all-powerful” or “ineffectual.” A game (or upset or filter) on a set of players $X$ is a set $F$ of coalitions $A \subseteq X$ closed under inclusion. The coalition $A$ is winning (resp. losing, blocking) if $A \in F$ (resp. $A \notin F$, $X \setminus A \notin F$). The game $F$ is simple (resp. strong) if winning implies blocking (resp. blocking implies winning).

**Distributed computing.** The set $\min(F)$ of minimal elements of an intersecting family $F$ is called a coterie whereas if $F$ is maximal, then $\min(F)$ is called a non-dominated coterie [8]. (Recall that a MIF is determined by its minimal elements.) They are used in mutual exclusion protocols (to limit access to a protected resource) and replication protocols (to manage a distributed memory system).

**Logic or linear programming.** The characteristic function $\xi(F)$ of a MIF $F$ is a self-dual monotone boolean function. Conversely, given any self-dual monotone boolean function $f$, the preimage $f^{-1}(\text{True})$ is a MIF. (This is related to Dedekind’s problem [6] of enumerating all monotone boolean functions.)

**Category theory.** The ipsodual elements of the free distributive lattice [24] and the elements of the free median set [21] generated by $X$ correspond to MIF’s.
Social science. Arrow’s impossibility theorem [2] states that nondictatorial, unanimous social choice functions independent of irrelevant alternatives exist only when the public faces at most two choices. Using a MIF to determine winning coalitions gives an effective voting scheme when there are exactly two choices [10].

Graph theory. A coloring of a hypergraph is an assignment of colors to vertices such that each nontrivial edge contains at least two colors. The minimal sets of a MIF can be regarded as the edges of a critical tripartite hypergraph $H$. That is to say, $H$ is 3-colorable, and if any edge is removed from $H$ then it would be 2-colorable.

Reliability theory. Games are thought of as semi-coherent structure functions [11, 22].

Each rediscovery of a theory gives birth to alternate notation and terminology. An attempt has been made here to choose a consistent terminology which makes our results as clear as possible. The above references are useful in adapting our results to other fields of interest.

In Section 2, we define homomorphisms or quotients of MIFs. Voters in the same orbit of the automorphism group of a MIF can be said to play the equivalent roles when the MIF is thought of as a voting system.

This notion of equivalence is used in Section 3 as a measure of fairness. The remainder of the paper is devoted to the enumeration of MIFs whose automorphism groups act transitively on the set of voters, so that all voters play the same role.

In Section 4, we enumerate all $207,650,662,008$ transitive MIFs on up to 12 voters. For completeness, we survey previous results on MIFs with up to 7 voters before giving the classification of larger MIFs. Such a list is important in applications, since the “best” transitive MIF can be selected from it, depending upon your personal criteria that define what is best [3, 8].

Finally in Section 5, we explain the search techniques used in our research. We believe that similar techniques can be helpful in the enumeration of other combinatorial objects according to their symmetries.

2. QUOTIENTS AND ISOMORPHISMS

Let $\mathcal{F}$ be a MIF on $X$, and let $\sigma : X \to Y$ be some function. It is easy to define the quotient voting scheme $\sigma(\mathcal{F}) = \{ A \subseteq Y : \sigma^{-1}(A) \in \mathcal{F} \}$. Note that $\min(\sigma(\mathcal{F}))$ is equal to $\sigma(\min(\mathcal{F}))$ in the usual sense.
Proposition 1. Let $F$ be a MIF on $X$, and let $\sigma: X \to Y$ be some function. Then $\sigma(F)$ is a MIF on $Y$.

Proof. $\sigma^{-1}$ is a monotone function from $2^Y$ to $2^X$, thus $\sigma(F)$ is a game. Suppose $A \in \sigma(F)$. Then $\sigma^{-1}(A) \in F$. Hence, $\sigma^{-1}(Y - A) = X - \sigma^{-1}(A) \notin F$. Thus, $Y - A \notin \sigma(F)$, so $\sigma(F)$ is simple.

Similarly, suppose $B \notin \sigma(F)$. Then $\sigma^{-1}(B) \notin F$. Hence, $\sigma^{-1}(Y - A) = X - \sigma^{-1}(A) \in F$. Thus, $Y - A \in \sigma(F)$, so $\sigma(F)$ is strong.

$X$ can be thought of as a set of offices and $Y$ as a set of voters. $\sigma$ describes which offices are held by which voters. If $\sigma$ is non-surjective, then certain voters will hold no office, and thus are powerless (dummies). If $\sigma$ is non-injective, then certain voters will combine the functions of several offices. The single vote of each such voter is then taken into account as the vote of each of his offices.

If $\sigma$ is bijective, then $F$ and $\sigma(F)$ are said to be isomorphic. Furthermore, if $F = \sigma(F)$, then $\sigma$ is said to be an automorphism of $F$. An automorphism is a permutation of $X$ taking winning sets into winning sets.

Let Aut($F$) be the set of automorphisms of $F$.

Theorem 2. Let $F$ be a MIF on $X$. Then Aut($F$) is a permutation group of $X$.

Proof. Observe that $(\tau \circ \sigma)(F) = \tau(\sigma(F))$. Thus, the composition of two automorphisms or the inverse of an automorphism is again an automorphism.

A permutation group containing only one permutation (the identity) is said to be trivial. For large $n$, most MIFs have trivial automorphism groups.

Theorem 3. Let $b_n$ be the number of MIFs with trivial automorphism groups on an $n$-element set. ($b_n/n!$ is the number of isomorphism classes of such MIFs.) Then the fraction of MIFs (resp. isomorphism classes of MIFs) with trivial automorphism groups tends to $1$ as $n$ tends to infinity.

$$\lim_{n \to \infty} \frac{b_n}{a_n} = 1, \quad \lim_{n \to \infty} \frac{b_n/n!}{a_n} = 1.$$ 

Sketch of proof. From [15], we recall that all but a vanishingly small fraction of all MIFs have all but a vanishingly small fraction of their minimal sets of cardinality $n/2$ for $n$ even. That is, for all $\varepsilon < 1$ there is an $N$ such that for $n > N$, over $\varepsilon a_n$ MIFs on an $n$ element set have over $1 - \varepsilon$ of their minimal sets of cardinality $n/2$. 


The \( \binom{n}{2} \) such sets are divided by a non-trivial permutation group \( G \) into a number of orbits not exceeding

\[
\begin{align*}
\epsilon_n &= \binom{n-2}{n/2} + \binom{n-2}{(n/2)-2} + \binom{n-2}{(n/2)-1} \\
&= \frac{n}{(n/2)} - \frac{n-2}{(n/2)-1} \\
&\approx \frac{3}{4} \frac{n}{n/2}.
\end{align*}
\]

For \( n \) even, the orbits form complementary pairs (otherwise there is no MIF with automorphism group \( G \)) and we must choose one orbit from each such pair. Thus, the logarithm (base two) of the number of MIFs with automorphism group \( G \) is asymptotically bounded by \( \frac{3}{4} \binom{n}{n/2} \) whereas \( \log_2 a_n \sim \frac{1}{3} \binom{n}{n/2} \).

For \( n \) odd, all but a vanishingly small fraction of MIFs have all their sets of cardinality at least \( i = (n-1)/2 \) and at most \( \binom{n}{(n/2)} \) sets of cardinality exactly \( i \). In fact these MIFs are uniquely determined by their sets of cardinality \( i \). As above, the \( \binom{n}{2} \) such sets are divided by a non-trivial permutation group \( G \) into at most about \( \frac{3}{4} \binom{n}{n/2} \) orbits. Hence, \( \log_2 b_n / \log_2 a_n \) is asymptotically bounded by

\[
\frac{3}{4} \binom{n}{(n/2)} 2^{-n/2} \ll 1.
\]

It is difficult to find such examples for small \( n \). For \( n \leq 6 \), the only example is the trivial one-voter MIF. Nonetheless, already for \( n = 7 \), there are 498,960 MIFs with trivial automorphism groups. They can be divided into 99

<table>
<thead>
<tr>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
<th>( v_4 )</th>
<th>( v_5 )</th>
<th>( v_6 )</th>
<th>( v_7 )</th>
<th>Quota</th>
<th>( \ell )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>15</td>
<td>155</td>
</tr>
<tr>
<td>9</td>
<td>7</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>16</td>
<td>78</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>18</td>
<td>299</td>
</tr>
</tbody>
</table>

Note. \( v_i \): Number of votes to be cast by player \( i \); Quota: Number of votes needed to win; \( \ell \): Line number in [13, Tables 2, 3].
automorphism classes and constitute over 35% of the seven-voter MIFs [3]. Three of the 99 classes involve weighted majority games (or threshold functions or quota games) [13]. (See Table II and Section 4.1.)

Such MIFs are maximally unfair in the sense that no two voters play the same role. That is, each element of \( X \) is in a separate orbit under the action of the permutation group \( \text{Aut}(\mathcal{F}) \).

Conversely, if all of the elements of \( X \) are in the same orbit, then \( \text{Aut}(\mathcal{F}) \) is a transitive subgroup of \( \text{Sym}(X) \), and we will say that \( \mathcal{F} \) is a transitive MIF (or fair game or homogeneous game).

3. FAIRNESS

Depending on the interpretation chosen, different measures of fairness are appropriate.

- In a democratic country, each voter should play the same role in the system of vote adopted.
- In a game, each player should have the same possibilities of winning.
- In a distributed system, load should be equally divided among all of the processors.

One might require:

- (Regularity [5].) All voters belong to the same number of winning coalitions.
- (Equal Banzhaf index [7].) All voters belong to the same number of minimal winning coalitions.
- (Equal Shapley-Shubik index [7].) All voters have an equal probability of being the pivot voter given a random alignment of the voters in order of their enthusiasm for a proposal under consideration.

However, we will retain the notion of transitivity as a measure of “fairness” since it is stricter than any of the others mentioned above.

The main result of this paper is the enumeration of all transitive MIFs for \( n < 13 \). Such a list is important in applications, since the “best” transitive MIF can be selected from it, depending upon your personal criteria that define what is best [3, 8].

Since a permutation group on a set \( X \) is defined to be \( k \)-transitive \( (k \leq n) \) if it acts transitively on the set of \( k \)-tuples of distinct elements of \( X \), we can go further and discuss \( k \)-transitive MIFs. Presumably, a \( k \)-transitive MIF is somehow more “fair” than a 1-transitive MIF, since it does not distinguish among \( k \)-tuples of players.
However, the natural object of study is not \( k \)-tuples of players, but rather coalitions, that is, unordered sets of players. We will therefore define a permutation group on \( X \) to be \( k \)-homogeneous \((k \leq n)\) if it acts transitively on the set \( \binom{X}{j} = \{ A \subseteq X : |A| = j \} \) of \( j \) element subsets of \( X \) for each \( j \) \((0 \leq j \leq k)\).

**Proposition 4.** Let \( G \) be a permutation group on \( X \). (\(|X| = n\).)

1. If \( G \) is \( k \)-transitive, then it is also \( k \)-homogeneous.

2. If \( G \) is \( \frac{1}{2}n \)-homogeneous, then \( G \) is \( n \)-homogeneous.

**Proof.** 1. Let \((x_1, \ldots, x_k)\) and \((y_1, \ldots, y_k)\) be \( k \)-tuples of distinct elements of \( X \). By hypothesis, there exists \( \sigma \in G \) such that \( \sigma x_i = y_i \). Thus, \( \sigma \{ x_1, \ldots, x_k \} = \sigma \{ y_1, \ldots, y_k \} \).

2. If \( \sigma A = B \), then \( \sigma(X - A) = X - B \). Thus, if \( G \) acts transitively on \( \binom{X}{j} \), then it also acts transitively on \( \binom{X}{j} \).

Several authors have studied the set \( A \) of numbers of voters \( n \) such that there exists a transitive MIF. One might think that there is no such game having an even number of players; however, see below for explicit examples in the cases \( n = 6 \) (Section 4.2) and \( n = 10 \) (Section 4.5).

See Table III for a list of (possible) non-elements of \( A \). The first few values of \( n \) whose membership in \( A \) is still in doubt are 40, 72, 80, and 88.

<table>
<thead>
<tr>
<th>( c )</th>
<th>( 2c )</th>
<th>( 4c )</th>
<th>( 8c )</th>
<th>( 16c )</th>
<th>( 32c )</th>
<th>64c</th>
<th>( c2^8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>( 64 )</td>
<td>( 2^6 ) for all ( k \geq 1 )</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>24</td>
<td>48</td>
<td>96</td>
<td>192</td>
<td>( 320 )</td>
<td>( 3 \cdot 2^6 ) for all ( k \geq 2 )</td>
</tr>
<tr>
<td>5</td>
<td>( 40^* )</td>
<td>( 80^* )</td>
<td>160</td>
<td>( 320^* )</td>
<td>( 640^* )</td>
<td>( 1280^* )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>7</td>
<td>( 72^* )</td>
<td>144</td>
<td>288</td>
<td>576</td>
<td>( 1152^* )</td>
<td>( 2304^* )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>9</td>
<td>( 88^* )</td>
<td>176</td>
<td>352</td>
<td>704</td>
<td>( 1408^* )</td>
<td>( 2816^* )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>11</td>
<td>( 104^* )</td>
<td>208</td>
<td>416</td>
<td>832</td>
<td>( 1664^* )</td>
<td>( 3328^* )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>15</td>
<td>( 480^* )</td>
<td>960</td>
<td>1920</td>
<td>( 3840^* )</td>
<td>( 7680^* )</td>
<td>( 15360^* )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>17</td>
<td>( 136^* )</td>
<td>272</td>
<td>544</td>
<td>1088</td>
<td>( 2176^* )</td>
<td>( 4352^* )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>19</td>
<td>( 152^* )</td>
<td>304</td>
<td>608</td>
<td>1216</td>
<td>( 2432^* )</td>
<td>( 4864^* )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>21</td>
<td>( 224^* )</td>
<td>( 448^* )</td>
<td>( 896^* )</td>
<td>( 1792^* )</td>
<td>( 3584^* )</td>
<td>( 7168^* )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>23</td>
<td>( 368^* )</td>
<td>( 736^* )</td>
<td>( 1472^* )</td>
<td>( 2944^* )</td>
<td>( 5888^* )</td>
<td>( 11776^* )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>25</td>
<td>( 800^* )</td>
<td>( 1600^* )</td>
<td>( 3200^* )</td>
<td>( 6400^* )</td>
<td>( 12800^* )</td>
<td>( 25600^* )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>27</td>
<td>( 432^* )</td>
<td>( 864^* )</td>
<td>1728</td>
<td>( 3456^* )</td>
<td>( 6912^* )</td>
<td>( 13824^* )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>29</td>
<td>( 232^* )</td>
<td>( 464^* )</td>
<td>928</td>
<td>( 1856^* )</td>
<td>( 3712^* )</td>
<td>( 7424^* )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>31</td>
<td>( 15872^* )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>
Theorem 5. 1. \([14, I: \text{Lemma 1}]\) \(n \in A\) if and only if there is a transitive permutation group of degree \(n\) containing no fixed-point free 2-element.

2. \([5]\) \(A\) is multiplicatively closed.

3. \([5]\) \(A\) contains all non-multiples of 8 (with the exception of 2, 4, and 12).

4. \([5]\) \(A\) has density 1.

4. EXAMPLES OF TRANSITIVE MIFs

4.1. Democracy

One of the simplest ways to define a MIF is to attribute weights to each of the voters \(w: X \to \mathbb{N}\). A coalition wins if its total weight is greater than the total weight of its complement.

\[ A \in S_w \quad \text{if and only if} \quad \sum_{a \in A} w(a) > \frac{1}{2} \sum_{v \in \mathbb{N}} w(v). \]

To enforce duality, the total weight can be taken to be odd.

For \(n\) odd, we have the true democracy

\[ \text{Dem}_n = \left\{ \begin{array}{c} 1 \ 1 \ \cdots \ 1 \end{array} \right\}_{(n+1)/2} \]

\[ = \{ A \subseteq X : |A| > n/2 \}. \]

Note that the weights are not uniquely defined by the MIF. For example, the weights \([1, 1, 1]\) and \([2, 2, 1]\) both give the democratic voting scheme \(\text{Dem}_3\).

Note however the following proposition.

Proposition 6. Let \(w: V \to X\) be a weight function. Then there are alternative weight functions \(w': V \to \mathbb{N}\) constant on all orbits of \(\text{Aut}(S_w)\), and zero on all dummies such that \(S_w = S_{w'}\).

Conversely, if \(w(a) = w(b)\), then \(a\) and \(b\) lie in the same orbit of \(S_w\), and if \(w(a) = 0\), then \(a\) is a dummy.

Proof. Without loss of generality, the weight of all dummies is 0. Let \(\text{Aut}(S) \subseteq \text{Sym}(V)\) be the automorphism group of \(S\). Then \(w'(v) = \sum_{g \in \text{Aut}(S)} w(g(v))\) is the required weight function.

The converse is evident. 

Clearly, the democracy has the full group of symmetries \(\text{Aut}(\text{Dem}_n) = S_n\).

Moreover, we have the following results.
### TABLE IV

Transitive MIFs for \( n \leq 13 \)

| \( n \) | \( a_n \) | \( \tilde{a}_n \) | \( t_n \) | \( \tilde{t}_n \) | \( \min(|A|) \) |
|-------|-------|-------|-------|-------|-------|
| 1     | 1     | 1     | 1     | 1     |       |
| 2     | 2     | 1     | 0     | 0     |       |
| 3     | 4     | 2     | 1     | 1     | 2     |
| 4     | 12    | 3     | 0     | 0     |       |
| 5     | 81    | 7     | 1     | 1     | 3     |
| 6     | 2646  | 30    | 12    | 1     | 3     |
| 7     | 1,422,564 | 716 | 31    | 2     | 3–4   |
| 8     | 229,809,982,112 | 0 | 0     | 0     |       |
| 9     | \( \sim 9 \times 10^{23} \) | 570,361 | 24    | 4–5   |       |
| 10    | \( \sim 3 \times 10^{40} \) | 1,441,440 | 28    | 4–5   |       |
| 11    | \( \sim 6 \times 10^{40} \) | 207,648,650,161 | 57,259 | 4–7   |       |
| 12    | \( \sim 5 \times 10^{45} \) | 0     | 0     |       |       |
| 13    | \( \sim 5 \times 10^{596} \) |       |       |       |       |

Note. \( a_n \): Number of maximal intersecting families on an \( n \)-element set; \( \tilde{a}_n \): number of isomorphism classes of maximal intersecting families on an \( n \)-element set; \( t_n \): number of transitive maximal intersecting families on an \( n \)-element set; \( \tilde{t}_n \): number of isomorphism classes of transitive maximal intersecting families on an \( n \)-element set; \( \min(|A|) \): minimal number of elements in a winning coalition of a transitive MIF.

#### Corollary 7.

\( \text{Dem}_n \) is the only transitive strong simple weighted MIF.

**Proof.** By Proposition 6, we must be able to assign the same weight to all voters.

#### Proposition 8.

\( \text{Dem}_n \) is the only MIF whose automorphism group is \( (n–1)/2 \)-homogeneous.

**Proof.** Suppose \( \text{Aut}(\mathcal{F}) \) is \( (n–1)/2 \)-homogeneous. Then by Proposition 4, \( \mathcal{F} \) is \( n \)-transitive. Thus, all sets of equal cardinality lie in the same orbit. Since there must be at least one winning coalition of \( (n+1)/2 \) elements, they are all winning.

#### Proposition 9.

Every strong simple majority game is a quotient of \( \text{Dem}_n \) for some \( n \).

**Proof.** Let \( \mathcal{F} = \{ v_1 v_2 \cdots v_k \} \) be a strong simple majority game. Consider the set \( X = \{ (i, j): 1 \leq i \leq k, 1 \leq j \leq v_i \} \) and the function \( f: (i, j) \mapsto i \). Then \( \mathcal{F} \) is the quotient of democracy on \( X \) with the function \( f \).
### TABLE V
List of Transitive MIFs: Part I \((n \leq 9)\)

<table>
<thead>
<tr>
<th>(n)</th>
<th>(t)</th>
<th>MWC</th>
<th>Name</th>
<th>(t')</th>
<th>(\text{Aut}(\mathcal{F}))</th>
<th>Generators</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(a)</td>
<td>Dem(_1) = Dict(_1)</td>
<td>1</td>
<td>1</td>
<td>(S_1)</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>(ab)</td>
<td>Dem(_3)</td>
<td>3</td>
<td>3</td>
<td>(3T2 = S_3)</td>
<td>((abc),(ab))</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>(abc)</td>
<td>Dem(_5)</td>
<td>5</td>
<td>5</td>
<td>(5T5 = S_5)</td>
<td>((abcde),(ab))</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>(abc)</td>
<td>Icos</td>
<td>2</td>
<td>2</td>
<td>(6T12 = L(2, 5))</td>
<td>((abcde),(af)(bd))</td>
</tr>
<tr>
<td>7</td>
<td>716</td>
<td>(abcd)</td>
<td>Dem(_7)</td>
<td>7</td>
<td>7</td>
<td>(7T5 = S_7)</td>
<td>((abcdefg)(ab))</td>
</tr>
<tr>
<td>7</td>
<td>713</td>
<td>(abd)</td>
<td>Fano</td>
<td>2</td>
<td>2</td>
<td>(7T7 = L(3, 2))</td>
<td>((abcdefg)(bc)(dg))</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>31</td>
</tr>
</tbody>
</table>

**Note.** \(n\): Number of voters. \(n = |X|\); \(t\): Line number in \([4, \text{Tables 2, 3}]\); MWC: A list of representatives of the orbits of the set of minimal winning minority coalitions under the action of \(\text{Aut}(\mathcal{F})\); Name: Notation used to denote \(\mathcal{F}\); \(t\): Degree of transitivity of \(\text{Aut}(\mathcal{F})\); \(t'\): Degree of homogeneity of \(\text{Aut}(\mathcal{F})\); \(\text{Aut}(\mathcal{F})\): Designation of the automorphism group of \(\mathcal{F}\) using the notation of \([4]\) and any common name (see Table VI); Generators: A minimal set of generators of the group \(\text{Aut}(\mathcal{F})\); \#: The number of MIF on \(X\) which are isomorphic to \(\mathcal{F}\) (only one MIF is listed for each isomorphism class).

### 4.2. \(n = 6\): Icosahedral MIF

As part of the enumeration of six-player games, Gurk and Isbell \([9]\) discovered a transitive MIF which they described by its minimal winning coalitions:

\[
\text{Icos} = \{abc, acf, aef, ade, abd, bce, cef, bef, ecf, edf, bdf\}.
\]

**Icos** is the smallest transitive MIF on an even number of voters. Note that all majority coalitions are winning and all minority coalitions are losing.

### TABLE VI
Groups Appearing in Tables V, VII, VIII, and IX

<table>
<thead>
<tr>
<th>Group</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_n)</td>
<td>Symmetric group</td>
</tr>
<tr>
<td>(L(2, q))</td>
<td>Group of rational linear maps on a (q)-element field</td>
</tr>
<tr>
<td>(L(3, q))</td>
<td>Group of invertible (3 \times 3) matrices over a (q)-element field</td>
</tr>
<tr>
<td>(C_n)</td>
<td>Cyclic group</td>
</tr>
<tr>
<td>(D_{2n})</td>
<td>Dihedral group</td>
</tr>
<tr>
<td>(A_n)</td>
<td>Alternating group</td>
</tr>
<tr>
<td>(M_{11})</td>
<td>Matthieu group</td>
</tr>
<tr>
<td>(G \times H)</td>
<td>Direct product of groups (G) and (H)</td>
</tr>
<tr>
<td>(G \wr H)</td>
<td>Wreath product of (G) and (H)</td>
</tr>
</tbody>
</table>
A more intuitive characterization which highlights the 2-transitivity of the underlying automorphism group was found by Dmitri Zvonkin. Consider an icosahedron and identify opposite vertices (the resulting map is $K_6$ drawn on the projective plane!). Define the voting scheme $Icos$ to be the collection of all sets of vertices which includes a face. It is easy to see that $Icos$ is a MIF. Since the icosahedron is a platonic solid, $\text{Aut}(Icos)$ acts transitively on the edges of the icosahedron; that is, $\text{Aut}(Icos)$ is 2-transitive.

4.3. \( n = 7 \): Fano MIF

Let $\mathcal{P}_k$ be a projective plane of order $k$ on a set $X (|X| = k^2 + k + 1)$. We can define an intersecting family consisting of all collections of points which include a line $\mathcal{F}_X = \{A \subseteq X : \exists \ell \in \mathcal{P} \text{ such that } \ell \cap A\}$. This projective plane is not maximal unless $k = 2$ [23, Theorem 1], in which case $\mathcal{F} = \mathcal{P}_2$ is the Fano plane. Any two points in a projective plane determine a line, and all lines are mapped to each other by the automorphism group $L(3, 2)$. Thus, $\mathcal{F}$ is 2-transitive.

For $k > 2$, $\mathcal{F}_X$ can be extended to a transitive MIF in a number of ways (for example, by including all sets with over half the elements whose complement does not contain a line). These transitive MIFs are distinguished by the fact that they have minimal winning coalitions containing as few as $a$ elements where $n = a^2 - a + 1$. More precisely, we have the following result:

**Proposition 10 ([5, Theorem 3.a]).** Let $\mathcal{F}$ be a transitive MIF on $n$ voters.
FIG. 2. Transitive MIF: $\text{Aut}(\mathcal{F}) = 7T5$.

\[ (n > 1) \quad \text{Let } a = \min_{A \in \mathcal{F}} |A|. \text{ Then} \]
\[ a \geq 1 + \left\lfloor \sqrt{n} \right\rfloor. \]

\textbf{Proof.} Let $A \in \mathcal{F}$. Consider the orbit of $A$ under the action of $\text{Aut}(\mathcal{F})$. What is the average size $E$ of the intersection of two sets in this orbit? On one hand, $E$ is at least one, $\mathcal{F}$ is an intersecting set, so all pairs of sets intersect. Actually, $E > 1$, since $A \cup A > 1$. On the other hand, $E$ must be exactly $a \times (a/n)$ since each of the $a$ members of $A$ is mapped equally often to each of the $n$ members of $X$.)
Hence, $a \geq 1$ and $a > \sqrt{n}$. \hfill \Box

4.4. $n = 9$

4.4.1. \textit{New results.} All of the transitive MIFs mentioned above had already been known prior to our work. For $n = 8$, there are no transitive MIFs. (Any MIF on 8 voters includes 35 winning 4-element coalitions. To be transitive, each voter would have to be a member of exactly $35 \times 4/8 = 17.5$ of them, which is of course impossible.

Thus, the transitive MIFs for $n = 9$ listed in Table VII represent our first new results. (See Table VII for legend.)

For example, we see that there is a single MIF $\mathcal{F}$ with $\text{Aut}(\mathcal{F}) = 9T8$, where $9T8$ is the 8th transitive permutation group on 9 letters listed in [4]. To determine the winning coalitions in $\mathcal{F}$, we consult the column “MWC.” In this column it is indicated that $C = \{a, d, e, h\}$ is a minimal winning coalition. Acting the group $9T8$ on $C$, we find that the entire orbit

\[ [C] = \{ade, afgi, bdeg, acfg, cefh, befj, acdi, bdgh, adfi, aegi, bghi, bcei, cehi, bcfh, cdfg, cdgi, abch, abeg\} \]


<table>
<thead>
<tr>
<th>MWC Name</th>
<th>Number of missing lines</th>
<th>9T Name</th>
<th>Generators</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>See Figure 3 for 10 missing lines aefg</td>
<td>1 1</td>
<td>9T1 = C9</td>
<td>(aehbfcidg)</td>
<td>10 × 40,320</td>
</tr>
<tr>
<td>dghi</td>
<td>1 1</td>
<td>9T3 = D18</td>
<td>(aehbfcidg)</td>
<td>20,160</td>
</tr>
<tr>
<td>See Figure 5 for 5 missing lines adeh</td>
<td>1 1</td>
<td>9T4 = C7 × S3</td>
<td>(adg)(beh)(cifi)</td>
<td>5 × 20,160</td>
</tr>
<tr>
<td>aefg dghi</td>
<td>1 1</td>
<td>9T13</td>
<td>(adg)(beh)(cifi)</td>
<td>6,720</td>
</tr>
<tr>
<td>abfi</td>
<td>1 1</td>
<td>9T16 = C7 × D18</td>
<td>(abc)(def)(ghi)</td>
<td>5,040</td>
</tr>
<tr>
<td>aefg</td>
<td>1 1</td>
<td>9T18</td>
<td>(adg)(beh)(cifi)</td>
<td>3,360</td>
</tr>
<tr>
<td>bdef</td>
<td>1 1</td>
<td>9T28</td>
<td>(abc, bc)</td>
<td>560</td>
</tr>
<tr>
<td>bcde</td>
<td>Dem</td>
<td>1 1</td>
<td>9T31 = S3 × S3</td>
<td>(abc, bc)</td>
</tr>
<tr>
<td>abcd e Dem9</td>
<td>9 9</td>
<td>9T34 = S9</td>
<td>(abcdefghi)</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td>570,361</td>
</tr>
</tbody>
</table>

of minimal winning coalitions. There are other minimal winning coalitions, namely \{b, c, d, g, i\}, \{a, b, c, f, i\}, \{a, b, c, h, i\} and their orbits under the action of 9T8. However, only minimal winning coalitions containing at most half the number of voters are listed. Given the winning minority coalitions, the winning majority coalitions are simply those majority coalitions whose complement is losing.

Due to space constraints, the transitive MIFs with symmetry 9T1, 9T3, and 9T4 are listed in Figs. 3–5.
In Fig. 3, orbits are represented up to rotation by an oriented cycle \((aehbfcidg)\). (This oriented cycle is used instead of \((abedfghc)\) in order to conform to the standard notation given in [19].)

For each isomorphism class of maximal intersecting families of sets with the indicated group of symmetries, one MIF is depicted by representing its orbits (under the action of the group) of minimal winning coalitions containing at most half of the elements of \(X\). The complete MIF can then be reconstituted by symmetry, inclusion, and duality.

In Fig. 4, orbits are represented up to rotation and flip by an unoriented cycle.

In Fig. 4, Transitive MIFs: \(\text{Aut}(\mathcal{F}) = 9T3\) (dihedral symmetry).
In Fig 5, orbits are represented by a “tic-tac-toe” graph drawn on a torus, up to rotation along both axes, and reflection about the vertical axis. (It is understood that edges going off one edge of the diagram reappear on the other side.)

In Fig 6, the orbit \([ C \) in the MIF mentioned above is represented by a “tic-tac-toe” graph drawn on a torus, up to rotation and reflection about both axes.

In Fig 7, orbits are represented by the complete symmetry group of the “tic-tac-toe” graph. When drawn on the torus, this includes rotation and reflection about both axes and exchange of axes.

4.4.2. Composition. Given a MIF \( F \) on \( n \) voters, and \( n \) MIFs \( G_1, \ldots, G_n \) on disjoint sets of voters \( X_1, \ldots, X_n \), respectively. Then one can define the composition of \( F \) with \( G_1, \ldots, G_n \) to be the set of subsets \( A \) of \( X = X_1 \cup \cdots \cup X_n \) such that \( \{ t : A \cap X_t \in G_t \} \in F \). In other words, \( F [ G_1, \ldots, G_n ] \) is the voting scheme in which the voters vote by committee. Each committee votes according to its own rules \( G_t \), and results are combined via the voting scheme \( F \).

If \( F \) is a transitive game on \( n \) voters, and \( G \) is a transitive game on \( m \) voters, then
\[
F [ G_1, \ldots, G_n ]
\]
is a transitive game on \( nm \) voters. (This is essentially the proof of part 2 of Theorem 5 given by Cameron, Frankl, and Kantor [5].)

**Proposition 11** [17]. Let \( S, T_1, \ldots, T_n \) be strong simple games. If \( S [ T_1, \ldots, T_n ] \) is transitive, then \( S \) is also transitive, and the games \( T_1, \ldots, T_n \) must all be isomorphic transitive strong simple games.
FIG. 6. Transitive MIFs: $\text{Aut}(\mathcal{F}) = 9T_8$ (unoriented toroidal symmetry).

FIG. 7. Transitive MIFs: $\text{Aut}(\mathcal{F}) = 9T_{16}$ (total toroidal symmetry).

TABLE VIII
List of Transitive MIFs: Part III ($n = 10$)

<table>
<thead>
<tr>
<th>MWC</th>
<th>$t$</th>
<th>$t'$</th>
<th>$\text{Aut}(\mathcal{F})$</th>
<th>Generators</th>
<th>$#$</th>
</tr>
</thead>
<tbody>
<tr>
<td>See Figure 5</td>
<td>1</td>
<td>1</td>
<td>$10T_7 = A_5$</td>
<td>$(bf)(ce)(dg)(ij)$</td>
<td>$16 \times 60,480$</td>
</tr>
<tr>
<td>16 missing lines</td>
<td></td>
<td></td>
<td>$(aeg)(bad)(chj)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>See Figure 9 for 10 missing lines</td>
<td>1</td>
<td>1</td>
<td>$10T_8$</td>
<td>$(ab)(cd)$</td>
<td>$10 \times 45,360$</td>
</tr>
<tr>
<td>$dghj, abej$</td>
<td>2</td>
<td>2</td>
<td>$10T_{26} = L(2, 9)$</td>
<td>$(aceg)(bdhfj)$</td>
<td>$10,080$</td>
</tr>
<tr>
<td>$abef, abfi$</td>
<td>2</td>
<td>2</td>
<td></td>
<td>$(abc)(def)(ghi)$</td>
<td>$10,080$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$(bdg)(eefh)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$1,441,440$</td>
</tr>
</tbody>
</table>

FIG. 8. Transitive MIFs: $\text{Aut}(\mathcal{F}) = 10T_7$. 
In fact, the symmetries of such a composition Aut(S[T, T, ..., T]) is given by the wreath product Aut(S) ∘ Aut(T), since each committee can be permuted by Aut(T) or the committee can be permuted with each other by Aut(S).

Thus, \( \text{Dem}_2 = \text{Dem}_3[\text{Dem}_3, \text{Dem}_3, \text{Dem}_3] \) is a simply transitive MIF on 9 voters with \( \text{Aut(} \text{Dem}_2) = S_3 \setminus S_3 \).

All MIFs can be expressed as trivial compositions

\[
\mathcal{F} = \text{Dem}_3[\mathcal{F}]
= \mathcal{F}[\text{Dem}_1, ..., \text{Dem}_3]
\]

**TABLE IX**

List of Transitive MIFs: Part IV (\( n = 11 \))

<table>
<thead>
<tr>
<th>MWC</th>
<th>Name</th>
<th>( t )</th>
<th>( m )</th>
<th>Aut(( \mathcal{F} ))</th>
<th>Generators</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>57,196 missing lines</td>
<td>1</td>
<td>1</td>
<td>11T1 = C_{11}</td>
<td>(abcdefgfhj)</td>
<td>57,196 \times 3,628,800</td>
<td>43 \times 1,814,400</td>
</tr>
<tr>
<td>See Figure 10 for 48 missing lines</td>
<td>1</td>
<td>1</td>
<td>11T2 = D_{22}</td>
<td>(abcdefgfhj) (bk,i,cj,di,eh,if,fg)</td>
<td>11 \times 725,760</td>
<td>2 \times 362,880</td>
</tr>
<tr>
<td>See Figure 11 for 11 missing lines</td>
<td>1</td>
<td>2</td>
<td>11T3 = C_5 \times C_{11}</td>
<td>(abcdefgfhj) (beful)calk,hj</td>
<td>11 \times 725,760</td>
<td>2 \times 362,880</td>
</tr>
<tr>
<td>bdefh</td>
<td>1</td>
<td>2</td>
<td>11T4 = C_{10} \times C_{11}</td>
<td>(abcdefgfhj) (bceijfjhidg)</td>
<td>2 \times 362,880</td>
<td>5040</td>
</tr>
<tr>
<td>adefh</td>
<td>2</td>
<td>4</td>
<td>11T6 = M_{11}</td>
<td>(abc)(def)</td>
<td>ghi (bakeg)j(ef)hi</td>
<td>207,688,650,161</td>
</tr>
<tr>
<td>abdef</td>
<td>\text{Dem}_{11}</td>
<td>11</td>
<td>11</td>
<td>11T8 = S_{11}</td>
<td>(abcdefgfhj)</td>
<td>1</td>
</tr>
</tbody>
</table>

Total | 207,688,650,161 |
FIG. 10. Transitive MIFs: Aut(\(\mathcal{F}\)) = 11T2.

FIG. 11. Transitive MIFs: Aut(\(\mathcal{F}\)) = 11T3.
FIG. 12. Lattice of orbits $G = 9T4$. 
involving a single “committee” or a large number of one-person “committees.”
A MIF is said to be prime if it cannot be expressed as a composition \( F[\%_1, \ldots, \%_n] \) in any other way [17].

**Proposition 12** [17]. *All 2-transitive strong-simple games are prime strong-simple games.*

4.5. \( n = 10 \)

Transitive MIFs for \( n = 10 \) are tabulated in Table VIII. (See Table V for legend.) As we saw before in the case of Icos (for \( n = 6 \)), an even number of voters is not necessarily a barrier to the existence of a transitive MIF. In fact, there are even 12 classes of transitive MIFs with at least one winning minority coalition.

Due to space constraints, the transitive MIFs with symmetries 10\( T^7 \) and 10\( T^8 \) are listed in Figs. 8 and 9. They are classified according to how many minority winning coalition they include (4 elements).

4.6. \( n = 11 \)

Transitive MIFs for \( n = 11 \) are tabulated in Table IX. (See Table V for legend.) Due to space constraints, the 57,196 isomorphism classes of transitive MIFs with symmetry 11\( T^1 \) have been omitted. They each have 11-fold cyclic symmetry.

Transitive MIFs with symmetries 11\( T^2 \) and 11\( T^3 \) are listed in Figs. 10 and 11, respectively.

In Fig. 12, orbits are represented by an unoriented cycle up to rotation and reflection.

5. SEARCH TECHNIQUES

In this section, we give details concerning our computer search for transitive MIFs. We believe that other symmetrical combinatorial structures can be enumerated by similar techniques. For example, McKay and others [19, 20, 27] have enumerated graphs with vertex transitive automorphism groups and up to 26 vertices.

5.1. Tree Search

We first identified and eliminated the transitive groups which were liable to be the automorphism group of some MIF. We began with the catalog of transitive groups of degree up to 11 by Butler and McKay [4]. (For 11 \( < n < 23 \), one can use the program Gap. For \( n = 25 \) or 26, see A. Hulpke [28].) Groups which contained a fixed-point free 2-element
were then identified by using the table of group elements according to cyclic decomposition type.

A program was written in Caml Special Light (CSL) [12] that given a group $G$ would generate the set $S_G$ of all MIFs $\mathcal{F}$ with $G \subseteq \text{Aut}(\mathcal{F})$. To do this, the computer calculated the orbits of the action of $G$ on the power-set of $X$ by applying the generators of $G$ repeatedly to the subsets of $X$. Given an orbit $\alpha$, define its dual $\alpha^*$ to be the orbit generated by $[X \setminus A]$, where $A \in \alpha$.

We never have $\alpha = \alpha^*$ since $G$ is assumed to contain no fixed-point free 2-elements. There is no point in studying such groups as they are not the automorphism groups of any MIF. For each pair $(\alpha, \alpha^*)$ we must decide whether $\alpha$ or $\alpha^*$ will be in our MIF.

We write $\alpha \supseteq \beta$ if there is some $A \in \alpha$ and $B \in \beta$ such that $B \subseteq A$. (See Fig. 12.) If $\alpha \supseteq \alpha^*$, then $\alpha$ is a subset of every MIF in $S_G$.

Our main algorithm considers an orbit $\alpha = [A]$, where $A$ has minimal cardinality among those orbits still under consideration. If $\alpha \leq \alpha^*$, then $\alpha$ must be rejected and $\alpha^*$ must be included as above. Otherwise, we either reject $\alpha$ and accept $\alpha^*$, or else we accept $\alpha$ and we reject all orbits $\beta \leq \alpha^*$.

In both cases, a recursive call to the algorithm allows us to determine the possible ways to treat the remaining orbits.

If only the number of solutions $|S_G|$ is required, then a dynamic programming algorithm can be used by creating a hash table (remember table) and using it to treat most of the recursive calls. Using a 50-Mb table, and taking advantage of the obvious 8-fold symmetry, we were able to compute all 8-voter MIFs in only 12 min., whereas an analogous algorithm by Bioch and Ibaraki [3] required 6 months.

5.2. Inclusion–Exclusion

The set $S_G$ includes all MIFs with at least the group $G$ as symmetry. To compute the number of MIFs with exactly the group $G$ as symmetry, inclusion–exclusion techniques (Möbius inversion) are required.

$S_n$ acts on its subgroups by conjugation. Given a permutation group $G \subseteq S_n$, let $Z'(G) = \{ \sigma \in S_n : \sigma G = G \sigma \}$. $Z'(G)$ includes the center $Z(G)$ (and in fact $G$ itself) but also possibly other elements since we require not that $\sigma$ commute with each element of $G$ but simply that $\sigma$ commute with $G$ itself. Thus, there are $n!/|Z'(G)|$ conjugate copies of the group $G$ in $S_n$.

By restricting this action to a group $H (G \subseteq H \subseteq S_n)$, the stabilizer is $Z'(G) \cap H$. We thus deduce that there are $|H|/|Z'(G) \cap H|$ conjugate copies of the group $G$ in $H$.

For each pair of groups $(G, H)$ under consideration for which we found at least one nondemocratic MIF, we calculated $|Z'(G) \cap H|$ using a short program in CSL. The generators of $G$ were conjugated the generators of $H$. The result was then compared to a list of elements of $G$. 
Since the \( n!/|Z'(H)| \) copies of \( H \) contain \( |H|/|Z'(G) \cap H| \) copies of \( G \) and there are \( n!/|Z'(G)| \) copies of \( G \), it then follows that each conjugate copy of \( G \) is contained in exactly

\[
M_{GH} = \frac{n!}{|Z'(H)|} \times \frac{|H|}{|Z'(G) \cap H|} \times \frac{n!}{|Z'(G)|} = \frac{|H| \times |Z'(G)|}{|Z'(H)| \times |Z'(G) \cap H|}
\]

copies of \( H \).

**Proposition 13.** Given the unitriangular matrix \( M \) above and the vector \( v = v_G \) (indexed by certain subgroups of \( S_n \)), we have the identity

\[
v = M_w, \tag{1}
\]

where \( w_G \) is the number of MIF \( \mathcal{F} \) with \( \text{Aut}(\mathcal{F}) = G \).

**Proof.**

\[
v_G = \sum_{\mathcal{F} \in S_n} 1 = \sum_{G \in \text{Aut}(\mathcal{F})} 1 = \sum_{G \in H \in \text{Aut}(\mathcal{F})} \sum_{H} 1 = M_{GH} w_H.
\]

Using the Maple `linalg` package, we solved (1) for \( w \). For example, in the case \( n = 11 \) (see Table IX), we solved

\[
\begin{bmatrix}
572227 \\
243 \\
27 \\
3 \\
2 \\
1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 2 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 2 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4 \\
w_5 \\
w_6
\end{bmatrix}
\]

to find \( w = [571960, 240, 22, 2, 1, 1]^T \).

Among the \( v_G \) MIFs we found with symmetry at least \( G \), there are \( w_G \) with symmetry exactly \( G \). However, some of these may be isomorphic.

**Proposition 14.** The MIFs with automorphism group \( G \) are divided into \( w_G |G|/|Z'(G)| \) isomorphism classes each of size \( |Z'(G)|/|G| \).
Proof. Let $\text{Aut}(\mathcal{F}) = G$. The $n!/[G]$ isomorphic images $\sigma \mathcal{F}$ of $\mathcal{F}$ have automorphism groups $\text{Aut}(\sigma \mathcal{F}) = \sigma G \sigma^{-1}$ which are conjugates of $G$. There are $n!/[Z'(G)]$ conjugates of $G$ which are all identical up to permutations of $X$. Thus, there are $|Z'(G)|/[G]$ isomorphic images of $\mathcal{F}$ with automorphism group exactly equal to $G$. □

Dividing by the appropriate quantities we know, for example, that the group $1171$ is the automorphism group of $57,196$ collections of $10$ isomorphic MIFs.

5.3. Statistics

To identify the various isomorphism classes, we used a collection of Maple routines to apply various statistics (invariant under permutation of $X$) to the MIFs generated. Typical statistics included:

- The distribution of winning coalitions according to coalition size.
- The distribution of minimal winning coalitions according to coalition size.
- The set of numbers $c_j (i' < j \leq n)$, where $c_j$ is the number of winning coalitions containing $1, 2, 3, \ldots, i'$ and $j$, and $i'$ is the homogeneity of the group.

In a handful of cases, no easily computed statistic could distinguish certain isomorphism classes. In those cases, coset representatives were found for the quotient $Z'(G)/G$ using the cosets function in the Maple group package. By applying these permutations to these MIFs using the procedure subs, we were able to compute the isomorphism classes as they are simply the orbits of the action of $Z'(G)$ on these MIFs.

REFERENCES

27. H. P. Yap, Point-symmetric graphs with $P \leq 13$ points, Nanta Math. 6 (1973), 8-20.