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A new semidirect product for association schemes

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ABSTRACT

We propose a new definition for the semidirect product of association schemes, generalizing work of Bang, Hirasaka, and Song. We define an action of one scheme on another, and show how one can use such an action to construct all semidirect products satisfying a certain technical condition.

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1. Introduction

We begin with an overview of the theory of association schemes, and the connection of that theory with groups. Details for what follows can be found in Zieschang's Theory of Association Schemes [1].

Given a set X , an association scheme on X , or a scheme on X for short, is a set S consisting of nonempty subsets of $X \times X$, satisfying the following axioms:

1. S is a partition of $X \times X$.
2. The diagonal subset $1_X \subseteq X \times X$, defined as $\{(x, x) : x \in X\}$, is an element in S .
3. For each element $s \in S$, the set $s^* = \{(y, x) \in X \times X : (x, y) \in s\}$ is an element in S .
4. For each triple $p, q, r \in S$, and each element $(x, y) \in r$, the cardinality of the set of elements $z \in X$ such that $(x, z) \in p$ and $(z, y) \in q$ depends only on p, q , and r (and not on x and y). The cardinal number for this set is denoted a_{pqr} .

We will assume throughout that X is a finite set, so that each a_{pqr} is a nonnegative integer.

For example, suppose given a finite group G . For each $g \in G$, let s_g be the set of all pairs $(x, y) \in G \times G$ such that $y = xg$. Let $S_G = \{s_g : g \in G\}$. Then:

1. S_G is a partition of $G \times G$, since for each pair $(x, y) \in G$, we have $(x, y) \in s_g$ if and only if $g = x^{-1}y$.

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2. 1_G is equal to s_1 , where $1 \in G$ is the identity element.
3. For each $g \in G$, $s_g^* = s_{g^{-1}}$, since $y = xg$ if and only if $x = yg^{-1}$.
4. For each triple $g_1, g_2, g_3 \in G$, and each pair $(x, y) \in s_{g_3}$, the number of elements $z \in G$ such that $(x, z) \in s_{g_1}$ and $(z, y) \in s_{g_2}$ is 1 if $g_1g_2 = g_3$ and 0 otherwise.

The scheme S_G on G has the property that for each $p \in S$ and $x \in G$, there is exactly one $y \in G$ such that $(x, y) \in p$. Schemes with this property are called thin schemes, and any thin scheme is isomorphic to a scheme obtained from a group as described above. Thus, schemes may be viewed as a generalization of groups.

Many concepts in group theory, like products, subgroups, and quotient groups, have corresponding generalizations to scheme theory. One can define the product pq of two elements p and q in a scheme S , but in this case, pq is not an element of S , but a subset of S ; that is pq is the set of all r such that $a_{pqr} > 0$. Similarly, given subsets P and Q in S , we can define PQ to be the union over all pairs $p \in P, q \in Q$ of the sets pq . We also write pQ for $\bigcup_{q \in Q} pq$ and Pq for $\bigcup_{p \in P} pq$.

Corresponding to subgroups of a group are the closed subsets of a scheme. A subset $T \subseteq S$ is said to be a closed subset if $T^*T \subseteq T$, where $T^* = \{t^* \in S : t \in T\}$. If S_G is the thin scheme arising from a group G , then the closed subsets of S_G are precisely the sets of the form $S_H = \{s_g \in S_G : g \in H\}$, where H is a subgroup of G . As with groups, we say that a closed subset T is normal if $pT = Tp$ for all $p \in S$.

Just as subgroups partition groups into cosets, a closed subset T of a scheme S on a set X partitions X into a set of cosets. For $x \in X$ and $t \in T$, we write xt for the set of all $y \in X$ such that $(x, y) \in t$. We then write xT for $\bigcup_{t \in T} xt$. Thus, x_1 and x_2 are in the same coset if the pair (x_1, x_2) belongs to an element in T . We denote the set of all cosets of T in X as X/T . One obtains a scheme T_{xT} on xT , called *the subscheme of S defined by xT* , whose elements are the sets $t \cap (xT \times xT)$, where $t \in T$.

For each $s \in S$, we define s^T to be the set of all pairs $(x_1T, x_2T) \in X/T \times X/T$ such that $(x'_1, x'_2) \in s$ for some $x'_1 \in x_1T$ and $x'_2 \in x_2T$. The set of all such sets s^T then forms a scheme $S//T$ on X/T . In contrast to group theory, a scheme admits a quotient for any closed subset T , not just for normal closed subsets.

Finally, we can define a morphism between two schemes. Given schemes S on X and S' on X' , a morphism $\phi : S \rightarrow S'$ is a function $\phi_X : X \rightarrow X'$ such that if (x_1, x_2) and (x'_1, x'_2) belong to the same scheme element in S , then $(x_1\phi_X, x_2\phi_X)$ and $(x'_1\phi_X, x'_2\phi_X)$ belong to the same scheme element in S' . Note that ϕ_X gives rise to a function ϕ_S from the set of scheme elements S to the set of scheme elements S' : given $s \in S$, choose $(x_1, x_2) \in s$, and define $s\phi_S$ to be the scheme element in S' containing $(x_1\phi_X, x_2\phi_X)$. Our definition ensures that ϕ_S is well defined. As an example, if T is a closed subset of a scheme S defined on X , then there is a canonical morphism $\pi : S \rightarrow S//T$. The function $\pi_X : X \rightarrow X//T$ takes a given $x \in X$ to its coset xT . The function π_S takes s to s^T . Clearly, if $(x, y) \in s$, then $(xT, yT) \in s^T$. If $\phi : S \rightarrow S'$ is a morphism of schemes, we define the kernel of ϕ to be the set of $s \in S$ such that $s\phi = 1_{X'}$. As with groups, there is an induced morphism $\bar{\phi} : S//T \rightarrow S'$, given by $(xT)\bar{\phi}_{X//T} = x\phi$ and $(s^T)\bar{\phi}_{S//T} = s\phi_S$. Thus, $\pi\bar{\phi} = \phi$. Given a morphism of schemes $\phi : S \rightarrow S'$, we will often use ϕ to denote ϕ_X or ϕ_S , relying on context to make the meaning clear.

Since schemes generalize groups, it is natural to look for appropriate generalizations of various concepts from group theory to scheme theory. One classical problem in group theory is the extension problem. Given a pair of groups N and H , one seeks to classify those groups G with a given normal subgroup \tilde{N} isomorphic to N , and quotient G/\tilde{N} isomorphic to H . Such a group is called an extension of N by H . In the same way, one could pose the following problem: given two schemes T and U , classify the schemes S which are equipped with a closed subset \tilde{T} , such that the quotient scheme $S//\tilde{T}$ is isomorphic to U , and such that the subscheme of S defined by one of the cosets of \tilde{T} is isomorphic to the scheme T .

The simplest class of extensions of a group N by another group H is the class of semidirect products of N and H . In considering the extension problem for schemes, it is natural to seek a generalization of the semidirect product, which should satisfy a certain collection of properties similar to those of a semidirect product of groups. Bang, Hirasaka, and Song [2] have already proposed one construction for semidirect products. Our goal in this paper is to propose a more general construction.

To understand our motivation, we must examine semidirect products of groups more closely. In group theory, given a pair of groups H and N together with a homomorphism $\zeta : H \rightarrow \text{Aut}(N)$, one defines the semidirect product $G = N \rtimes_{\zeta} H$ as the group whose underlying set of elements is the Cartesian product $N \times H$, with product defined by

$$(n_1, h_1)(n_2, h_2) = (n_1(n_2(h_1\zeta)), h_1h_2).$$

Then G contains a normal subgroup $\tilde{N} = \{(n, 1_H) : n \in N\}$, which is isomorphic to N , and G/\tilde{N} is isomorphic to H . We have the following properties:

1. G contains a subgroup \tilde{H} isomorphic to H , $G = \tilde{N}\tilde{H}$ and $\tilde{H} \cap \tilde{N} = \{1\}$.
2. The homomorphism $\pi : G \rightarrow G/\tilde{N}$ splits, in the sense that there is a homomorphism $i : H \rightarrow G$ such that $i\pi : H \rightarrow G/\tilde{N}$ is an isomorphism.

In fact, given a group G containing a normal subgroup \tilde{N} isomorphic to N , such that G/\tilde{N} is isomorphic to H , then properties (1) and (2) are equivalent, and each implies that G is isomorphic to $N \rtimes_{\zeta} H$ for some $\zeta : H \rightarrow \text{Aut}(N)$.

Returning to schemes, there are then two approaches one might take to generalize semidirect products. On the one hand, one could say that a scheme S on a set X is a semidirect product of T by U if S contains a normal closed subset \tilde{T} with a subscheme isomorphic to T , as well as a closed subset \tilde{U} isomorphic to U , such that $\tilde{U} \cap \tilde{T} = \{1_X\}$ and $\tilde{T}\tilde{U} = S$. Bang, Hirasaka, and Song [2] describe a construction that takes two schemes T and U , together with a kind of twisting map ζ , and outputs a scheme $S = T \rtimes_{\zeta} U$ with the above properties. Zieschang [1, 7.4] proves that any scheme satisfying the conditions above, together with one extra condition (which holds automatically if T is thin), is isomorphic to a scheme arising from their construction. The extra required condition is

Condition 1.1. For any $u \in \tilde{U}$ and $t \in \tilde{T}$, we have $|tu| = 1$.

We now present a second approach to defining semidirect products, which we will adopt in this paper.

Definition 1.2. A scheme S on a set X is a *semidirect product of T by U* if S contains a closed subset \tilde{T} , there is a morphism $i : U \rightarrow S$ such that $i\pi : U \rightarrow S//\tilde{T}$ is an isomorphism, and the subscheme of S defined by one of the cosets of \tilde{T} is isomorphic to T .

Note that the image of a scheme morphism $i : U \rightarrow S$ need not determine a closed subset of S ; that is, the image of $i_U : U \rightarrow S$ need not be closed. On the other hand, if S contains a normal closed subset \tilde{T} , as well as a closed subset \tilde{U} isomorphic to U , such that $\tilde{U} \cap \tilde{T} = \{1_X\}$, and $\tilde{T}\tilde{U} = S$, then the inclusion of the subscheme of S defined by any of the cosets of \tilde{U} , composed with the projection $\pi : S \rightarrow S//\tilde{T}$, is an isomorphism of schemes, and thus determines a splitting of π . Therefore, Definition 1.2 is more general than the semidirect product defined by Bang, Hirasaka, and Song.

Our goal in this paper is to show how one can take two schemes T and U , together with a kind of twisting map ζ , and construct an extension S of T by U which is a semidirect product in the sense of Definition 1.2. We will show that every semidirect product satisfying one extra condition (analogous to Condition 1.1 above) is isomorphic to a scheme arising from our construction. Our extra condition is

Condition 1.3. For any $u \in U$ and $t \in \tilde{T}$, we have $|t(ui)| = 1$.

Bang and Song [3] also give a generalization of their own semidirect product. As we show in Section 6, it is possible to describe a semidirect product using our approach that cannot be obtained as a generalized semidirect product in their sense. On the other hand, any generalized semidirect product in their sense would arise from our construction.

In Sections 2 and 3, we work through some preliminary definitions which we will need in order to construct our semidirect product. In Section 4, we define an action of one scheme on another,

and show how this can be used to construct a semidirect product, much the way one can use a homomorphism from a group G to the automorphisms of a group N to construct a semidirect product in group theory. We discuss properties of our semidirect product in Section 5 and provide an example in Section 6. In the final section, Section 7, we show how, given Condition 1.3, one can obtain an action of one scheme on another from a given semidirect product, and then reconstruct this given semidirect product up to isomorphism using our construction.

2. A category of schemes

A based set X is a set together with a distinguished element $x_* \in X$, called the basepoint. A based scheme is a scheme T on a based set X . If T and U are based schemes (on based sets X and Y), then a morphism of schemes $\phi : T \rightarrow U$ is a based morphism if it takes the basepoint of X to the basepoint of Y . Any quotient of a based scheme is naturally a based scheme, where we take the basepoint of the quotient to be the coset containing the basepoint of the original scheme.

Of course, one could define a category whose objects were based schemes and whose morphisms were the above morphisms of based schemes. In particular, this is the category \mathcal{AS}_0 discussed by Hanaki in [4]. We now define a different category \mathcal{C} of based schemes. An object of \mathcal{C} is a based scheme T on X . Given two based schemes T and U , a morphism $\phi \in \text{Hom}_{\mathcal{C}}(T, U)$ consists of

1. a normal closed subset T_ϕ in T ,
2. a normal closed subset U_ϕ in U ,
3. a based isomorphism of schemes $\tilde{\phi} : T//T_\phi \rightarrow U//U_\phi$.

To define composition of morphisms in \mathcal{C} , suppose given $\phi \in \text{Hom}_{\mathcal{C}}(T, U)$ and $\psi \in \text{Hom}_{\mathcal{C}}(U, V)$, where T, U , and V are schemes on X, Y , and Z respectively. We define

$$T_{\phi\psi} = \{t \in T : \text{for some } u \in U \text{ we have } t^{T_\phi} \tilde{\phi} = u^{U_\phi} \text{ and } u^{U_\psi} \tilde{\psi} = 1_Z^{V_\psi}\},$$

$$V_{\phi\psi} = \{v \in V : \text{for some } u \in U \text{ we have } 1_X^{T_\phi} \tilde{\phi} = u^{U_\phi} \text{ and } u^{U_\psi} \tilde{\psi} = v^{V_\psi}\}.$$

In Lemma 2.1 below, we will show that $T_{\phi\psi}$ and $V_{\phi\psi}$ are closed subsets. Now, we define a function $\widehat{\phi\psi}_{X/T_{\phi\psi}} : X/T_{\phi\psi} \rightarrow Z/V_{\phi\psi}$ as follows. Given $xT_{\phi\psi} \in X/T_{\phi\psi}$, choose a coset representative y so that $(xT_\phi)\tilde{\phi} = yU_\phi$. Then, choose a coset representative z so that $(yU_\psi)\tilde{\psi} = zV_\psi$. Let $(xT_{\phi\psi})(\widehat{\phi\psi}_{X/T_{\phi\psi}}) = zV_{\phi\psi}$. We will let $\phi\psi \in \text{Hom}_{\mathcal{C}}(T, V)$ be the morphism determined by $T_{\phi\psi}$, $V_{\phi\psi}$ and $\widehat{\phi\psi}$. The following lemma assures us that $\phi\psi$ is actually in $\text{Hom}_{\mathcal{C}}(T, V)$.

Lemma 2.1. *The sets $T_{\phi\psi}$ and $V_{\phi\psi}$ defined above are normal closed subsets of T and V . The function $\widehat{\phi\psi}_{X/T_{\phi\psi}}$ defined above is well defined and determines a based isomorphism of schemes $\tilde{\phi\psi} : T//T_{\phi\psi} \rightarrow V//V_{\phi\psi}$. Finally, we have $t^{T_{\phi\psi}} \widehat{\phi\psi} = v^{V_{\phi\psi}}$ if and only if there exists a $u \in U$ such that $t^{T_\phi} \tilde{\phi} = u^{U_\phi}$ and $u^{U_\psi} \tilde{\psi} = v^{V_\psi}$.*

Proof. We first recall two facts about quotients of schemes (see Lemmas 4.1.7 and 4.2.4 in Zieschang’s work [1]). First, whenever R is a nonempty subset of a scheme S with a closed subset T , the quotient $R//T$ is closed if and only if R is closed. Also, if T and U are closed subsets of S with $T \subseteq U$, then $U//T$ is normal in $S//T$ if and only if U is normal in S .

Since U_ϕ and U_ψ are normal and closed, then $U_\phi U_\psi$ is normal and closed, whence by the above comments, $(U_\phi U_\psi)//U_\phi$ is normal and closed in $U//U_\phi$. Now, observe that $T_{\phi\psi}$ contains T_ϕ , and $T_{\phi\psi}//T_\phi$ coincides precisely with the preimage of $(U_\phi U_\psi)//U_\phi$ under $\tilde{\phi}$. Since $\tilde{\phi}$ is an isomorphism, it follows that $T_{\phi\psi}//T_\phi$ is normal and closed in $T//T_\phi$, so $T_{\phi\psi}$ is normal and closed in T . A similar argument shows that $V_{\phi\psi}$ is normal and closed.

Now, $T_{\phi\psi}$ is the kernel of the composition

$$T \rightarrow T//T_\phi \xrightarrow{\tilde{\phi}} U//U_\phi \rightarrow U//U_\phi U_\psi$$

where the first and third morphisms are the natural quotient morphisms. There is then an induced isomorphism of schemes

$$\bar{\phi} : T // T_{\phi\psi} \rightarrow U // U_{\phi} U_{\psi}.$$

The function $\bar{\phi}_{X/T_{\phi\psi}}$ takes an element $xT_{\phi\psi} \in X/T_{\phi\psi}$ to the element $y(U_{\phi}U_{\psi}) \in Y/(U_{\phi}U_{\psi})$, where $y \in Y$ is any coset representative of $(xT_{\phi\psi})\bar{\phi} \in Y/U_{\phi}$. Likewise, the function $\bar{\phi}_{T//T_{\phi\psi}}$ takes $t^{T_{\phi\psi}} \in T//T_{\phi\psi}$ to the element $u^{U_{\phi}U_{\psi}}$ where $u^{U_{\phi}} = t^{T_{\phi\psi}}\bar{\phi}$. Since $\bar{\phi}$ is a based morphism, $\bar{\phi}$ preserves the basepoint.

Similarly, $V_{\phi\psi}$ is the kernel of the composition

$$V \rightarrow V // V_{\psi} \xrightarrow{\bar{\psi}^{-1}} U // U_{\psi} \rightarrow U // U_{\phi} U_{\psi}.$$

There is again an induced isomorphism

$$\overline{\psi}^{-1} : V // V_{\phi\psi} \rightarrow U // U_{\phi} U_{\psi}.$$

If we then let $\bar{\psi}$ denote the inverse of $\overline{\psi}^{-1}$, then $\bar{\psi}_{Y/U_{\phi}U_{\psi}}$ takes an element $y(U_{\phi}U_{\psi})$ to $zV_{\phi\psi}$, where z is any coset representative of $(yU_{\psi})\bar{\psi}$. Likewise, $\bar{\psi}$ takes $u^{U_{\phi}U_{\psi}}$ to $v^{V_{\phi\psi}}$ where $u^{U_{\psi}}\bar{\psi} = v^{V_{\psi}}$. As above, $\bar{\psi}$ preserves the basepoint.

Now notice that $(\bar{\phi}\bar{\psi})_{X/T_{\phi\psi}}$ coincides with $\widetilde{\phi\psi}_{X/T_{\phi\psi}}$. The second statement of the lemma therefore follows since $\bar{\phi}\bar{\psi}$ is a based isomorphism of schemes. For the final statement, we have $t^{T_{\phi\psi}}\bar{\phi}\bar{\psi} = v^{V_{\phi\psi}}$ if and only if $t^{T_{\phi\psi}}\bar{\phi}\bar{\psi} = v^{V_{\phi\psi}}$. If there is a $u \in U$ such that $t^{T_{\phi\psi}}\bar{\phi} = u^{U_{\phi}}$ and $u^{U_{\psi}}\bar{\psi} = v^{V_{\psi}}$, then by our above descriptions of $\bar{\phi}_{T//T_{\phi\psi}}$ and $\bar{\psi}_{U//U_{\phi}U_{\psi}}$, we have $t^{T_{\phi\psi}}\bar{\phi} = u^{U_{\phi}}$ and $u^{U_{\phi}U_{\psi}}\bar{\psi} = v^{V_{\phi\psi}}$, so indeed $t^{T_{\phi\psi}}\bar{\phi}\bar{\psi} = v^{V_{\phi\psi}}$. On the other hand, if $t^{T_{\phi\psi}}\bar{\phi}\bar{\psi} = v^{V_{\phi\psi}}$, then for some $u_0 \in U$ and $v_0 \in V$, we have $t^{T_{\phi\psi}}\bar{\phi} = u_0^{U_{\phi}}$ and $u_0^{U_{\psi}}\bar{\psi} = v_0^{V_{\psi}}$, where $v_0^{V_{\phi\psi}} = v^{V_{\phi\psi}}$. Since $V_{\phi\psi}$ is normal, then $v \in v_0v_1$ for some $v_1 \in V_{\phi\psi}$. By definition of $V_{\phi\psi}$, there is a $u_1 \in U_{\phi}$ such that $u_1^{U_{\psi}}\bar{\psi} = v_1^{V_{\psi}}$. Thus, $(u_0^{U_{\psi}}u_1^{U_{\psi}})\bar{\psi}$ contains $v^{V_{\psi}}$, so we can find $u \in u_0u_1$ such that $u^{U_{\psi}}\bar{\psi} = v^{V_{\psi}}$. Since $u_1 \in U_{\phi}$, we have $u^{U_{\phi}} = u_0^{U_{\phi}}$, so $t^{T_{\phi\psi}}\bar{\phi} = u^{U_{\phi}}$ and $u^{U_{\psi}}\bar{\psi} = v^{V_{\psi}}$, as needed. \square

The following lemma now implies that C as defined above is a category.

Lemma 2.2. *The composition defined above is associative. Also, if T is a scheme on X , and we let T_{id_T} be $\{1_X\}$ and id_T denote the identity morphism on $T//\{1_X\}$, then id_T is the identity morphism for T in C .*

Proof. The second statement is easy, so we only prove the first. We suppose that S, T, U and V are schemes on based sets W, X, Y , and Z respectively, and ϕ, χ and ψ are morphisms in $\text{Hom}_C(S, T)$, $\text{Hom}_C(T, U)$, and $\text{Hom}_C(U, V)$ respectively. We must show that $(\phi\chi)\psi = \phi(\chi\psi)$. Using the third statement of Lemma 2.1, it is straightforward to show that both $S_{(\phi\chi)\psi}$ and $S_{\phi(\chi\psi)}$ coincide with the set

$$\{s \in S : \text{for some } t \in T \text{ and } u \in U \text{ we have } s^{S_{\phi\psi}}\bar{\phi} = t^{T_{\phi}}, t^{T_{\chi}}\bar{\chi} = u^{U_{\chi}} \text{ and } u^{U_{\psi}}\bar{\psi} = 1_Z^{V_{\psi}}\}.$$

Thus, $S_{(\phi\chi)\psi} = S_{\phi(\chi\psi)}$. By a similar argument, $V_{(\phi\chi)\psi} = V_{\phi(\chi\psi)}$. Finally, we must show $(\widetilde{\phi\chi})\bar{\psi} = \widetilde{\phi(\chi\psi)}$, and for this, it suffices to show that $(\widetilde{\phi\chi})\bar{\psi}_{W/S_{(\phi\chi)\psi}} = \widetilde{\phi(\chi\psi)}_{W/S_{\phi(\chi\psi)}}$. But given $w \in W$, we may choose $x \in X, y \in Y$, and $z \in Z$ such that $(wS_{\phi})\bar{\phi} = xT_{\phi}$, $(xT_{\chi})\bar{\chi} = yU_{\chi}$, and $(yU_{\psi})\bar{\psi} = zV_{\psi}$. Then by the definition we gave immediately before the statement of Lemma 2.1, both $(\widetilde{\phi\chi})\bar{\psi}_{W/S_{(\phi\chi)\psi}}$ and $\widetilde{\phi(\chi\psi)}_{W/S_{\phi(\chi\psi)}}$ take w to $zV_{(\phi\chi)\psi} = zV_{\phi(\chi\psi)}$. \square

Given two based schemes T and U on based sets X and Y , the set $\text{Hom}_{\mathcal{C}}(T, U)$ can be given the structure of a partially ordered set. Given two morphisms $\phi, \psi \in \text{Hom}_{\mathcal{C}}(T, U)$, we will say that $\phi \leq \psi$ if $T_{\phi} \subseteq T_{\psi}$, $U_{\phi} \subseteq U_{\psi}$, and for each $x \in X$, $(xT_{\phi})\tilde{\phi} \subseteq (xT_{\psi})\tilde{\psi}$.

Notation 2.3. If $\phi \in \text{Hom}_{\mathcal{C}}(T, U)$, then we have a morphism $\phi^* \in \text{Hom}_{\mathcal{C}}(U, T)$ defined as follows: $T_{\phi^*} = T_{\phi}$, $U_{\phi^*} = U_{\phi}$, and $\tilde{\phi}^* = \tilde{\phi}^{-1}$.

Note that if $\phi \in \text{Hom}_{\mathcal{C}}(T, U)$, then the composition $\phi\phi^* \in \text{Hom}_{\mathcal{C}}(T, T)$ will not typically be the identity. Instead, both normal closed subsets of $\phi\phi^*$ coincide with T_{ϕ} , and $\phi\phi^*$ is the identity morphism on the scheme $T//T_{\phi}$. Thus, $\text{id}_T \leq \phi\phi^*$.

3. Labeling sets

Definition 3.1. A *weak labeling set* is a set τ equipped with an involution $*$: $\tau \rightarrow \tau$, a distinguished element $1 \in \tau$ fixed by $*$, and for each triple $p, q, r \in \tau$, a nonnegative integer a_{pqr}^{τ} . We write p^* for the image of p under $*$.

Definition 3.2. Given a weak labeling set τ , a τ -*scheme* is a pair (T, α) , where T is a scheme on a based set X , and $\alpha : \tau \rightarrow T$ is a bijection satisfying the following:

1. $1\alpha = 1_X$,
2. for each $p \in \tau$, $(p^*)\alpha = (p\alpha)^*$,
3. for each triple $p, q, r \in \tau$, $a_{pqr}^{\tau} = a_{(p\alpha)(q\alpha)(r\alpha)}$.

Definition 3.3. A *labeling set* is a weak labeling set τ for which there exists a τ -scheme.

For example, a scheme T on a set X determines a labeling set τ_T by forgetting about the underlying set X . That is, the set τ_T is equal to the set T , the involution in τ_T is equal to the involution on T , and the distinguished element $1 \in \tau_T$ is equal to 1_X . Finally, if $p, q, r \in \tau_T = T$, then a_{pqr}^{τ} is the corresponding structure constant for T . As another example, if S is a scheme on a set X , and T is a closed subset of S , and $\tau = \tau_T$, then for each $x \in X$, the subscheme of S defined by xT is a τ -scheme: the bijection α takes $t \in \tau_T = T$ to $t \cap (xT \times xT)$.

Remark 3.4. If (T, α) and (U, β) are both τ -schemes, and $r, p, q \in \tau$, then $r\alpha \in (p\alpha)(q\alpha)$ if and only if $r\beta \in (p\beta)(q\beta)$. Indeed, $r\alpha \in (p\alpha)(q\alpha)$ if and only if $a_{(p\alpha)(q\alpha)(r\alpha)} > 0$, which holds if and only if $a_{pqr}^{\tau} > 0$. Similarly, $r\beta \in (p\beta)(q\beta)$ if and only if $a_{pqr}^{\tau} > 0$.

Definition 3.5. Given a labeling set τ , and given $P \subseteq \tau$ and $Q \subseteq \tau$, we let

$$PQ = \{r \in \tau : a_{pqr}^{\tau} > 0 \text{ for some } p \in P, q \in Q\}.$$

If τ' is a subset of τ and $p, q \in \tau$, we define $p\tau' = \{p\}\tau'$, $\tau'q = \tau'\{q\}$ and $pq = \{p\}\{q\}$.

To see that the product on subsets of τ is associative, choose some τ -scheme (T, α) . Then by Condition (3) of Definition 3.2, we have $(PQ)\alpha = (P\alpha)(Q\alpha)$. Since the complex product for association schemes is associative, we have for any $P, Q, R \subseteq \tau$

$$((PQ)R)\alpha = ((P\alpha)(Q\alpha))R\alpha = P\alpha((Q\alpha)(R\alpha)) = (P(QR))\alpha,$$

so $(PQ)R = P(QR)$ since α is a bijection.

Definition 3.6. A subset $\tau' \subseteq \tau$ is *closed* if $pq^* \subseteq \tau'$ whenever $p, q \in \tau'$. A subset $\tau' \subseteq \tau$ is *normal* if $p\tau' = \tau'p$ whenever $p \in \tau$.

Remark 3.7. If (T, α) is any τ -scheme, then α induces a one-to-one correspondence between the closed subsets of τ and the closed subsets of T . Indeed, if τ' is closed, and $p\alpha, q\alpha$ are arbitrary elements in $\tau'\alpha$, then as in Remark 3.4, for any $r\alpha \in (p\alpha)(q\alpha)^* = (p\alpha)(q^*\alpha)$, we must have $a_{pq^*r}^T > 0$, so $r \in p^*q$. This implies $r \in \tau'$, so $r\alpha \in \tau'\alpha$, and thus $\tau'\alpha$ is closed. Conversely, if $\tau'\alpha$ is closed, and $p, q \in \tau'$, then for any $r \in pq^*$, we have $r\alpha \in (p\alpha)(q^*\alpha) = (p\alpha)(q\alpha)^* \subseteq \tau'\alpha$, so $r \in \tau'$, whence τ' is closed.

By a similar argument, if (T, α) is a τ -scheme, then α induces a one-to-one correspondence between the normal subsets of τ and the normal subsets of T . Thus, if τ_1 and τ_2 are two normal closed subsets of a labeling set τ , then $\tau_1\tau_2$ must be a normal closed subset of τ . To see this, just choose a τ -scheme (T, α) ; then the image of $\tau_1\tau_2$ under α corresponds to $(\tau_1\alpha)(\tau_2\alpha)$, which is a normal closed subset of T .

Finally, if τ' is closed and normal in τ , and (T, α) is a τ -scheme, then the partition of T induced by $\tau'\alpha$ corresponds under α to a partition of τ . Thus, the cosets $p\tau'$, with $p \in \tau$, form a partition of τ .

Notation 3.8. Let $\mathcal{C}(\tau)$ denote the category whose objects are τ -schemes, and whose morphisms are morphisms in the underlying category \mathcal{C} . That is, if (T, α) and (U, β) are τ -schemes, then $\text{Hom}_{\mathcal{C}(\tau)}((T, \alpha), (U, \beta)) = \text{Hom}_{\mathcal{C}}(T, U)$.

In the following definition, we let $\mathcal{P}(\tau)$ denote the power set of τ .

Definition 3.9. Given $\phi \in \text{Hom}_{\mathcal{C}(\tau)}((T, \alpha), (U, \beta))$, let $\phi(\tau): \mathcal{P}(\tau) \rightarrow \mathcal{P}(\tau)$ be the function taking $P \subseteq \tau$ to

$$\{u \in \tau: \text{for some } t \in P, (t\alpha)^T \tilde{\phi} = (u\beta)^U \phi\}.$$

Remark 3.10. In general, given a function $\alpha: \mathcal{P}(\tau) \rightarrow \mathcal{P}(\tau)$, we define

$$\ker(\alpha) = \{t \in \tau: \{t\}\alpha = \{1\}\alpha\}.$$

Then if $\phi \in \text{Hom}_{\mathcal{C}(\tau)}((T, \alpha), (U, \beta))$, it is easy to check that $U_\phi = (\{1\}\phi(\tau))\beta$ and $T_\phi = \ker(\phi(\tau))\alpha$. That is, we can recover T_ϕ and U_ϕ from $\phi(\tau)$, α , and β .

4. Semidirect products from actions

Definition 4.1. Suppose T is a scheme on a based set X , U a scheme on a based set Y , and $\tau = \tau_T$ is the labeling set corresponding to T . Then an *action* ζ of U on T consists of a τ -scheme $\zeta_y = (T_y, \alpha^y)$ on X for each $y \in Y$, and a morphism $\zeta_{y_1}^{y_2} \in \text{Hom}_{\mathcal{C}}(T_{y_1}, T_{y_2})$ for each pair $y_1, y_2 \in Y$. We require the following properties to hold:

1. If $y_* \in Y$ is the basepoint in Y , then $T_{y_*} = T$ and $\alpha^{y_*}: \tau_T \rightarrow T_{y_*} = T$ is the identity. (Recall that the underlying set of τ_T is defined to be T .)
2. For each $y \in Y$, ζ_y^y is the identity morphism on T_y .
3. For each pair $y_1, y_2 \in Y$, $\zeta_{y_1}^{y_2} = (\zeta_{y_1}^{y_2})^*$ (see Notation 2.3).
4. The function $\zeta_{y_1}^{y_2}(\tau): \mathcal{P}(\tau) \rightarrow \mathcal{P}(\tau)$ depends only on the scheme element $u \in U$ containing (y_1, y_2) .
5. For each triple $y_1, y_2, y_3 \in Y$, we have $\zeta_{y_1}^{y_3} \leq \zeta_{y_1}^{y_2} \zeta_{y_2}^{y_3}$.

Notation 4.2. We will often denote $(T_{y_1})_{\zeta_{y_1}^{y_2}}$ as $T'_{y_1 y_2}$ and $(T_{y_2})_{\zeta_{y_1}^{y_2}}$ as $T''_{y_1 y_2}$. Thus, $\tilde{\zeta}_{y_1}^{y_2}$ is an isomorphism of schemes from $T_{y_1} // T'_{y_1 y_2}$ to $T_{y_2} // T''_{y_1 y_2}$. We will let $\zeta_u = \zeta_{y_1}^{y_2}(\tau)$, for some $(y_1, y_2) \in u$. By Condition (4), ζ_u is well defined. Let $\tau'_u = \ker \zeta_u$ and $\tau''_u = \{1\}\zeta_u$. Note that by Remark 3.10, the normal closed subsets $T'_{y_1 y_2}$ and $T''_{y_1 y_2}$ coincide with $(\tau'_u)\alpha^{y_1}$ and $(\tau''_u)\alpha^{y_2}$.

Given an action ζ of U on T , where U is a based scheme on Y and T is a based scheme on X , we will define a corresponding scheme $U \ltimes_{\zeta} T$ on the set $Y \times X$. Given $u \in U$ and $t \in \tau = \tau_T$, we let $[u, t]$ denote the set of all pairs $((y_1, x_1), (y_2, x_2)) \in (Y \times X)^{\times 2}$ such that $(y_1, y_2) \in u$ and

$$((x_1 T'_{y_1 y_2}) \tilde{\zeta}_{y_1}^{y_2}, x_2 T''_{y_1 y_2}) \in (t\alpha^{y_2})^{T''_{y_1 y_2}}.$$

Lemma 4.3. *As u ranges over U and t over τ , the sets $[u, t]$ form a partition of $(Y \times X)^{\times 2}$.*

Proof. Given an arbitrary pair $((y_1, x_1), (y_2, x_2))$, we have $(y_1, y_2) \in u$ for some $u \in U$, and

$$((x_1 T'_{y_1 y_2}) \tilde{\zeta}_{y_1}^{y_2}, x_2 T''_{y_1 y_2}) \in (t\alpha^{y_2})^{T''_{y_1 y_2}}$$

for some $t \in \tau$, since $\alpha^{y_2} : \tau \rightarrow T_{y_2}$ is a bijection. Thus every element in $(X \times Y)^{\times 2}$ belongs to $[u, t]$ for some $u \in U, t \in \tau$.

If $((y_1, x_1), (y_2, x_2)) \in [u, t] \cap [\hat{u}, \hat{t}]$, then $(y_1, y_2) \in u \cap \hat{u}$, so $u = \hat{u}$ since U is a partition of $Y \times Y$. Similarly $(t\alpha^{y_2})^{T''_{y_2}} = (\hat{t}\alpha^{y_2})^{T''_{y_2}}$; equivalently (since T''_{y_2} is normal and α^{y_2} is a bijection), we can find $s \in T''_{y_1 y_2} (\alpha^{y_2})^{-1}$ such that $t\alpha^{y_2} \in (\hat{t}\alpha^{y_2})(s\alpha^{y_2})$. We claim $[u, t] = [u, \hat{t}]$. If $((y_3, x_3), (y_4, x_4)) \in [u, \hat{t}]$, then $(y_3, y_4) \in u$. Also, since

$$s \in T''_{y_1 y_2} (\alpha^{y_2})^{-1} = \tau_u'' = T''_{y_3 y_4} (\alpha^{y_4})^{-1},$$

we have $s\alpha^{y_4} \in T''_{y_3 y_4}$. By Remark 3.4, $t\alpha^{y_2} \in (\hat{t}\alpha^{y_2})(s\alpha^{y_2})$ implies that $t\alpha^{y_4} \in (\hat{t}\alpha^{y_4})(s\alpha^{y_4})$, so $(t\alpha^{y_4})^{T''_{y_3 y_4}} = (\hat{t}\alpha^{y_4})^{T''_{y_3 y_4}}$, which in turn implies that $((y_3, x_3), (y_4, x_4)) \in [u, \hat{t}]$. Thus, $[u, t] \subseteq [u, \hat{t}]$, and by a symmetric argument, $[u, \hat{t}] \subseteq [u, t]$. \square

Notation 4.4. If ζ is an action of U on T , where U and T are schemes on based sets Y and X , we let $U \ltimes_{\zeta} T$ denote the set of all $[u, t] \subseteq (Y \times X)^{\times 2}$ as u ranges over U and t over $\tau = \tau_T$.

Lemma 4.5. *The set $U \ltimes_{\zeta} T$ contains*

$$1_{Y \times X} = \{((y, x), (y, x)): y \in Y \text{ and } x \in X\}.$$

Proof. We claim $[1_Y, 1] = 1_{Y \times X}$, where 1 denotes the distinguished element in τ . Given any $((y_1, x_1), (y_2, x_2)) \in (Y \times X)^{\times 2}$, we have $(y_1, y_2) \in 1_Y$ if and only if $y_1 = y_2$. By Condition (1) in Definition 4.1, $\zeta_{y_1}^{y_1}$ is the identity morphism in $\text{Hom}_{\mathcal{C}}(T_{y_1}, T_{y_1})$. Thus, $T'_{y_1 y_1} = T''_{y_1 y_1} = 1_{\{X\}}$, and $\tilde{\zeta}_{y_1}^{y_1}$ is the identity morphism of schemes. Since $1\alpha^{y_2} = 1_X$ by Condition (1) of Definition 3.2, $((y_1, x_1), (y_2, x_2)) \in [1_Y, 1]$ if and only if $y_1 = y_2$ and $x_1 = x_2$. \square

Lemma 4.6. *Suppose $[u, t] \in U \ltimes_{\zeta} T$. Let \hat{t}^* be any element in $\{t^*\} \zeta_u \subseteq \tau$. Then $[u, t]^* = [u^*, \hat{t}^*]$. In particular, $[u, t]^* \in U \ltimes_{\zeta} T$.*

Proof. First, $((y_1, x_1), (y_2, x_2)) \in [u, t]$ if and only if $(y_1, y_2) \in u$ and

$$((x_1 T'_{y_1 y_2}) \tilde{\zeta}_{y_1}^{y_2}, x_2 T''_{y_1 y_2}) \in (t\alpha^{y_2})^{T''_{y_1 y_2}}.$$

Equivalently, $(y_2, y_1) \in u^*$ and (by Condition (2) of Definition 3.2),

$$(x_2 T''_{y_1 y_2}, (x_1 T'_{y_1 y_2}) \tilde{\zeta}_{y_1}^{y_2}) \in (t^* \alpha^{y_2})^{T''_{y_1 y_2}}.$$

Since $\tilde{\zeta}_{y_1}^{y_2}$ is an isomorphism of schemes, this is equivalent to saying $(y_2, y_1) \in u^*$ and

$$((x_2 T''_{y_1 y_2})(\tilde{\zeta}_{y_1}^{y_2})^{-1}, x_1 T'_{y_1 y_2}) \in ((t^* \alpha^{y_2})^{T''_{y_1 y_2}})(\tilde{\zeta}_{y_1}^{y_2})^{-1}.$$

By Condition (3) of Definition 4.1, $\zeta_{y_1}^{y_2} = (\zeta_{y_1}^{y_2})^*$, so the above is equivalent to saying $(y_2, y_1) \in u^*$ and

$$((x_2 T'_{y_2 y_1})\tilde{\zeta}_{y_2}^{y_1}, x_1 T''_{y_2 y_1}) \in ((t^* \alpha^{y_2})^{T'_{y_2 y_1}})(\tilde{\zeta}_{y_2}^{y_1}).$$

From the definition of \hat{t}^* and ζ_{u^*} , we have

$$((t^* \alpha^{y_2})^{T'_{y_2 y_1}})(\tilde{\zeta}_{y_2}^{y_1}) = (\hat{t}^* \alpha^{y_1})^{T''_{y_2 y_1}}.$$

Thus, $((y_1, x_1), (y_2, x_2)) \in [u, t]$ if and only if $((y_2, x_2), (y_1, x_1)) \in [u^*, \hat{t}^*]$. \square

In order to prove the third scheme condition for $U \times_{\zeta} T$, we need the following preliminary lemma.

Lemma 4.7. *Suppose that τ is a labeling set, τ_1 and τ_2 are normal closed subsets of τ (see Definition 3.6), and p, q, r are elements in τ . If (T, α) is any τ -scheme and $(y, z) \in r\alpha$, then $|y(p\alpha)(\tau_1\alpha) \cap z(q\alpha)^*(\tau_2\alpha)|$ depends only on τ_1, τ_2, p, q and r . That is, this number does not depend on (T, α) or (y, z) .*

Proof. First, let $T_1 = \tau_1\alpha$ and $T_2 = \tau_2\alpha$. As observed in Remark 3.7, T_1 and T_2 are normal closed subsets of T . We have $x \in y(p\alpha)T_1 \cap z(q\alpha)^*T_2$ if and only if $x \in yp'\alpha \cap z(q'\alpha)^*$ for some $p'\alpha \in (p\alpha)T_1$ and some $q'\alpha \in ((q\alpha)^*T_2)^*$. Since T_2 is closed and normal, $((q\alpha)^*T_2)^* = T_2(q\alpha) = (q\alpha)T_2$. Therefore,

$$|y(p\alpha)T_1 \cap z(q\alpha)^*T_2| = \sum_{p'\alpha \in (p\alpha)T_1} \sum_{q'\alpha \in (q\alpha)T_2} a_{(p'\alpha)(q'\alpha)(r\alpha)} = \sum_{p' \in p\tau_1} \sum_{q' \in q\tau_2} a_{p'q'r}^{\tau}.$$

This number depends only on τ_1, τ_2, p, q and r . \square

Lemma 4.8. *Suppose $p_i = [u_i, t_i] \in U \times_{\zeta} T$ for $i = 1, 2, 3$. Then there is a nonnegative integer $a_{p_1 p_2 p_3}$ such that for any $((y_1, x_1), (y_2, x_2)) \in p_3$, $|y_1 p_1 \cap y_2 p_2| = a_{p_1 p_2 p_3}$.*

Proof. First, suppose $u_3 \notin u_1 u_2$. Then given $((y_1, x_1), (y_2, x_2)) \in p_3$, $(y_1, x_1)p_1 \cap (y_2, x_2)p_2^*$ must be empty: if it contained (y, x) , then we would have $(y_1, y_2) \in u_3$, $(y_1, y) \in u_1$ and $(y, y_2) \in u_2$ so $u_3 \in u_1 u_2$. Thus, if $u_3 \notin u_1 u_2$, then $a_{p_1 p_2 p_3} = 0$. We may therefore just consider the case when $u_3 \in u_1 u_2$.

Suppose given $((y_1, x_1), (y_2, x_2)) \in p_3$. Fix an element $y \in y_1 u_1 \cap y_2 u_2^*$. We will show that the cardinality

$$|\{x \in X: (y, x) \in (y_1, x_1)p_1 \cap (y_2, x_2)p_2^*\}| \tag{1}$$

depends only on p_1, p_2 , and p_3 . Since this cardinality is independent of y , multiplying it by $a_{u_1 u_2 u_3} = |y_1 u_1 \cap y_2 u_2^*|$ then yields the needed value for $a_{p_1 p_2 p_3}$.

Since $((y_1, x_1), (y_2, x_2)) \in [u_3, t_3]$, we have

$$((x_1 T'_{y_1 y_2})\tilde{\zeta}_{y_1}^{y_2}, x_2 T''_{y_1 y_2}) \in (t_3 \alpha^{y_2})^{T''_{y_1 y_2}}.$$

Since $T''_{y_1 y_2}$ is normal, we may choose a coset representative \bar{x}_1 of $(x_1 T'_{y_1 y_2})\tilde{\zeta}_{y_1}^{y_2}$ such that $(\bar{x}_1, x_2) \in t_3 \alpha^{y_2}$, so

$$(\bar{x}_1 T'_{y_2 y}, x_2 T'_{y_2 y}) \in (t_3 \alpha^{y_2})^{T'_{y_2 y}}. \tag{2}$$

Since $\zeta_{y_1}^{y_2} \leq \zeta_{y_1}^y \zeta_{y_1}^{y_2}$, the scheme isomorphism $\widetilde{\zeta_{y_1}^y \zeta_{y_1}^{y_2}}$ takes the coset of x_1 to the coset of \bar{x}_1 , so by the definition of $\widetilde{\zeta_{y_1}^y \zeta_{y_1}^{y_2}}$ given in the paragraph preceding Lemma 2.1, we can choose \hat{x}_1 such that

$$(x_1 T'_{y_1 y}) (\widetilde{\zeta_{y_1}^y}) = \hat{x}_1 T''_{y_1 y} \tag{3}$$

and

$$(\hat{x}_1 T'_{y_2 y}) (\widetilde{\zeta_y^{y_2}}) = \bar{x}_1 T''_{y_2 y}. \tag{4}$$

Since $\zeta_y^{y_2} = (\zeta_{y_2}^y)^*$, we have $T''_{y_2 y} = T'_{y_2 y}$, and

$$\hat{x}_1 T'_{y_2 y} = (\bar{x}_1 T'_{y_2 y}) (\widetilde{\zeta_{y_2}^y}). \tag{5}$$

Since $(y_2, y) \in u_2^*$, it follows from the definition of $\zeta_{u_2}^*$ that for any $r \in \{t_3\} \zeta_{u_2}^*$, we have

$$(r \alpha^y)^{T''_{y_2 y}} = (t_3 \alpha^{y_2})^{T'_{y_2 y}} \widetilde{\zeta_{y_2}^y},$$

which, together with Eq. (2) yields

$$((\bar{x}_1 T'_{y_2 y}) (\widetilde{\zeta_{y_2}^y}), (x_2 T'_{y_2 y}) (\widetilde{\zeta_{y_2}^y})) \in (r \alpha^y)^{T''_{y_2 y}} = (r \alpha^y)^{T'_{y_2 y}}.$$

Thus from Eq. (5),

$$(\hat{x}_1 T'_{y_2 y}, (x_2 T'_{y_2 y}) (\widetilde{\zeta_{y_2}^y})) \in (r \alpha^y)^{T'_{y_2 y}}.$$

Since $T'_{y_2 y}$ is normal, we may choose a coset representative $\hat{x}_2 \in (x_2 T'_{y_2 y}) (\widetilde{\zeta_{y_2}^y})$ such that $(\hat{x}_1, \hat{x}_2) \in r \alpha^y$.

Now, recalling that y is fixed, we have $(y, x) \in (y_1, x_1) p_1$ if and only if

$$((x_1 T'_{y_1 y}) \widetilde{\zeta_{y_1}^y}, x T''_{y_1 y_2}) \in (t_1 \alpha^y)^{T'_{y_1 y}}.$$

From Eq. (3), this holds if and only if

$$(\hat{x}_1 T''_{y_1 y}, x T''_{y_1 y}) \in (t_1 \alpha^y)^{T'_{y_1 y}}.$$

Since $T''_{y_1 y}$ is normal in T_y , this holds if and only if $x \in \hat{x}_1 (t_1 \alpha^y) T''_{y_1 y}$. By a similar argument, together with Lemma 4.6 and Condition (2) of Definition 3.2, we have $(y, x) \in (y_2, x_2) p_2^*$ if and only if $x \in \hat{x}_2 (\hat{t}_2 \alpha^y)^* T''_{y_2 y}$, where $\hat{t}_2^* \in \tau$ belongs to $\{t_2^*\} \zeta_{u_2}^*$. Now, recall from Notation 4.2 that $T''_{y_1 y} = \tau''_{u_1} \alpha^y$ and $T''_{y_2 y} = T'_{y_2 y} = \tau'_{u_2} \alpha^y$. By Lemma 4.7,

$$|\hat{x}_1 (t_1 \alpha^y) (\tau''_{u_1} \alpha^y) \cap \hat{x}_2 (\hat{t}_2 \alpha^y)^* (\tau'_{u_2} \alpha^y)| \tag{6}$$

depends only on $t_1, \hat{t}_2, \tau''_{u_1}, \tau'_{u_2}$ and r , hence only on t_1, \hat{t}_2, u_1, u_2 and r . But since \hat{t}_2 is any element in $(\{t_2^*\} \zeta_{u_2}^*)^*$ and r is any element in $\{t_3\} \zeta_{u_2}^*$, \hat{t}_2 and r depend only on t_2, t_3 and u_2 . Thus, the cardinality of the set (6), and hence of set (1), depends only on p_1, p_2 and p_3 . \square

Now, if Y and X are based sets, then $Y \times X$ has a natural basepoint (y_*, x_*) . The following corollary now follows from Lemmas 4.5, 4.6, and 4.8.

Corollary 4.9. *The set $U \times_{\zeta} T$ is a based scheme on $Y \times X$.*

5. Properties of the semidirect product

Suppose U and T are based schemes on X and Y , $\tau = \tau_T$, and ζ is an action of U on T . In this section, we show that the scheme $U \ltimes_{\zeta} T$ contains a normal closed subset $\tilde{T} := \{[1_Y, t] : t \in \tau\}$, such that the subscheme of $U \ltimes_{\zeta} T$ defined by $(y_*, x_*)\tilde{T}$ is isomorphic to T . We also produce a morphism of schemes $i : U \rightarrow U \ltimes_{\zeta} T$ which splits the natural quotient morphism $\pi : U \ltimes_{\zeta} T \rightarrow (U \ltimes_{\zeta} T) // \tilde{T}$. This justifies calling $U \ltimes_{\zeta} T$ a semidirect product. We also show that Condition 1.3 holds.

Proposition 5.1. *The subset \tilde{T} is a normal closed subset of $U \ltimes_{\zeta} T$. Moreover, the subscheme of $U \ltimes_{\zeta} T$ defined by $(y_*, x_*)\tilde{T}$ is isomorphic to T .*

Proof. First, given $[1, t] \in \tilde{T}$, we have $((y_1, x_1), (y_2, x_2)) \in [1, t]$ if and only if $y_2 = y_1$ and $(x_1, x_2) \in t\alpha^{y_1}$, since $\zeta_{y_1}^{y_1}$ is the identity. It follows from Definitions 3.2 and 3.5 that $[1, t_1][1, t_2] = \bigcup_{t \in t_1 t_2} [1, t]$ and $[1, t_1]^* = [1, t_1^*]$, so \tilde{T} is closed. Now, to show that \tilde{T} is normal, it suffices to show $\tilde{T}[u, t] \subseteq [u, t]\tilde{T}$ for any $[u, t] \in U \ltimes_{\zeta} T$. Suppose given $(y_1, x_1), (y_2, x_2), (y_3, x_3)$, where $((y_1, x_1), (y_2, x_2)) \in [1, t']$ for some $t' \in \tau$, and $((y_2, x_2), (y_3, x_3)) \in [u, t]$. Then $y_1 = y_2$ and $(y_2, y_3) \in u$, so $(y_1, y_3) \in u$. Also $(x_1, x_2) \in t'\alpha^{y_2}$ and

$$((x_2 T'_{y_2 y_3}) \tilde{\zeta}_{y_2}^{y_3}, x_3 T''_{y_2 y_3}) \in (t\alpha^{y_3})^{T''_{y_2 y_3}}.$$

Now let $y_4 = y_3$, and choose x_4 so that

$$((x_1 T'_{y_1 y_4}) \tilde{\zeta}_{y_1}^{y_4}, x_4 T''_{y_1 y_4}) \in (t\alpha^{y_4})^{T''_{y_1 y_4}}.$$

Then $((y_1, x_1), (y_4, x_4)) \in [u, t]$. Choose $\tilde{t} \in \tau$ so that $(x_4, x_3) \in \tilde{t}\alpha^{y_4}$. Then

$$((y_4, x_4), (y_3, x_3)) \in [1, \tilde{t}] \in \tilde{T}.$$

Thus, $\tilde{T}[u, t] \subseteq [u, t]\tilde{T}$.

For the final statement, note that $(y_*, x_*)\tilde{T} = \{y_*\} \times X \subseteq Y \times X$. Define $f_X : X \rightarrow \{y_*\} \times X$ by $x f_X = (y_*, x)$. If $(x_1, x_2) \in t \in T$, then it follows from Conditions (1) and (2) of Definition 4.1 that

$$(x_1 f_X, x_2 f_X) = ((y_*, x_1), (y_*, x_2)) \in [1, t\alpha^{y_*}] = [1, t].$$

Thus, f_X determines a morphism of schemes $f : T \rightarrow \tilde{T}_{(y_*, x_*)\tilde{T}}$; in particular, $t f_T = [1, t]$ for any $t \in T = \tau_T$. Clearly, f_X is a bijection, and f_T is surjective. If $t f_T = t' f_T$, then $[1, t] = [1, t']$, so for some $x_1, x_2 \in X$, we have $((y_*, x_1), (y_*, x_2)) \in [1, t] \cap [1, t']$. Thus, $(x_1, x_2) \in t \cap t'$, so $t = t'$, so f_T is injective. \square

Now define $i_Y : Y \rightarrow Y \times X$ by $y i_Y = (y, x_*)$, and let $\pi : U \ltimes_{\zeta} T \rightarrow (U \ltimes_{\zeta} T) // \tilde{T}$ denote the natural quotient morphism.

Proposition 5.2. *The function i_Y defines a based morphism of schemes $i : U \rightarrow U \ltimes_{\zeta} T$, and $i\pi$ is an isomorphism of schemes.*

Proof. Suppose $(y_1, y_2) \in u$. Since $\tilde{\zeta}_{y_1}^{y_2}$ is based, we have $(x_* T'_{y_1 y_2}) \tilde{\zeta}_{y_1}^{y_2} = x_* T''_{y_1 y_2}$. Since $1\alpha^{y_2} = 1_X$, we then have

$$((x_* T'_{y_1 y_2}) \tilde{\zeta}_{y_1}^{y_2}, x_* T''_{y_1 y_2}) \in (1\alpha^{y_2})^{T''_{y_1 y_2}}.$$

Therefore, $((y_1, x_*), (y_2, x_*)) \in [u, 1]$. Since $[u, 1]$ does not depend on (y_1, y_2) , and since $y_*i_Y = (y_*, x_*)$, i_Y determines a based morphism $i: U \rightarrow U \times_{\zeta} T$ of schemes. Note that if $u \in U$, then $ui_U = [u, 1]$.

Now, to show that $i\tau$ is an isomorphism, it suffices to show that $(i\tau)_Y$ is a bijection and $(i\pi)_U$ is an injection, since if $(i\pi)_Y$ is surjective, then $(i\pi)_U$ is immediately surjective. If $y_1i\pi = y_2i\pi$, then $(y_1, x_*)\tilde{T} = (y_2, x_*)\tilde{T}$, so $(y_1, x_*) \in (y_2, x_*)[1, t]$ for some $t \in \tau$, implying $y_1 = y_2$. On the other hand, given a coset $(y, x)\tilde{T}$, we have $(y, x) \in (y, x_*)[1, t]$ for some $t \in \tau$, so $(y, x)\tilde{T} = (y, x_*)\tilde{T} = yi\pi$. Thus, $(i\tau)_Y$ is a bijection.

Finally, suppose $u(i\pi)_U = v(i\pi)_U$, where $u, v \in U$. Choose $(y_1, y_2) \in u$. Then

$$((y_1, x_*)\tilde{T}, (y_2, x_*)\tilde{T}) \in [u, 1]^{\tilde{T}} = [v, 1]^{\tilde{T}}.$$

Thus, since \tilde{T} is normal, there is some $t \in \tau$ so that

$$((y_1, x_*), (y_2, x_*)) \in [v, 1][1_Y, t].$$

That is, we can find some (y, x) so that

$$((y_1, x_*), (y, x)) \in [v, 1] \quad \text{and} \quad ((y, x), (y_2, x_*)) \in [1_Y, t].$$

Therefore $(y_1, y_2) = (y_1, y) \in v$. But since $(y_1, y_2) \in u$, we must have $u = v$, so $(i\pi)_U$ is injective. \square

The following proposition shows that $U \times_{\zeta} T$ satisfies Condition 1.3.

Proposition 5.3. *For each $u \in U$ and each $\tilde{t} \in \tilde{T}$, $|(ui)\tilde{t}| = |\tilde{t}(ui)| = 1$.*

Proof. We must show $|[u, 1][1_Y, t]| = |[1_Y, t][u, 1]| = 1$ whenever $u \in U$ and $t \in \tau$. We first show that $[u, 1][1_Y, t] = [u, t]$. Suppose $((y_1, x_1), (y_2, x_2)) \in [u, 1]$ and $((y_2, x_2), (y_3, x_3)) \in [1_Y, t]$. Then, we must have $y_3 = y_2 \in y_1u$ and (since $\zeta_{y_2}^{y_3} = \zeta_{y_2}^{y_2}$ is the identity morphism), $x_3 \in x_2(t\alpha^{y_3})$. Also,

$$x_2T''_{y_1y_2} = (x_1T'_{y_1y_2})\tilde{\zeta}_{y_1}^{y_2} = (x_1T'_{y_1y_3})\tilde{\zeta}_{y_1}^{y_3}.$$

Thus,

$$x_3T''_{y_1y_3} = x_2T''_{y_1y_2} \in (x_2T''_{y_1y_2})(t\alpha^{y_3})^{T''_{y_1y_2}} = (x_1T'_{y_1y_3})\tilde{\zeta}_{y_1}^{y_3}(t\alpha^{y_3})^{T''_{y_1y_3}}.$$

Thus, $((x_1, y_1), (x_3, y_3)) \in [u, t]$, so $[u, 1][1_Y, t] = [u, t]$. That is, $|(ui)\tilde{t}| = 1$ for all $u \in U$ and $\tilde{t} \in \tilde{T}$.

Therefore, given $u \in U$ and $\tilde{t} \in \tilde{T}$, we have $(\tilde{t}(ui))^* = (ui)^*\tilde{t}^* = (u^*i)\tilde{t}^*$. By Proposition 5.1, \tilde{T} is closed, so $\tilde{t}^* \in \tilde{T}$. As we have just seen, $|(u^*i)\tilde{t}^*| = 1$, so $|\tilde{t}(ui)| = |(\tilde{t}(ui))^*| = 1$. \square

6. Example

We now provide an example to illustrate the definition above. We let $X = \{1, 2, 3\}$, with basepoint 1, and let $T = \{1, t\}$ be the unique scheme on X with 2 elements. That is, the adjacency matrix for 1 is the identity, and the adjacency matrix for t is

$$\sigma_t = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then $\tau = \tau_T = \{1, t\}$. Let Y be the set $\{a, b, c, d\}$, with basepoint a , and let U be the thin scheme on Y corresponding to the group $\mathbb{Z}/4$. Thus, the adjacency matrices for the elements of Y are

$$\sigma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

For each $y \in Y$, we let $T_y = T$ and α^y be the identity; then $\zeta^y = (T_y, \alpha^y)$.

Now, we define $\zeta_a^b \in \text{Hom}_{\mathcal{C}}(T_a, T_b)$ as follows. The normal closed subsets $(T_a)_{\zeta_a^b}$ and $(T_b)_{\zeta_a^b}$ are both T , and $\tilde{\zeta}_a^b$ is the identity on $T//T$. We define the seven morphisms $\zeta_b^c, \zeta_c^d, \zeta_d^a, \zeta_b^a, \zeta_c^b, \zeta_d^c, \zeta_a^d$ in the same way. Next, we define $\zeta_a^c \in \text{Hom}_{\mathcal{C}}(T_a, T_b)$ as follows. The normal closed subsets $(T_a)_{\zeta_a^c}$ and $(T_c)_{\zeta_a^c}$ are both $\{1\}$, and $\tilde{\zeta}_a^c$ is the identity on $T//\{1\} \cong T$. We define the three morphisms $\zeta_b^d, \zeta_c^a, \zeta_d^b$ in the same way. Finally, we define ζ_y^y to be the identity for each $y \in \{a, b, c, d\}$. The first four conditions of Definition 4.1 are immediate. The fifth condition is also easy. Indeed, if either (y_1, y_2) or (y_2, y_3) belong to scheme elements 1 or 3, then the composition $\zeta_{y_1}^{y_2} \zeta_{y_2}^{y_3}$ is the greatest element in the partial ordering of $\text{Hom}_{\mathcal{C}}(T_{y_1}, T_{y_3})$. If (y_1, y_2) and (y_2, y_3) both belong to 2, then $y_1 = y_3$, and $\zeta_{y_1}^{y_2} \zeta_{y_2}^{y_3}$ is the identity in $\text{Hom}_{\mathcal{C}}(T_{y_1}, T_{y_3})$, and so is equal to $\zeta_{y_1}^{y_3}$. Finally, if $y_1 = y_2$ or $y_2 = y_3$, then $\zeta_{y_1}^{y_2} \zeta_{y_2}^{y_3} = \zeta_{y_1}^{y_3}$.

Let $S = U \times_{\zeta} T$. Then $X \times Y$ has 12 elements, and S has 6 elements:

$$[0, 1], \quad [0, t], \quad [1, 1], \quad [2, 1], \quad [2, t], \quad [3, 1].$$

Indeed, $[1, t] = [1, 1]$ and $[3, t] = [3, 1]$. The elements $[0, 1]$ and $[2, 1]$ are both thin. The valencies of $[0, t]$ and $[2, t]$ are both 2, and the valencies of $[1, 1]$ and $[3, 1]$ are both 3. The elements of valency 1 and 2 are symmetric, while $[1, 1] = [3, 1]^*$. Using Hanaki and Miyamoto’s classification [5], we see that there is only one scheme of order 12 satisfying these conditions, scheme No. 34. Indeed, we can give an explicit isomorphism between scheme No. 34 and our semidirect product. On underlying sets, this isomorphism would take the elements in the first row of the matrix below to the elements directly below them in $X \times Y$:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ (a, 1) & (c, 1) & (a, 2) & (a, 3) & (c, 2) & (c, 3) & (b, 1) & (b, 2) & (b, 3) & (d, 1) & (d, 2) & (d, 3) \end{pmatrix}.$$

On scheme elements, the isomorphism can be described by the matrix below:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ [0, 1] & [2, 1] & [0, t] & [2, t] & [1, 1] & [3, 1] \end{pmatrix}.$$

We next show that S cannot be obtained as a generalized semidirect product in the sense of Bang and Song [3], and for the remainder of this section, we will assume the reader is familiar with that paper. First, we observe that the scheme S has exactly three proper nontrivial closed subsets:

- $K_1 = \{[0, 1], [2, 1]\}$,
- $K_2 = \{[0, 1], [0, t]\}$,
- $K_3 = \{[0, 1], [0, t], [2, 1], [2, t]\}$.

To see this, note first that $[1, 1]$ and $[3, 1]$ both generate S , so any proper closed subset can only contain the other four elements. Moreover, of the three elements $[0, t]$, $[2, 1]$, and $[2, t]$, each is in the complex product of the other two.

If $i = 1$ or $i = 2$, then K_i has only two elements, so that $\text{Aut}(K_i)$ is trivial, and K_i has no nontrivial proper closed subsets. It follows that the only generalized semidirect products of K_i

by another scheme are the ordinary product and the wreath product. Suppose H were a scheme whose elements had valencies $h_1 = 1, h_2, \dots, h_m$. Then the valencies of the wreath product of H and K_1 would be $1, 1, 2h_2, 2h_3, \dots, 2h_m$, while those of the wreath product of H and K_2 would be $1, 2, 3h_2, 3h_3, \dots, 3h_m$. In either case, it is impossible to obtain valencies $1, 1, 2, 2, 3, 3$. On the other hand, the valencies of the direct product $H \times K_2$ would be $h_1, h_2, \dots, h_m, 2h_1, 2h_2, \dots, 2h_m$. Again, we cannot obtain $1, 1, 2, 2, 3, 3$. We could obtain this sequence of valencies from $H \times K_1$ if the valencies of H were $1, 2, 3$. However, such a scheme must clearly be symmetric, so $H \times K_1$ would be symmetric. But S is not symmetric, since the two elements of valency 3 are conjugates.

Now, we claim $\text{Aut}(K_3)$ is also trivial. Indeed, any automorphism must send $[0, 1]$ to itself, since this represents the identity. Since $[2, 1]$ is the only other thin element, any automorphism must send $[2, 1]$ to itself. Thus, the only possible nontrivial automorphism would transpose $[0, t]$ and $[2, t]$. However, if we let $p = [0, t]$ and $q = [2, t]$, then $a_{ppp} = 1$, while $a_{qqq} = 0$. Thus, transposing $[0, t]$ and $[2, t]$ does not determine an automorphism of K_3 .

Thus, any semidirect product in the sense of Bang, Hirasaka, and Song [2] of K_3 by another scheme must be a direct product. In order that such a semidirect product have order 12, the other scheme would have to be the unique scheme of order 2, since the valency of K_3 is 6. Since K_3 and the scheme of order 2 are both symmetric, such a direct product would be symmetric. Since the generalized semidirect product of Bang and Song [3] is a fusion of the semidirect product in the sense of Bang, Hirasaka and Song [2] by Theorem 2.1 in the former paper, and any fusion of a symmetric scheme is symmetric, we cannot obtain S by taking a generalized semidirect product of K_3 by a scheme of order 2.

7. Actions obtained from semidirect products

In this section, we suppose given two based schemes T on X and U on Y . We suppose S is a based scheme on a set Z (with basepoint z_*), equipped with a closed subset $\tilde{T} \subseteq S$ such that there is a based isomorphism from the subscheme of S defined by $z_*\tilde{T}$ to the scheme T . We also assume that there is a based morphism $i: U \rightarrow S$ such that the composition $i\pi: U \rightarrow S//\tilde{T}$ is an isomorphism, where $\pi: S \rightarrow S//\tilde{T}$ denotes the natural morphism. Finally, we assume Condition 1.3: for any $u \in U$ and $t \in \tilde{T}$, we have $|t(ui)| = 1$. Our goal in this final section is to show that one can construct an action ζ of U on T such that $U \times_{\zeta} T$ is isomorphic to S .

Lemma 7.1. *Given the conditions above, \tilde{T} is normal.*

Proof. Let $z = yi$ for some $y \in Y$. To prove the claim, it suffices to show that for any $s \in S$, we have $\tilde{T}s \subseteq s\tilde{T}$. We first assume that $s = ui$ for some $u \in U$. If $z' \in z\tilde{T}(ui)$, then $z'\pi \in z\pi(ui\pi) = yi\pi(ui\pi)$. Since $i\pi$ is an isomorphism, $yi\pi(ui\pi) = (yu)i\pi \subseteq ((yi)(ui))\pi$. Thus, $z'\pi \in ((yi)(ui))\pi = z(ui)\pi$. This implies $z' \in z(ui)\tilde{T}$. Since z' was arbitrarily chosen from $z\tilde{T}(ui)$, we have $z\tilde{T}(ui) \subseteq z(ui)\tilde{T}$, whence $\tilde{T}(ui) \subseteq (ui)\tilde{T}$.

Now, given any $s \in S$, we have $s\pi = u(i\pi)$ for some $u \in U$, since $i\pi$ is an isomorphism. That is, $s\tilde{T} = (ui)\tilde{T}$, so $s \in \tilde{T}(ui)\tilde{T}$. Since $\tilde{T}(ui) \subseteq (ui)\tilde{T}$, we find $s \in (ui)\tilde{T}$. Thus, for some $t \in \tilde{T}$, $s \in (ui)t$. By Condition 1.3, $|(ui)t| = 1$, so $(ui)t = \{s\}$. This implies $(ui)\tilde{T} = s\tilde{T}$. Now, since $\tilde{T}(ui) \subseteq (ui)\tilde{T}$, we obtain $\tilde{T}s \subseteq \tilde{T}(ui)\tilde{T} \subseteq (ui)\tilde{T} = s\tilde{T}$. \square

Next, we wish to define an action ζ of U on T . For this, we will need a τ -scheme $\zeta_y = (T_y, \alpha^y)$ on X for each $y \in Y$, where $\tau = \tau_T$. For each $y \in Y$, we first let \tilde{T}_y denote the subscheme of S defined by $(yi)\tilde{T}$, with basepoint yi . That is, \tilde{T}_y consists of elements $t \cap (yi\tilde{T} \times yi\tilde{T})$, for $t \in \tilde{T}$. Let $\delta^y: \tilde{T}_y \rightarrow \tilde{T}$ be the bijection taking $t \cap (yi\tilde{T} \times yi\tilde{T})$ to t .

We had assumed at the beginning of this section that the subscheme of S defined by $z_*\tilde{T}$ is isomorphic to T . Thus, since $z_* = y_*i$, we may choose a based isomorphism $\gamma: T \rightarrow \tilde{T}_{y_*}$. (Thus, γ_T is a bijection from T to \tilde{T}_{y_*} , and γ_X is a bijection from X to $(y_*i)\tilde{T}$ taking x_* to y_*i .) For each $y \in Y$, we have $|(yi)\tilde{T}| = |z_*\tilde{T}| = |X|$, so we may choose a set of based bijections $\gamma_X^y: X \rightarrow (yi)\tilde{T}$, one for each $y \in Y$; we choose $\gamma_X^{y_*}$ to be γ_X .

Definition 7.2. Let T_y be the unique scheme on X defined by the requirement that $\gamma_X^y : X \rightarrow (yi)\tilde{T}$ determines a based isomorphism of schemes $\gamma^y : T_y \rightarrow \tilde{T}_y$. Let $\alpha^y : \tau_T \rightarrow T_y$ be the bijection defined by $(t\alpha^y)\gamma_{T_y}^y \delta^y = t\gamma_T \delta^{y*}$ for each $t \in \tau_T = T$. Let $\zeta_y = (T_y, \alpha^y)$.

Thus, the elements of T_y are the preimages of elements in \tilde{T}_y under the product bijection

$$\gamma_X^y \times \gamma_X^y : X \times X \rightarrow (yi)\tilde{T} \times (yi)\tilde{T}.$$

Since $\gamma : T \rightarrow \tilde{T}_{y_*}$ is already an isomorphism of schemes and $\gamma_X^{y_*} = \gamma_X$, it then follows that $T_{y_*} = T$. Also, since $\gamma_T = \gamma_{T_{y_*}}^{y_*}$, it follows that α^{y_*} is the identity. The following commutative diagram, in which y is an arbitrary element in Y and y_* is the basepoint, may be helpful in keeping these definitions straight:

$$\begin{array}{ccccc}
 & & T_y & \xrightarrow{\gamma_{T_y}^y} & \tilde{T}_y & & \\
 & \nearrow \alpha^y & & & & \searrow \delta^y & \\
 \tau & \xrightarrow{\alpha^{y_*}} & T_{y_*} & \xrightarrow{\gamma_T = \gamma_{T_{y_*}}^{y_*}} & \tilde{T}_{y_*} & \xrightarrow{\delta^{y_*}} & \tilde{T}.
 \end{array} \tag{7}$$

We next require morphisms $\zeta_{y_1}^{y_2} \in \text{Hom}_{\mathcal{C}}(T_{y_1}, T_{y_2})$ for each pair $y_1, y_2 \in Y$. Recall that such a morphism consists of normal closed subsets $T'_{y_1 y_2}$ in T_{y_1} and $T''_{y_1 y_2}$ in T_{y_2} , together with a based isomorphism of schemes $\tilde{\zeta}_{y_1}^{y_2}$ from $T_{y_1} // T'_{y_1 y_2}$ to $T_{y_2} // T''_{y_1 y_2}$. The following subsets of \tilde{T} will be useful in defining $T'_{y_1 y_2}$ and $T''_{y_1 y_2}$.

Definition 7.3. For a given $u \in U$, we define

$$\tilde{T}'_u = \{t \in \tilde{T} : t(ui) = \{ui\}\} \quad \text{and} \quad \tilde{T}''_u = \{t \in \tilde{T} : (ui)t = \{ui\}\}.$$

Let $\tau'_u = \{t \in \tau : t\gamma_T \delta^{y_*} \in \tilde{T}'_u\}$, and let $\tau''_u = \{t \in \tau : t\gamma_T \delta^{y_*} \in \tilde{T}''_u\}$.

Lemma 7.4. The sets \tilde{T}'_u and \tilde{T}''_u are closed in \tilde{T} .

Proof. Suppose $p, q \in \tilde{T}'_u$. Then $p(ui) = \{ui\}$ and $q(ui) = \{ui\}$. Since $ui \in p^*p(ui) = p^*(ui)$, and $p^*(ui)$ consists of precisely one element (by Condition 1.3), it follows that $p^*(ui) = \{ui\}$. Therefore, $p^*q(ui) = \{ui\}$, so $p^*q \subseteq \tilde{T}'_u$. Thus, \tilde{T}'_u is closed, and by a similar argument, \tilde{T}''_u is closed as well. \square

Definition 7.5. Suppose $(y_1, y_2) \in u$. Let $T'_{y_1 y_2} = \tau'_u \alpha^{y_1}$ and $T''_{y_1 y_2} = \tau''_u \alpha^{y_2}$. Also let $\tilde{T}'_{y_1 y_2} = T'_{y_1 y_2} \gamma_{T_{y_1}}^{y_1}$, and let $\tilde{T}''_{y_1 y_2} = T''_{y_1 y_2} \gamma_{T_{y_2}}^{y_2}$.

Remark 7.6. By the commutativity of diagram (7), and the fact that α^y and $\gamma_{T_y}^y$ are bijections for each $y \in Y$, it follows that if $(y_1, y_2) \in u$, then δ^{y_1} defines a bijection from $\tilde{T}'_{y_1 y_2}$ to \tilde{T}'_u and δ^{y_2} defines a bijection from $\tilde{T}''_{y_1 y_2}$ to \tilde{T}''_u . Also, from the way they are defined, it is clear that δ^{y_1} and δ^{y_2} preserve the involution and the complex product on subsets of $\tilde{T}'_{y_1 y_2}$ and $\tilde{T}''_{y_1 y_2}$. Since γ^{y_1} and γ^{y_2} are isomorphisms of schemes, it follows now from Lemma 7.4 that the sets $T'_{y_1 y_2}$ and $T''_{y_1 y_2}$ are closed subsets of T_{y_1} and T_{y_2} respectively.

In order to show that $T'_{y_1 y_2}$ and $T''_{y_1 y_2}$ are normal in T_{y_1} and T_{y_2} , we will need to show that \tilde{T}'_u and \tilde{T}''_u are normal in \tilde{T} , which is more difficult than proving that they are closed. We will require a few preliminary lemmas. First, we recall that if u is an element of a scheme U on a set Y , then the valency of u , denoted n_u , is equal to a_{uu^*1} . For any $y \in Y$, n_u is the number of elements in yu . Similarly, if $R \subseteq U$ is any subset, then $n_R = \sum_{u \in R} n_u$ is the number of elements in yR .

Lemma 7.7. *Suppose $u \in U$. Then $n_{ui} = n_u \cdot n_{\tilde{T}''_u}$.*

Proof. We will show that

$$(y_1 i)(ui) = \bigsqcup_{y \in y_1 u} (yi)\tilde{T}''_u.$$

That is, the set $(y_1 i)(ui)$ decomposes as a disjoint union of n_u sets, each having $n_{\tilde{T}''_u}$ elements. (Note that the union is disjoint since $i\pi$ is an injective and $\tilde{T}''_u \subseteq \tilde{T}$.) If $y \in y_1 u$, then $yi \in (y_1 i)(ui)$. But $(ui)t = \{ui\}$ for any $t \in \tilde{T}''_u$, so $(yi)\tilde{T}''_u \subseteq (y_1 i)(ui)$. Thus,

$$\bigsqcup_{y \in y_1 u} (yi)\tilde{T}''_u \subseteq (y_1 i)(ui).$$

Now, suppose $z \in (y_1 i)(ui)$. Since $i\pi$ is an isomorphism, we have $z\pi = yi\pi$ for some $y \in Y$, and also $z\pi \in (y_1 i\pi)(ui\pi) = (y_1 u)i\pi$. Thus, $y \in y_1 u$, so $yi \in (y_1 i)(ui)$. Now, since $z\pi = yi\pi$, $z \in (yi)t \subseteq (y_1 i)(ui)t$ for some $t \in \tilde{T}$. But we supposed to begin with that $z \in (y_1 i)(ui)$. Thus, we must have $ui \in (ui)t$, whence $(ui)t = \{ui\}$ by Condition 1.3. Therefore, $t \in \tilde{T}''_u$. Thus, $z \in (yi)\tilde{T}''_u$ for some $y \in y_1 u$. Since z was an arbitrary element in $(y_1 i)(ui)$, we have

$$(y_1 i)(ui) \subseteq \bigsqcup_{y \in y_1 u} (yi)\tilde{T}''_u. \quad \square$$

Lemma 7.8. *Suppose $(y_1, y_2) \in u$. If $z_1 \in (y_1 i)\tilde{T}$, then $z_1(ui) \cap (y_2 i)\tilde{T} = z_2\tilde{T}''_u$ for some $z_2 \in (y_2 i)\tilde{T}$. Moreover, if $z'_1 \in z_1\tilde{T}'_u$, then $z'_1(ui) \cap (y_2 i)\tilde{T} = z_1(ui) \cap (y_2 i)\tilde{T}$.*

Proof. First, since $z_1\tilde{T} = y_1 i\tilde{T}$, the cosets of \tilde{T} which contain elements in $z_1(ui)$ are the same as the cosets of \tilde{T} containing elements in $(y_1 u)i$, and there are n_u such cosets. By Lemma 7.7, $z_1(ui)$ contains $n_u \cdot n_{\tilde{T}''_u}$ elements. Moreover, $z_1(ui)$ decomposes into cosets of \tilde{T}''_u , each of which contains $n_{\tilde{T}''_u}$ elements. Thus, there must be exactly n_u cosets of \tilde{T}''_u in $z_1(ui)$, and each of these cosets is contained in a different coset of \tilde{T} since $\tilde{T}''_u \subseteq \tilde{T}$. Thus for any $y_2 \in y_1 u$, $z_1(ui) \cap (y_2 i)\tilde{T}$ must coincide with one of the cosets of \tilde{T}''_u . If we choose an element z_2 in this coset, then $z_1(ui) \cap (y_2 i)\tilde{T} = z_2\tilde{T}''_u$. For the last statement, if $z'_1 \in z_1\tilde{T}'_u$ for some $\tilde{t} \in \tilde{T}'_u$, then since $\tilde{t}(ui) = \{ui\}$, we have $z_1(ui) = z'_1(ui)$. \square

Lemma 7.9. *Suppose given $u \in U$, $t \in \tilde{T}$ and $s \in (ui)t$. If $\tilde{t} \in \tilde{T}''_u$, then $s\tilde{t} = \{s\}$.*

Proof. Choose $(y_1, y_2) \in u$, and let $z_1 = y_1 i$ and $z_2 = y_2 i$, so $(z_1, z_2) \in ui$. Since $s \in (ui)t$, we have $ui \in st^*$, so we can find $z_3 \in z_1 s \cap z_2 t$. Now, if $r \in s\tilde{t}$, then $s \in r\tilde{t}^*$, so we can find $z_4 \in z_1 r \cap z_3 \tilde{t}$. By Lemma 7.8 (applied to u^*), $z_3(ui)^* \cap z_1 \tilde{T}$ is nonempty, so we may choose $z_5 \in z_3(ui)^* \cap z_1 t'$ for some $t' \in \tilde{T}$. Since $z_3 \in z_1 s$ and $z_5 \in z_3(ui)^* \cap z_1 t'$, we have $s \in t'(ui)$, so by Condition 1.3, $t'(ui) = \{s\}$. Since $\tilde{t} \in \tilde{T}''_u$, we have $(ui)\tilde{t} = \{ui\}$, so we must have $z_4 \in z_5(ui)$. But then $z_4 \in z_1 t'(ui) = z_1 s$ and $z_4 \in z_1 r$, so $r = s$. Since $s\tilde{t}$ is nonempty and can only contain s , it must be equal to $\{s\}$. \square

Proposition 7.10. *For each $u \in U$, \tilde{T}''_u is normal in \tilde{T} .*

Proof. It will suffice to show that $\tilde{T}''_u \tilde{t} \subseteq \tilde{t} \tilde{T}''_u$ for any $\tilde{t} \in \tilde{T}$. Note that if $\tilde{t} \in \tilde{T}$, then $\tilde{t}^* \in \tilde{T}$ since \tilde{T} is closed. If $r \in \tilde{T}''_u \tilde{t}$, then $r \in \tilde{t}'' \tilde{t}$ for some $\tilde{t}'' \in \tilde{T}''_u$. Then we can choose $z_1, z_2, z_3 \in Z$ such that $z_3 \in z_1 r \cap z_2 \tilde{t}$ and $z_2 \in z_1 \tilde{t}''$, and we may assume $z_3 = y_3 i$ for some $y_3 \in Y$. Now, choose $y \in y_3 u^*$, so $y_3 \in yu$. Let $s \in S$ be the element containing (yi, z_2) . Then $s \in (ui) \tilde{t}^*$. By Lemma 7.9, $s(\tilde{t}'')^* = \{s\}$, so $z_1 \in (yi)s$. Since $s \in (ui) \tilde{t}^*$, we may find $z_4 \in (yi)(ui) \cap z_1 \tilde{t}$. Note that z_1, z_2, z_3 and z_4 are all in the same coset of \tilde{T} , so $(z_3, z_4) \in \tilde{t}'$ for some $\tilde{t}' \in \tilde{T}$. But then $ui \in (ui)(\tilde{t}')^*$, so $(ui)(\tilde{t}')^* = \{ui\}$, and $(\tilde{t}')^* \in \tilde{T}''_u$. Finally, $r \in \tilde{t}(\tilde{t}')^* \subseteq \tilde{t} \tilde{T}''_u$, as we see by considering z_1, z_3 and z_4 . So, $\tilde{T}''_u \tilde{t} \subseteq \tilde{t} \tilde{T}''_u$, as needed. \square

Corollary 7.11. \tilde{T}'_u is normal in \tilde{T} .

Proof. It is easy to check that $(\tilde{T}'_u)^* = \tilde{T}''_u$, and by Lemma 7.4, $(\tilde{T}'_u)^* = \tilde{T}'_u$, so the normality of \tilde{T}'_u follows from Proposition 7.10 applied to u^* . \square

The following corollary now follows by Remark 7.6, and since γ^{y_1} and γ^{y_2} are isomorphisms of schemes.

Corollary 7.12. For any $y_1, y_2 \in Y$, the sets $T'_{y_1 y_2}$ and $T''_{y_1 y_2}$ are normal in T_{y_1} and T_{y_2} .

Next, we need to define for each pair $y_1, y_2 \in Y$ a based isomorphism of schemes $\tilde{\xi}^{y_2}_{y_1}$ from $T_{y_1} // T'_{y_1 y_2}$ to $T_{y_2} // T''_{y_1 y_2}$. Using the isomorphisms γ^{y_1} and γ^{y_2} , it will suffice to define a based isomorphism $\tilde{\xi}^{y_2}_{y_1}$ from $\tilde{T}_{y_1} // \tilde{T}'_{y_1 y_2}$ to $\tilde{T}_{y_2} // \tilde{T}''_{y_1 y_2}$. We suppose $u \in U$ is the element containing (y_1, y_2) .

Definition 7.13. Let

$$\tilde{\xi}^{y_2}_{y_1} : (y_1 i) \tilde{T} / \tilde{T}'_{y_1 y_2} \rightarrow (y_2 i) \tilde{T} / \tilde{T}''_{y_1 y_2}$$

be the function which takes the coset $z_1 \tilde{T}'_{y_1 y_2}$ to the coset $z_2 \tilde{T}''_{y_1 y_2}$, where z_2 is any element in $z_1(ui) \cap (y_2 i) \tilde{T}$.

Lemma 7.14. The function $\tilde{\xi}^{y_2}_{y_1}$ is well defined.

Proof. If $z_1 \in (y_1 i) \tilde{T}$, then by Lemma 7.8, there is a $z_2 \in (y_2 i) \tilde{T}$ such that $z_1(ui) \cap (y_2 i) \tilde{T} = z_2 \tilde{T}''_u$, and the coset $z_2 \tilde{T}''_u$ only depends on the coset of \tilde{T}'_u containing z_1 . The lemma will follow if we can show that the coset of \tilde{T}'_u containing z_1 is the same as the coset of $\tilde{T}'_{y_1 y_2}$ containing z_1 , and that the coset of \tilde{T}''_u containing z_2 is the same as the coset of $\tilde{T}''_{y_1 y_2}$ containing z_2 . Indeed, restricting elements of \tilde{T}'_u and \tilde{T}''_u to $(y_1 i) \tilde{T}$ and $(y_2 i) \tilde{T}$ corresponds to taking the preimage under δ^{y_1} and δ^{y_2} . Thus, the restrictions of elements of \tilde{T}'_u and \tilde{T}''_u to $(y_1 i) \tilde{T}$ and $(y_2 i) \tilde{T}$ coincide with $\tilde{T}'_{y_1 y_2}$ and $\tilde{T}''_{y_1 y_2}$ by Definitions 7.3 and 7.5. \square

Lemma 7.15. The function $\tilde{\xi}^{y_2}_{y_1}$ determines a based morphism of schemes from $\tilde{T}_{y_1} // \tilde{T}'_{y_1 y_2}$ to $\tilde{T}_{y_2} // \tilde{T}''_{y_1 y_2}$. Moreover, for $v \in \tilde{T}'_{y_1}$, we have $v \tilde{T}'_{y_1 y_2} \tilde{\xi}^{y_2}_{y_1} = (v') \tilde{T}'_{y_1 y_2}$ if and only if $(v \delta^{y_1})(ui) = (ui)(v' \delta^{y_2})$.

Proof. Since $y_2 i \in (y_1 i)(ui)$ it follows that $\tilde{\xi}^{y_2}_{y_1}$ takes the coset of $\tilde{T}'_{y_1 y_2}$ containing $y_1 i$ to the coset of $\tilde{T}''_{y_1 y_2}$ containing $y_2 i$, so $\tilde{\xi}^{y_2}_{y_1}$ is based. Suppose that $z_1, z'_1 \in (y_1 i) \tilde{T}$, and let $v \in \tilde{T}_{y_1}$ be the scheme element containing (z_1, z'_1) . Choose z_2, z'_2 as above such that $z_1(ui) \cap (y_2 i) \tilde{T} = z_2 \tilde{T}''_u$ and $z'_1(ui) \cap (y_2 i) \tilde{T} = z'_2 \tilde{T}''_u$, and let $v' \in \tilde{T}_{y_2}$ be the scheme element containing (z_2, z'_2) . Then since $z_2 \in z_1(ui)$ and $z'_2 \in z'_1(ui)$, we have $(v \delta^{y_1})(ui) \cap (ui)(v' \delta^{y_2}) \neq \emptyset$. By Condition 1.3, $(v \delta^{y_1})(ui) = (ui)(v' \delta^{y_2})$. Note that if $(ui)r = (ui)r'$ for two elements $r, r' \in \tilde{T}$, then $ui \in (ui)r^* r^*$, so for some $p \in r^* r^* \subseteq \tilde{T}$, $ui \in (ui)p$,

whence $(ui)p = \{ui\}$, so $p \in \tilde{T}''_u$. Since $r' \in pr$, we have $(r)\tilde{T}''_u = (r')\tilde{T}''_u$. Thus, the equation $(v\delta^{y_1})(ui) = (ui)(v'\delta^{y_2})$ implies that $(v'\delta^{y_2})\tilde{T}''_u$ is uniquely determined by $v\delta^{y_1}$. By Remark 7.6, this in turn implies that $(v')\tilde{T}''_{y_1y_2}$ is uniquely determined by v . Similarly, if $r\tilde{T}''_u = r'\tilde{T}''_u$, then $r \in r't$ for some $t \in \tilde{T}''_u$ (since \tilde{T}''_u is normal), so $r(ui) = r'(ui)$. Thus, $(v')\tilde{T}''_{y_1y_2}$ is uniquely determined by $v\tilde{T}''_{y_1y_2}$, so $\tilde{\xi}^{y_2}_{y_1}$ determines a morphism of schemes.

For the second statement, if $v\tilde{T}''_{y_1y_2}\tilde{\xi}^{y_2}_{y_1} = (v')\tilde{T}''_{y_1y_2}$, then we can find $(z_1, z'_1) \in v$ and choose (z_2, z'_2) as above so that $(z_2, z'_2) \in v'$. But then, as shown above, we have $(v\delta^{y_1})(ui) = (ui)(v'\delta^{y_2})$. Conversely, if $(v\delta^{y_1})(ui) = (ui)(v'\delta^{y_2})$, then again, we may choose $(z_1, z'_1) \in v$. We choose z'_2 in $z'_1(ui) \cap (y_2i)\tilde{T}$. Then, $z'_2 \in z_1(v\delta^{y_1})(ui) = z_1(ui)(v'\delta^{y_2})$, so there must exist a $z_2 \in z_1(ui)$ with $(z_2, z'_2) \in (v'\delta^{y_2})$. Then $z_2 \in z_1(ui) \cap (y_2i)\tilde{T}$ since $(v'\delta^{y_2}) \in \tilde{T}$ and $z'_2 \in (y_2i)\tilde{T}$. But then, we see as above that $(v)\tilde{T}''_{y_1y_2}\tilde{\xi}^{y_2}_{y_1} = (v')\tilde{T}''_{y_1y_2}$. \square

Lemma 7.16. *The morphism $\tilde{\xi}^{y_2}_{y_1}$ is an isomorphism, with inverse $\tilde{\xi}^{y_1}_{y_2}$.*

Proof. First, we need to see that $\tilde{T}'_{y_2y_1} = \tilde{T}''_{y_1y_2}$ and $\tilde{T}''_{y_2y_1} = \tilde{T}'_{y_1y_2}$. Following Definition 7.3, we have

$$\tilde{T}'_{u^*} = \{t \in \tilde{T}: t(u^*i) = \{u^*i\}\} = \{t \in \tilde{T}: (ui)t^* = \{ui\}\} = (\tilde{T}''_u)^*.$$

Similarly, $\tilde{T}''_{u^*} = (\tilde{T}'_u)^*$. Since γ_T and δ^{y^*} preserve the involution, it follows that $\tau'_{u^*} = (\tau''_u)^*$ and $\tau''_{u^*} = (\tau'_u)^*$, and then since α^{y_1} and $\gamma^{y_1}_{T_1}$ preserve the involution and $\tilde{T}'_{y_1y_2}$ is closed, we have

$$\tilde{T}''_{y_2y_1} = \tau''_{u^*}\alpha^{y_1}\gamma^{y_1}_{T_1} = (\tau'_u\alpha^{y_1}\gamma^{y_1}_{T_1})^* = (\tilde{T}'_{y_1y_2})^* = \tilde{T}'_{y_1y_2}.$$

Similarly $\tilde{T}'_{y_2y_1} = \tilde{T}''_{y_1y_2}$.

Now, given $z_1 \in (y_1i)\tilde{T}$, we have $(z_1\tilde{T}'_{y_1y_2})\tilde{\xi}^{y_2}_{y_1} = z_2\tilde{T}''_{y_1y_2}$ if and only if $z_2 \in z_1(ui) \cap (y_2i)\tilde{T}$. But if $z_2 \in z_1(ui)$, then $z_1 \in z_2(u^*i)$, so $z_1 \in z_2(u^*i) \cap (y_1i)\tilde{T}$. Thus, if $(z_1\tilde{T}'_{y_1y_2})\tilde{\xi}^{y_2}_{y_1} = z_2\tilde{T}''_{y_1y_2}$, then $(z_2\tilde{T}'_{y_2y_1})\tilde{\xi}^{y_1}_{y_2} = z_1\tilde{T}''_{y_2y_1}$. The converse follows by reversing the rolls of y_1 and y_2 . Therefore, $\tilde{\xi}^{y_1}_{y_2}$ is the inverse of $\tilde{\xi}^{y_2}_{y_1}$. \square

As mentioned in the paragraph before Definition 7.13, the isomorphisms $\tilde{\xi}^{y_2}_{y_1}$ will induce the needed isomorphisms $\tilde{\xi}^{y_2}_{y_1}$, using the isomorphisms γ^{y_1} and γ^{y_2} . This is made precise in the following definition.

Definition 7.17. Given $y_1, y_2 \in Y$, let $\tilde{\xi}^{y_2}_{y_1}$ be the based isomorphism of schemes making the following diagram commute:

$$\begin{array}{ccc} T_{y_1} // T'_{y_1y_2} & \xrightarrow{\overline{\gamma^{y_1}}} & \tilde{T}_{y_1} // \tilde{T}'_{y_1y_2} \\ \downarrow \tilde{\xi}^{y_2}_{y_1} & & \downarrow \tilde{\xi}^{y_2}_{y_1} \\ T_{y_2} // T''_{y_1y_2} & \xrightarrow{\overline{\gamma^{y_2}}} & \tilde{T}_{y_2} // \tilde{T}''_{y_1y_2}. \end{array}$$

The horizontal arrows in the above diagram are the isomorphisms of quotient schemes induced by the isomorphisms γ^{y_1} and γ^{y_2} . (Recall that $\tilde{T}'_{y_1y_2}$ is defined simply as the image of $T'_{y_1y_2}$ under γ^{y_1} and $\tilde{T}''_{y_1y_2}$ is defined as the image of $T''_{y_1y_2}$ under γ^{y_2} .)

Having defined τ -schemes ζ_y for each $y \in Y$ and morphisms $\zeta_{y_1}^{y_2} \in \text{Hom}_{\mathcal{C}}(T_{y_1}, T_{y_2})$ for each pair of elements $y_1, y_2 \in Y$, we must now prove that ζ determines an action of U on T . That is, we need to verify the conditions of Definition 4.1. The following lemma will be needed.

Lemma 7.18. *Suppose $p, q, r \in U$, with $r \in pq$, and suppose $z_1, z_3 \in Z$, with $z_3 \in z_1(ri)$. Let y_1 and y_3 be the elements in Y such that $z_1\tilde{T} = (y_1i)\tilde{T}$ and $z_3\tilde{T} = (y_3i)\tilde{T}$. If $y_2 \in y_1p \cap y_3q^*$, there exists an element $z_2 \in z_1(pi) \cap z_3(qi)^*$ such that $z_2\tilde{T} = (y_2i)\tilde{T}$.*

Proof. First, we show that for each $y_2 \in y_1p \cap y_3q^*$, the number of elements $z_2 \in z_1(pi) \cap z_3(qi)^*$ such that $z_2\tilde{T} = (y_2i)\tilde{T}$ is either 0 or $n_{\tilde{T}''_p \cap \tilde{T}'_q}$. To see this, it suffices to show that if $z_1(pi) \cap z_3(qi)^* \cap (y_2i)\tilde{T}$ contains an element z_2 , then it coincides with the coset of $\tilde{T}''_p \cap \tilde{T}'_q$ containing z_2 . Suppose $t \in \tilde{T}''_p \cap \tilde{T}'_q$. Then $t^* \in \tilde{T}'_q$, so we have $z_2t \subseteq z_1(pi)t = z_1(pi)$ and $z_2t \subseteq z_3(qi)^*t = z_3(t^*(qi))^* = z_3(qi)^*$. Since $\tilde{T}''_p \cap \tilde{T}'_q \subseteq \tilde{T}$ and $z_2 \in (y_2i)\tilde{T}$, we have $z_2t \subseteq z_2\tilde{T} = (y_2i)\tilde{T}$. Thus, every element in the coset of $\tilde{T}''_p \cap \tilde{T}'_q$ containing z_2 is in $z_1(pi) \cap z_3(qi)^* \cap (y_2i)\tilde{T}$. Conversely, if $z \in z_1(pi) \cap z_3(qi)^* \cap (y_2i)\tilde{T}$, then let $t \in \tilde{T}$ be the element containing (z_2, z) . Considering the elements z_1, z_2 and z , we see that $pi \in (pi)t$, so $(pi)t = \{pi\}$, whence $t \in \tilde{T}''_p$; considering z_2, z , and z_3 , we see that $qi \in t(qi)$, so $t(qi) = \{qi\}$, whence $t \in \tilde{T}'_q$. Thus, z is in the coset of $\tilde{T}''_p \cap \tilde{T}'_q$ containing z_2 .

The argument in the previous paragraph applies when we choose $z_1 = y_1i$ and $z_3 = y_3i$, but in this case, $z_1(pi) \cap z_3(qi)^* \cap (y_2i)\tilde{T}$ always contains y_2i . Thus, for each $y_2 \in y_1p \cap y_3q^*$, there are $n_{\tilde{T}''_p \cap \tilde{T}'_q}$ elements $z_2 \in (y_1i)(pi) \cap (y_3i)(qi)^*$ with $z_2\tilde{T} = (y_2i)\tilde{T}$. Of course, if $y_2 \notin y_1p \cap y_3q^*$, then $(y_1i)(pi) \cap (y_3i)(qi)^* \cap (y_2i)\tilde{T}$ is empty since $i\pi$ is an isomorphism. Since the cardinality of $y_1p \cap y_3q^*$ is a_{pqr} , and the cardinality of $z_1(pi) \cap z_3(qi)^*$ is $a_{(pi)(qi)(ri)}$, we have shown that

$$a_{(pi)(qi)(ri)} = a_{pqr}n_{\tilde{T}''_p \cap \tilde{T}'_q}. \tag{8}$$

Returning to the general case, where $z_1 \in (y_1i)\tilde{T}$, $z_3 \in (y_3i)\tilde{T}$, and $z_3 \in z_1(ri)$, the cardinality $a_{(pi)(qi)(ri)}$ of $z_1(pi) \cap z_3(qi)^*$ is equal to the product of $n_{\tilde{T}''_p \cap \tilde{T}'_q}$ with the number of elements y_2 in $y_1p \cap y_3q^*$ for which $z_1(pi) \cap z_3(qi)^* \cap (y_2i)\tilde{T}$ is nonempty. By Eq. (8), there must be a_{pqr} such elements y_2 . Since $|y_1p \cap y_3q^*| = a_{pqr}$, it must be the case that for every $y_2 \in y_1p \cap y_3q^*$, we have $z_1(pi) \cap z_3(qi)^* \cap (y_2i)\tilde{T}$ is nonempty. \square

Theorem 7.19. *Definitions 7.2, 7.5, and 7.17 determine an action ζ of U on T .*

Proof. We must verify the five conditions of Definition 4.1.

Condition (1). We have seen in the paragraph following Definition 7.2 that Condition (1) is satisfied.

Condition (2). Suppose $y \in Y$, and let $1_Y \in U$ denote the scheme element containing (y, y) . Then since $1_Yi = 1_Z \in S$, it follows immediately from Definition 7.3 that $\tilde{T}'_{1_Y} = \tilde{T}''_{1_Y} = \{1_Z\}$, and, since γ_T is an isomorphism, and δ^y is a bijection taking $1_{(yi)\tilde{T}}$ to 1_Z , $\tau'_{1_Y} = \tau''_{1_Y} = \{1\}$. Now it follows from Definition 7.5 that $T'_{yy} = T''_{yy} = \{1_X\}$. Now, if $z_1 \in (yi)\tilde{T}$, then clearly, $z_1(1_Z) \cap (yi)\tilde{T} = z_1(1_Z)$, so $\tilde{\xi}^y$ is the identity. By Definition 7.17, $\tilde{\zeta}^y$ is then also the identity. Thus, ζ^y is the identity morphism in $\text{Hom}_{\mathcal{C}}(T_y, T_y)$, as needed for Condition (2).

Condition (3). If $y_1, y_2 \in Y$, then by Lemma 7.16, $\tilde{\xi}^{y_1}_{y_2} = (\tilde{\xi}^{y_2}_{y_1})^{-1}$, so by Definition 7.17, $\tilde{\zeta}^{y_1}_{y_2} = (\tilde{\zeta}^{y_2}_{y_1})^{-1}$. Therefore $\zeta^{y_1}_{y_2} = (\zeta^{y_2}_{y_1})^*$, as needed for Condition (3).

Condition (4). Suppose $(y_1, y_2) \in u$, and $P \subseteq \tau$. Recall from Definition 3.9 that

$$P(\zeta_{y_1}^{y_2}(\tau)) = \{v' \in \tau: \text{for some } v \in P, (v\alpha^{y_1})^{T'_{y_1 y_2}} \tilde{\zeta}_{y_1}^{y_2} = (v'\alpha^{y_2})^{T'_{y_1 y_2}}\}.$$

By Definition 7.17, Lemma 7.15, and Definition 7.2, we have

$$(v\alpha^{y_1})^{T'_{y_1 y_2}} \tilde{\zeta}_{y_1}^{y_2} = (v'\alpha^{y_2})^{T'_{y_1 y_2}}$$

if and only if

$$(v\gamma_T \delta^{y^*})(ui) = (ui)(v'\gamma_T \delta^{y^*}).$$

Thus, $\zeta_{y_1}^{y_2}(\tau)$ depends only on u , as needed for Condition (4).

Condition (5). Suppose $y_1, y_2, y_3 \in Y$, with $(y_1, y_2) \in p$, $(y_2, y_3) \in q$, and $(y_1, y_3) \in r$. We let

$$T'_{y_1 y_2 y_3} = \{t \in T: \text{for some } v \in T \text{ we have } t^{T'_{y_1 y_2}} \tilde{\zeta}_{y_1}^{y_2} = v^{T''_{y_1 y_2}} \text{ and } v^{T'_{y_2 y_3}} \tilde{\zeta}_{y_2}^{y_3} = 1_X^{T''_{y_2 y_3}}\},$$

$$T''_{y_1 y_2 y_3} = \{t \in T: \text{for some } v \in T \text{ we have } 1_X^{T'_{y_1 y_2}} \tilde{\zeta}_{y_1}^{y_2} = v^{T''_{y_2 y_3}} \text{ and } v^{T'_{y_2 y_3}} \tilde{\zeta}_{y_2}^{y_3} = t^{T''_{y_2 y_3}}\}.$$

Then $T'_{y_1 y_2 y_3}$ and $T''_{y_1 y_2 y_3}$ are the normal closed subsets of the composite $\zeta_{y_1}^{y_2} \zeta_{y_2}^{y_3}$, as defined in Section 2.

To show that $\zeta_{y_1}^{y_3} \leq \zeta_{y_1}^{y_2} \zeta_{y_2}^{y_3}$, we must show that $T'_{y_1 y_3} \subseteq T'_{y_1 y_2 y_3}$, that $T''_{y_1 y_3} \subseteq T''_{y_1 y_2 y_3}$, and that for each $x_1 \in X$,

$$(x_1 T'_{y_1 y_3}) \tilde{\zeta}_{y_1}^{y_3} \subseteq (x_1 T'_{y_1 y_2 y_3}) (\widetilde{\zeta_{y_1}^{y_2} \zeta_{y_2}^{y_3}}).$$

First, suppose $v \in T'_{y_1 y_3}$, and let $\tilde{v} = v\gamma_{T_{y_1}}^{y_1}$. Then $\tilde{v}\delta^{y_1} \in \tilde{T}'_r$ by Definitions 7.2, 7.3, and 7.5. That is, $(\tilde{v}\delta^{y_1})(ri) = \{ri\}$. Now, choose $z_1 \in (y_1 i)(\tilde{v}^*)$, so $(z_1, y_1 i) \in \tilde{v} \subseteq \tilde{v}\delta^{y_1}$. Thus, letting $z_3 = y_3 i$ we have

$$z_3 \in (y_1 i)(ri) \subseteq z_1(\tilde{v}\delta^{y_1})(ri) = z_1(ri).$$

By Lemma 7.18, there is a $z_2 \in z_1(pi) \cap z_3(qi)^*$ with $z_2 \tilde{T} = (y_2 i) \tilde{T}$. Now, let $\tilde{v}' \in \tilde{T}'_{y_2}$ be the scheme element containing $(z_2, y_2 i)$. Then $y_2 i \in z_1(pi)(\tilde{v}'\delta^{y_2})$ and $y_2 i \in z_1(\tilde{v}\delta^{y_1})(pi)$. Since $|(pi)(\tilde{v}'\delta^{y_2})| = |(\tilde{v}\delta^{y_1})(pi)| = 1$, this implies that $(pi)(\tilde{v}'\delta^{y_2}) = (\tilde{v}\delta^{y_1})(pi)$. By Lemma 7.15, we have $\tilde{v}'^{T'_{y_1 y_2}} \tilde{\zeta}_{y_1}^{y_2} = (\tilde{v}')^{T'_{y_1 y_2}}$. Moreover, since $y_3 i \in z_2(qi) \cap (y_2 i)(qi)$, we must have $\tilde{v}'(qi) = \{qi\}$, so $\tilde{v}'\delta^{y_2} \in \tilde{T}'_q$. Thus, $\tilde{v}'^{T'_{y_2 y_3}} \tilde{\zeta}_{y_2}^{y_3} = 1_X^{T''_{y_2 y_3}}$. Define v' by $v'\gamma_{T'_{y_2}}^{y_2} = \tilde{v}'$. Then by Definition 7.17, we have

$$v^{T'_{y_1 y_2}} \tilde{\zeta}_{y_1}^{y_2} = (v')^{T''_{y_1 y_2}} \text{ and } (v')^{T'_{y_2 y_3}} \tilde{\zeta}_{y_2}^{y_3} = 1_X^{T''_{y_2 y_3}}.$$

That is, $v \in T'_{y_1 y_2 y_3}$. Since $v \in T'_{y_1 y_3}$ was arbitrary, we have $T'_{y_1 y_3} \subseteq T'_{y_1 y_2 y_3}$. A similar argument shows that $T''_{y_1 y_3} \subseteq T''_{y_1 y_2 y_3}$.

Finally, given $x_1 \in X$, let $z_1 = x_1 \gamma_X^{y_1} \in (y_1 i) \tilde{T}$, and suppose $x_3 \in (x_1 T'_{y_1 y_3}) \tilde{\zeta}_{y_1}^{y_3}$. Let $z_3 = x_3 \gamma_X^{y_3} \in (y_3 i) \tilde{T}$. Then by Definition 7.17, $z_3 \in z_1 \tilde{T}'_{y_1 y_3} \tilde{\xi}_{y_1}^{y_3}$, so by Definition 7.13, we have $z_3 \in z_1(ri) \cap (y_3 i) \tilde{T}$. By Lemma 7.18, we may choose $z_2 \in z_1(pi) \cap z_3(qi)^*$ with $z_2 \tilde{T} = (y_2 i) \tilde{T}$. But then by Definition 7.13 again, $z_2 \tilde{T}'_{y_1 y_2} = (z_1 \tilde{T}'_{y_1 y_2}) \tilde{\xi}_{y_1}^{y_2}$ and $z_3 \tilde{T}'_{y_2 y_3} = (z_2 \tilde{T}'_{y_2 y_3}) \tilde{\xi}_{y_2}^{y_3}$. Letting $x_2 = z_2(\gamma_X^{y_2})^{-1}$, we see from Definition 7.17 that $x_2 T''_{y_1 y_2} = (x_1 T'_{y_1 y_2}) \tilde{\zeta}_{y_1}^{y_2}$ and $x_3 T''_{y_2 y_3} = (x_2 T'_{y_2 y_3}) \tilde{\zeta}_{y_2}^{y_3}$. By the definition of $\widetilde{\zeta_{y_1}^{y_2} \zeta_{y_2}^{y_3}}$

given in Section 2, it follows that $x_3 \in (x_1 T'_{y_1 y_2 y_3}) \widetilde{(\zeta_{y_1}^{y_2} \zeta_{y_2}^{y_3})}$. Since x_3 was an arbitrary element in $(x_1 T'_{y_1 y_3}) \widetilde{\zeta_{y_1}^{y_3}}$, we then have

$$(x_1 T'_{y_1 y_3}) \widetilde{\zeta_{y_1}^{y_3}} \subseteq (x_1 T'_{y_1 y_2 y_3}) \widetilde{(\zeta_{y_1}^{y_2} \zeta_{y_2}^{y_3})}.$$

This concludes the proof of Condition (5). \square

Theorem 7.20. *The scheme S on Z is isomorphic to the scheme $U \times_{\zeta} T$ on $Y \times X$.*

Proof. We will construct an isomorphism $\eta : S \rightarrow U \times_{\zeta} T$. We must first define $\eta_Z : Z \rightarrow Y \times X$. Since $i\pi$ and γ^y are isomorphisms, it follows that for each $z \in Z$, there is a unique $(y, x) \in Y \times X$ such that $yi\tilde{T} = z\tilde{T}$ and $x\gamma_X^y = z$; we let $z\eta_Z = (y, x)$. Since i and γ^{y*} are based morphisms, $y_*i\tilde{T} = z_*\tilde{T}$ and $x_*\gamma^{y*} = z_*$, so $z_*\eta_Z = (y_*, x_*)$. That is, η_Z is a based function.

Now, suppose given $s \in S$. Since $i\pi$ is an isomorphism, $s^{\tilde{T}} = (ui)^{\tilde{T}}$ for a unique $u \in U$, and so, since \tilde{T} is normal by Lemma 7.1, we must have $s \in (ui)\tilde{t}$ for some $\tilde{t} \in \tilde{T}$. Let $t \in \tau$ be defined by $t\gamma_T\delta^{y*} = \tilde{t}$. We will now show that if $(z_1, z_2) \in s$, then $(z_1\eta_Z, z_2\eta_Z) \in [u, t]$, which implies that η_Z is a morphism of schemes. Let $(y_1, x_1) = z_1\eta_Z$ and $(y_2, x_2) = z_2\eta_Z$. Then

$$(y_2i)\tilde{T} = z_2\tilde{T} \in (z_1\tilde{T})s^{\tilde{T}} = (y_1i\tilde{T})(ui)^{\tilde{T}}.$$

That is, $y_2i\pi \in (y_1i\pi)(ui\pi)$, so $y_2 \in y_1u$, since $i\pi$ is an isomorphism. Since $s \in (ui)\tilde{t}$, we may choose $z'_2 \in z_1(ui) \cap z_2\tilde{t}^*$, so in particular $z'_2 \in z_1(ui) \cap z_2\tilde{T}$. Now, by Definition 7.13, we have $(z_1\tilde{T}'_{y_1 y_2}) \widetilde{\zeta_{y_1}^{y_2}} = z'_2\tilde{T}'_{y_1 y_2}$. If we let $x'_2 = z'_2(\gamma_X^{y_2})^{-1} \in X$, then by Definition 7.17, $(x_1 T'_{y_1 y_2}) \widetilde{\zeta_{y_1}^{y_2}} = x'_2 T''_{y_1 y_2}$. Since $(z'_2, z_2) \in \tilde{t}$, we get $(z'_2\tilde{T}''_u, z_2\tilde{T}''_u) \in \tilde{t}^u$. Now $\tilde{T}''_{y_1 y_2}$ is the preimage of \tilde{T}''_u under δ^{y_2} , so the elements of $\tilde{T}''_{y_1 y_2}$ are simply the restrictions of the elements in \tilde{T}''_u to $(y_2i)\tilde{T} \times (y_2i)\tilde{T}$. Thus, $(z'_2\tilde{T}''_{y_1 y_2}, z_2\tilde{T}''_{y_1 y_2}) \in \tilde{t}^{\tilde{T}''_{y_1 y_2}}$. We now have

$$((x_1 T'_{y_1 y_2}) \widetilde{\zeta_{y_1}^{y_2}} \overline{\gamma^{y_2}}, x_2 T''_{y_1 y_2} \overline{\gamma^{y_2}}) = (x'_2 T''_{y_1 y_2} \overline{\gamma^{y_2}}, x_2 T''_{y_1 y_2} \overline{\gamma^{y_2}}) = (z'_2 \tilde{T}''_{y_1 y_2}, z_2 \tilde{T}''_{y_1 y_2}) \in \tilde{t}^{\tilde{T}''_{y_1 y_2}}.$$

Here, as in Definition 7.17, $\overline{\gamma^{y_2}}$ is the isomorphism from $T//T''_{y_1 y_2}$ to $\tilde{T}//\tilde{T}''_{y_1 y_2}$ induced by γ^{y_2} . Since $\tilde{t} = t\gamma_T\delta^{y*}$, we have $\tilde{t} = t\alpha^{y_2}\gamma_{T_{y_2}}^{y_2}\delta^{y_2}$ by Definition 7.2. That is, the restriction of \tilde{t} to $(y_2i)\tilde{T} \times (y_2i)\tilde{T}$ is equal to $t\alpha^{y_2}\gamma_{T_{y_2}}$. We thus have

$$((x_1 T'_{y_1 y_2}) \widetilde{\zeta_{y_1}^{y_2}} \overline{\gamma^{y_2}}, x_2 T''_{y_1 y_2} \overline{\gamma^{y_2}}) \in (t\alpha^{y_2}\gamma_{T_{y_2}})^{\tilde{T}''_{y_1 y_2}} = ((t\alpha^{y_2})^{T''_{y_1 y_2}}) \overline{\gamma^{y_2}}.$$

Since $\overline{\gamma^{y_2}}$ is an isomorphism of schemes, we get

$$((x_1 T'_{y_1 y_2}) \widetilde{\zeta_{y_1}^{y_2}}, x_2 T''_{y_1 y_2}) \in (t\alpha^{y_2})^{T''_{y_1 y_2}}.$$

Since we have already shown that $(y_1, y_2) \in u$, then by the definition of $[u, t]$ given in Section 4, we have

$$(z_1\eta_Z, z_2\eta_Z) = ((y_1, x_1), (y_2, x_2)) \in [u, t],$$

so η_Z is a morphism of schemes.

Finally, we show that η is an isomorphism; for this it suffices to show that η_Z and η_S are bijections. We have $z\eta_Z = (y, x)$ if and only if $z = x\gamma_X^y$, so η_Z is a bijection. It follows at once that

η_S is surjective. Finally, suppose $s\eta_S = [u, t]$ and $s'\eta_S = [u', t']$, where $[u, t] = [u', t']$. It follows immediately from the definitions that $u = u'$. If we let $\tilde{t} = t\gamma_T\delta^{y^*}$ and $\tilde{t}' = t'\gamma_T\delta^{y^*}$, then we must also have $\tilde{t}''_u = \tilde{t}'''_u$, so $\tilde{t} \in \tilde{T}''_u\tilde{t}'$, since \tilde{T}''_u is normal. Now, $s' \in (ui)\tilde{t}' = (ui)\tilde{T}''_u\tilde{t}' \supseteq (ui)\tilde{t}$ and $s \in (ui)\tilde{t}$. By Condition 1.3, $|(ui)\tilde{t}| = |(ui)\tilde{t}'| = 1$, so we must have $s = s'$. That is, η_S is injective, and the proof is complete. \square

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