Nonlinear Volterra Integrodifferential Systems with $L^1$-Kernels*

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I. INTRODUCTION

In this paper we are concerned with properties of integrodifferential systems of the form

$$x'(t) = Ax(t) + \int_0^t B(t - s)x(s) \, ds$$

$$+ \int_0^t L(t - s)x(s) \, ds + hx(t) + F(t - \tau),$$

(\text{N})

where $t \geq \tau$ and where $x(t) = f(t)$ on the interval $0 \leq t \leq \tau$. Here $\tau \geq 0$ is a given constant, $f$ is a given continuous vector-valued function, $A$, $B(t)$, and $L(t)$ are square matrices and $h$ is a functional which is "small" in a sense to be made precise later. A solution to (N) with initial values $(\tau, f)$ will be denoted by $x(t, \tau, f)$. If $\tau = 0$, then the function $f$ reduces to an initial vector $f(0) = x_0$ and $x(t, \tau, f) = x(t, 0, x_0)$.

Equations of this type were studied by the authors in [1] for the case $\tau = 0$. At first it might seem contrived to consider initial value problems of the form (N) when $\tau > 0$. Integrodifferential systems almost always occur in applications with $\tau = 0$. However, as was shown by Miller in [2], con-
siderable information concerning the special case $\tau = 0$ can be gleaned from consideration of the more general initial value problem. In [2], Miller studied the following linearized version of (N)

$$y'(t) - Ay(t) + \int_0^t B(t - s)y(s) \, ds,$$

with initial values $y(t) = f(t)$ on the interval $0 \leq t \leq \tau$. He showed that if $B(t) \in L^1(0, \infty)$ and if $R(t)$ is the resolvent associated with the matrices $A$ and $B(t)$ of (L), then $R(t)$ is of class $L^1(0, \infty)$ if and only if the trivial solution of (L) is uniformly asymptotically stable. He also obtained some sufficient conditions for uniform asymptotic stability.

The integrability of $R(t)$ is a more crucial fact than any stability properties of system (L). In this paper we prove that if $B(t) \in L^1(0, \infty)$ and if $\hat{B}(s)$ is the Laplace transformation of the function $B(t)$, then $R(t) \in L^1(0, \infty)$ if and only if $\det(s - A - \hat{B}(s)) \neq 0$ for $\Re s > 0$. This theorem contains and generalizes the corresponding results in [2]. It is also the analog for integro-differential systems of a theorem of Paley and Wiener [3, p. 60] which characterizes the integrability properties of resolvents of integral equations.

The remainder of this paper is organized as follows. Section II contains preliminary definitions and theorems. Section III contains the statement and proof of the theorem stated in the previous paragraph. In Section IV we apply the results in Section III together with a method developed by Grossman [4] to study stability properties of systems of the form (N). The last section contains a detailed comparison of the theorem of Paley and Wiener and our results on integrability properties of resolvents.

II. Preliminaries

Consider the initial value problem (L) where $\tau \geq 0$ is a real constant and $f(t)$ is a continuous vector valued function defined on the interval $0 \leq t \leq \tau$.

**Definition 2.1.** The differential resolvent $R(t)$ associated with the linear system (L) is the unique solution of the matrix equation

$$R'(t) = AR(t) + \int_0^t B(t - s)R(s) \, ds, \quad R(0) = I, \quad \text{(R)}$$

where $I$ is the identity matrix.

Consider the following system

$$y'(t) = Ay(t) + \int_0^t B(t - s)y(s) \, ds + P(t, y(\cdot)), \quad y(0) = y_0, \quad (2.1)$$
where \( P(t, y) \) is a given functional and \( B(t) \) is locally integrable on \([0, \infty)\), that is, \( \int_0^T |B(t)| \, dt < \infty \) for each \( T > 0 \). It was shown in [1] that (2.1) can be written in the equivalent “variation of constant form”

\[
y(t) = R(t)y_0 + \int_0^t R(t - s)P(s, y(s)) \, ds \quad (t > 0).
\]

Given an initial pair \((\tau, f)\) let \( y(t, \tau, f) \) be the unique solution of (L). Since \( B(t) \) is locally integrable the solution \( y(t, \tau, f) \) is known to exist locally (see Driver [5] for details). Then

\[
y'(t + \tau, \tau, f) = Ay(t + \tau, \tau, f) + \int_0^t B(t - s)y(s + \tau, \tau, f) \, ds
\]

for \( t \geq 0 \) and \( y(\tau, \tau, f) = f(\tau) \). Apply the variation of constants formula to this equation to obtain

\[
y(t + \tau, \tau, f) = R(t) f(\tau) + \int_0^t R(t - s) \left\{ \int_0^s B(s + \tau - u) f(u) \, du \right\} ds. \quad (2.3)
\]

We shall work with this form of (L) in the sequel.

Let \( M^n \) denote the set of all \( n \times n \) constant matrices. Let \( R^+ \) denote the interval \( 0 \leq t < \infty \) and let \( C(R^+) \) denote the set of all continuous functions \( \varphi: R^+ \rightarrow \mathbb{R}^n \). Given \( f \in C(R^+) \) and \( \tau > 0 \) let \( \|f\|_\tau = \max\{|f(t)| : 0 \leq t \leq \tau\} \).

**Definition 2.2.** Suppose \( B(t) \) is locally integrable on \( R^+ \). Consider the system (L) with initial conditions \((\tau, f) \in R^+ \times C(R^+)\). The trivial solution \( x = 0 \) is called: (i) stable if given any \( \tau \geq 0 \) and any \( \epsilon > 0 \) there exists a number \( \delta \) (depending on \( \epsilon \) and \( \tau \)) such that whenever \( f \in C(R^+) \) and \( \|f\|_\tau \leq \delta \), then the solution \( y(t, \tau, f) \) of (L) exists for all \( t \geq \tau \) and satisfies \( |x(t, \tau, f)| \leq \epsilon \);

(ii) uniformly stable if it is stable and \( \delta \) can be chosen independent of \( \tau \geq 0 \); or

(iii) uniformly asymptotically stable if it is uniformly stable and if given any \( \epsilon > 0 \) and any constant \( A > 0 \) there exists a number \( T(\epsilon) > 0 \) such that \( |y(t + T(\epsilon), \tau, f)| \leq \epsilon \) uniformly for all \( t \geq \tau \), all \( \tau \geq 0 \) and all functions \( f \) with \( \|f\|_\tau \leq A \).

**Definition 2.3.** Let \( A \) be a given constant \( n \times n \) matrix. Suppose \( B(t) \) is a locally integrable function whose Laplace transformation \( \hat{B}(s) \) is defined for all \( s \) in the half plane \( \text{Re}\, s \geq 0 \). Then the pair \((A, B(t))\) is said to satisfy condition (D) if and only if \( \det(s - A - \hat{B}(s)) \neq 0 \) for \( \text{Re}\, s \geq 0 \).
The following theorems are proved in [2].

**Theorem 2.4.** Suppose $B(t) \in L^1(R^+)$. If the trivial solution of (L) is uniformly asymptotically stable, then condition (D) is satisfied for the pair $(A, R(t))$.

**Theorem 2.5.** Suppose $B(t)$ and $R(t)$ are both in $L^1(R^+)$. Then (i) $R(t)$ and $R'(t) \in L^p(R^+)$ for $1 \leq p \leq \infty$ and both $R(t)$ and $R'(t) \to 0$ as $t \to \infty$; (ii) the trivial solution of (L) is uniformly asymptotically stable; and (iii) for any initial value $(\tau, f)$ in $R^+ \times C(R^+)$, the solution $y(t, \tau, f)$ of (L) is in $L^p(R^+)$ for $1 \leq p \leq \infty$.

**Theorem 2.6.** If $B(t) \in L^1(R^+)$ and if (L) is uniformly asymptotically stable, then the resolvent $R(t)$ associated with (L) is in $L^1(R^+)$. 

**Theorem 2.7.** Suppose $B(t) \in L^1(R^+)$, condition (D) is true for the pair $(A, B(t))$ and

$$
\int_0^\infty \int_s^\infty |B(u)| \, du \, ds < \infty.
$$

Then (L) is uniformly asymptotically stable.

Consider the integral equation

$$
x(t) = f(t) + \int_0^t a(t - s)x(s) \, ds, \quad (2.5)
$$

where $a(t)$ is locally integrable on $R^+$.

**Definition 2.8.** The integral resolvent $r(t)$ associated with (2.5) is the solution of the equation

$$
r(t) = -a(t) + \int_0^t a(t - s)r(s) \, ds. \quad (r)
$$

The integral resolvent can be used to solve (2.5), indeed

$$
x(t) = f(t) - \int_0^t r(t - s)f(s) \, ds. \quad (2.6)
$$

If $a(t)$ has a locally integrable derivative, then (2.5) can also be expressed in the form

$$
x'(t) = f'(t) + Ax(t) + \int_0^t B(t - s)x(s) \, ds, \quad x(0) = f(0),
$$
where $A = a(0)$ and $B(t) = a'(t)$. In this special case the differential resolvent $R(t)$ is also defined. The two types of resolvents are related by the formula

$$R(t) = I - \int_0^t r(s) \, ds.$$ 

Paley and Wiener [3, p. 60] proved the following result.

**Theorem 2.8.** If $a(t) \in L^1(R^+)$, then the integral resolvent $r(t) \in L^1(R^+)$ if and only if $\det(I - a(s)) \neq 0$ for $\text{Re} \, s \geq 0$.

### III. The Main Result

The purpose of this section is to precisely state and prove the main result of this paper, Theorem 3.5. We shall also prove that the set of all pairs $(A, B(t))$ which have resolvent $R(t) \in L^1(0, \infty)$ is an open set in $M^\pi \times L^1(0, \infty)$. This is Theorem 3.6. We begin with a simple definition. Given any $T > 0$ and the function $B(t)$ let

$$B_1(t) = B(t) \chi_{[0, T]}(t) = \begin{cases} B(t) & \text{on } 0 \leq t \leq T, \\ 0 & \text{on } T < t < \infty. \end{cases}$$

Rewrite system (L) as

$$y'(t) = Ay(t) + \int_0^t B_1(t - s)y(s) \, ds + \int_0^t B_2(t - s)y(s) \, ds, \quad (3.1)$$

where $B_2(t) = B(t) \chi_{[T, \infty)}(t) = B(t) - B_1(t)$.

**Lemma 3.1.** Suppose condition (D) is true for the pair $(A, B(t))$ and suppose $B(t) \in L^1(R^+)$. Then for all sufficiently large values of $T$ condition (D) is true for the pair $(A, B_1(t))$.

**Proof.** Suppose the lemma is false. Then there exists a sequence of positive, increasing, real numbers $T_n \to \infty$ with

$$\int_{T_n}^{\infty} |B(t)| \, dt < \frac{1}{n},$$

and there exists a sequence of complex numbers $s_n$ with $\text{Re} \, s_n \geq 0$ such that

$$\det(s_n - A - \int_0^{T_n} \exp(-s_n t)B(t) \, dt) = 0.$$
By the choice of $T_n$ and $s_n$ it follows that

$$\det(s_n - A - B(s_n)) = \epsilon_n, \quad \epsilon_n = O(1/n). \quad (3.2)$$

Now $\hat{B}(s) \to 0$ as $|s| \to \infty$ with $\Re s \geq 0$. This is trivial if $\Re s \to \infty$. It is essentially the Riemann–Lebesgue lemma if $\Re s$ is bounded but $|\Im s| \to \infty$. Using this fact in (3.2) it follows that the sequence $\{s_n\}$ must be bounded. Hence, there is a subsequence, again denoted by $\{s_n\}$, such that $s_n \to s$ for some $s$ with $\Re s \geq 0$. Since $\hat{B}(s)$ is continuous, then (3.2) implies that as $n \to \infty$

$$\lim_{n \to \infty} \det(s_n - A_n - B(s_n)) = \det(s - A - \hat{B}(s)) = 0.$$ 

This contradicts condition (D) so the lemma is established.

**Lemma 3.2.** \(\int_0^\infty \int_s^\infty B_1(u) \, du \, ds < \infty.\)

**Proof.** Since $B_1(t) = 0$ for $t \geq T$, then

$$\int_0^\infty \int_s^\infty |B_1(u)| \, du \, ds = \int_0^\infty \int_0^u |B_1(u)| \, ds \, du = \int_0^\infty u |B_1(u)| \, du,$$

$$= \int_0^T u |B(u)| \, du < \infty.$$

Now consider the truncated system

$$y'(t) = Ay(t) + \int_0^s B_1(t - s) y(s) \, ds \quad (t \geq \tau), \quad (3.3)$$

with $y(t) = f(t)$ on the interval $0 \leq t \leq \tau$. The constant $T$ used in the definition of $B_1(t)$ is chosen large enough to satisfy the conclusion of Lemma 3.1. Let $R_1(t)$ be the differential resolvent associated with system (3.3), that is

$$R_1'(t) = -m(t) + \int_0^t B_1(t - s) r(s) \, ds, \quad R_1(0) = I. \quad (3.4)$$

**Lemma 3.3.** $R_1(t) \in L^1(R^+)$. 

**Proof.** The last two lemmas show that the hypotheses of Theorem 2.7 are satisfied. Theorems 2.6 and 2.7 give the desired conclusion.

Let $v(t) = \int_0^t R_1(t - s) B_3(s) \, ds$, where $B_3(t)$ is the remainder $B_3(t) = B(t) - B_4(t)$ as defined before. Suppose $S(t)$ is the integral resolvent of $v(t)$, that is

$$S(t) = -v(t) + \int_0^t v(t - u) S(u) \, du. \quad (3.5)$$
If * is used to denote convolution multiplication and if \( \delta \) is used to denote the identity in this convolution algebra, then we can write \( v = R_1 \ast B_2 \). Moreover, (3.5) is equivalent to the relation

\[
\delta = (\delta - R_1 \ast B_2) \ast (\delta - S). \tag{3.6}
\]

Let \( y(t, \tau, f) \) be the solution of the truncated system (3.3) and \( x(t, \tau, f) \) the solution of (3.1). Then we have

\[
x'(t + \tau, \tau, f) = Ax(t + \tau, \tau, f) + \int_0^t B_1(t - s) x(s + \tau, \tau, f) \, ds
\]

\[
\quad \quad + \int_0^t B_2(t + \tau - s) f(s) \, ds + \int_0^t B_2(t - s) x(s + \tau, \tau, f) \, ds
\]

\[
\quad \quad + \int_0^t B_2(t + \tau - s) f(s) \, ds.
\]

On applying formula (2.3) we obtain

\[
x(t + \tau, \tau, f) = R_1(t) f(\tau) + \int_0^t R_1(t - s) \int_0^s B_1(s + \tau - u) f(u) \, du \, ds
\]

\[
\quad \quad + \int_0^t R_1(t - s) \left\{ \int_0^s B_2(s - u) x(s, \tau, f) \, du + \int_0^t B_2(s + \tau - u) f(u) \, du \right\} \, ds
\]

or

\[
x(t + \tau, \tau, f) = y(t + \tau, \tau, f) + R_1 \ast B_2 \ast x(t)
\]

\[
\quad \quad + R_1 \ast \left\{ \int_0^\tau B_2(t + \tau - s) f(s) \, ds \right\}.
\]

This together with (3.6) yields

\[
x(t + \tau, \tau, f)
\]

\[
= (\delta - S) \ast \left\{ y(t + \tau, \tau, f) + R_1 \ast \left( \int_0^\tau B_2(t + \tau - s) f(s) \, ds \right) \right\}. \tag{3.7}
\]

**Lemma 3.4.** If \( (A, B(t)) \) satisfies condition (D), \( B(t) \in L^1(0, \infty) \) and \( T \) is chosen sufficiently large, then the function \( S(t) \) defined by (3.5) is of class \( L^1(R^+) \).

**Proof.** Formula (3.5) or (3.6) indicates that \( S(t) \) is the integral resolvent of \( R_1 \ast B_2 \). Since both \( R_1(t) \) and \( B_2(t) \) are in \( L^1(0, \infty) \), then the convolution \( R_1 \ast B_2 \in L^1(R^+) \). By Theorem 2.8 above it is sufficient to show that

\[
\det(I - R_1 \ast \tilde{B}_2(s)) = \det(I - \tilde{R}_1(s) \tilde{B}_2(s)) \neq 0 \quad \text{for} \quad \text{Re} \, s \gg 0.
\]
Apply Laplace transforms to (3.4) to obtain
\[ s\hat{R}_1(s) - R_1(0) = A\hat{R}_1(s) + \hat{B}_1(s)\hat{R}_1(s) \]
or
\[ \hat{R}_1(s) = \{s - A - \hat{B}_1(s)\}^{-1} \quad (\text{Re } s \geq 0). \]
Therefore,
\[ I - \hat{R}_1(s)\hat{B}_2(s) = I - \{s - A - \hat{B}_1(s)\}^{-1}\hat{B}_2(s) \]
\[ = \{s - A - \hat{B}_1(s)\}^{-1}\{s - A - \hat{B}_1(s) - \hat{B}_2(s)\} \]
\[ = \{s - A - \hat{B}_1(s)\}^{-1}\{s - A - \hat{B}(s)\}. \]
This formula, condition (D), and Lemma 3.1 show that
\[ \det(I - \hat{R}_1(s)\hat{B}_2(s)) = \det(s - A - \hat{B}_1(s))^{-1}\det(s - A - \hat{B}(s)) \neq 0, \]
when \( \text{Re } s \geq 0 \). This proves Lemma 3.4.

The principle result of this paper is the following theorem.

**Theorem 3.5.** Given system (L) suppose \( B(t) \in L^1(\mathbb{R}^+) \). Then the differential resolvent \( R(t) \) is in \( L^1(\mathbb{R}^+) \) if and only if the pair \((A, B(t))\) satisfies condition (D).

**Proof.** The necessity of the theorem is Theorem 2.4. To prove sufficiency it is sufficient to show that the trivial solution of (L), or equivalently of (3.1), is uniformly asymptotically stable. Then Theorem 2.6 can be applied.

Fix \( T \) so large that Lemma 3.1 is true. For this \( T \) the functions \( R_1(t) \) and \( S(t) \) are well defined and integrable. Let \( \|R_1\|_1 \) and \( \|S\|_1 \) denote the \( L^1 \)-norms of \( R_1 \) and \( S \). From Theorem 2.5 we know that the trivial solution of (3.3) is uniformly asymptotically stable. Thus, there exists a constant \( A > 0 \) such that if \( y(t, \tau, f) \) is a solution of (3.3) then \( |y(t, \tau, f)| \leq A\|f\| \) uniformly for \( t \geq \tau \geq 0 \). Moreover, there exists \( \delta \) (independent of \( \epsilon > 0 \)) and \( T_1 = T_1(\epsilon_1) > 0 \) such that if \( \|f\|_\tau \leq \delta \) then \( |y(t, \tau, f)| \leq \epsilon_1 \) uniformly for \( t \geq T_1 + \tau \) and \( \tau \geq 0 \). Pick \( T_2 = T_2(\epsilon_1) \) so large that
\[ \int_{T_2}^{\infty} |S(t)| \, dt < \epsilon_1, \int_{T_2}^{\infty} |B_2(t)| \, dt < \epsilon_1 \]
and
\[ \int_{T_2}^{\infty} |\{R_1 \ast (\delta - S)\}(t)| \, dt < \epsilon_1. \]
To see that (3.1) is uniformly stable, use (3.7) to estimate
\[ |x(t + \tau, \tau, f)| \leq (1 + \|S\|_1)A\|f\|_\tau \]
\[ + \|R_1 \ast (\delta - S)\|_1 \max \left| \int_0^\tau B_2(t + \tau - s) f(s) \, ds \right| \]
\[ \leq \{(1 + \|S\|_1)A + \|R_1\|_1 (1 + \|S\|_1)\|B\|_1\|f\|_\tau \}. \]
This estimate implies uniform stability. To show uniform asymptotic stability we use (3.7) and the estimates for $\epsilon_1$ which were given previously. If $\tau \geq 0$, $\|f\|_r \leq \delta$ and $t \geq T^* = \max\{T, T_1(\epsilon_1), T_2(\epsilon_1)\}$, then

$$|x(t + \tau, \tau, f)| \leq |y(t + \tau, \tau, f)| + \int_0^T + \int_T^t \int_0^t |S(t - s)y(s + \tau, \tau, f)| ds$$

$$+ \int_0^T + \int_T^t \int_0^t \{|R_1 \ast (\delta - S)| (t - s) \int_0^T B_2(s + \tau - u) f(u) du\| ds$$

$$\leq \epsilon_1 + A \|f\|_r \int_0^\infty |S(t)| dt + \left(\int_0^\infty |S(t)| dt\right) \epsilon_1$$

$$+ \left(\int_T^\infty |\{R_1 \ast (\delta - S)\}(t)| dt\right) \|B_1\|_\infty \|f\|_r$$

$$+ \int_0^\infty |\{R_1 \ast (\delta - S)\}(t)| dt \times \int_T^\infty |B_2(t)| dt \|f\|_r$$

$$\leq \epsilon_1 + A \delta \epsilon_1 + \|S\|_1 \epsilon_1 + \epsilon_1 \|B_1\|_1 \delta + \|R_1 \ast (\delta - S)\|_1 \delta \epsilon_1 .$$

Hence, for a given $\epsilon$, we need only take

$$\epsilon_1 = \epsilon(1 + \delta A + \|S\|_1 + \delta \|B\|_1 + \delta \|R_1\|_l(1 + \|S\|_1))^{-1} .$$

If $t \geq T^*(\epsilon_1)$, then the solution $x(t, \tau, f)$ of (3.1) will satisfy

$$|x(t + \tau, \tau, f)| \leq \epsilon \quad \text{if} \quad \tau \geq 0, \quad \|f\|_r \leq \delta \quad \text{and} \quad t \geq T^* .$$

This proves the theorem.

We can give a variety of examples which satisfy the hypotheses of Theorem 2.5. For example if $B(t) = \exp(-Ct)$ where $C > 0$, then condition (D) is true if and only if $A < -C^{-1}$. If $B(t)$ is scalar valued and integrable, then the relation,

$$\text{Re}(i\omega - A - \dot{B}(i\omega)) > 0 \quad \text{for} \quad -\infty < \omega < \infty ,$$

will imply (D). If $A > 0$, this condition cannot be satisfied. If $A \leq 0$, the condition is $\text{Re} \dot{B}(i\omega) < -A$.

Before leaving this section we prove a topological result concerning the resolvent $R(t)$.

**Theorem 3.6.** Let $M^n = all n \times n$ constant matrices with norm $\|\|$ and let $X = M^n \times L^1(R^+)$ with the product topology. If $U = \{(A, B(t)) \in X: \text{the resolvent } R_{AB} \text{ associated with } (A, B(t)) \text{ is in } L^1(R^+)\}$, then $U$ is an open set in $X$. 
Proof. Given \((A, B(t)) \in X\), then \(R_{AB} \in L^1(R^+)\) if and only if
\[
\eta(A, B) = \min\{|\det(s - A - B^*(s))|: \Re s \geq 0\} > 0.
\]
Moreover, if \((A_n, B_n) \to (A, B)\) in \(X\), then \(B_n(s) \to B(s)\) uniformly for \(\Re s \geq 0\). Thus, \(\eta\) is a continuous function in \(X\).

IV. Existence Theorems

Since maps with kernels in \(L^1(R^+)\) are admissible with respect to a large class of Banach spaces, the results in the last section can be applied in a straightforward way to Theorems 4 and 5 in [1] in order to obtain new results. Instead of doing this directly we shall obtain some more general results by applying the results in the last section together with the techniques of [1] in order to analyze the nonlinear system \((N)\).

Let \(F = L^2(R^+)\) be the set of all functions \(\varphi: R^+ \to R^n\) such that for each \(T > 0\) the seminorm
\[
\int_0^T |\varphi(t)| \, dt < \infty.
\]
These seminorms generate a Fréchet space topology in \(L^2(R^+)\). As usual the set \(C(R^+)\) consists of all continuous functions \(\varphi: R^+ \to R^n\). This set can also be considered as a Fréchet space with seminorms
\[
\|\varphi\|_\tau = \max\{|\varphi(t)|: 0 \leq t \leq \tau\}.
\]

**Definition 4.1.** Let \(X\) be a Banach subspace of \(F\) with norm \(\|\|\).

(a) The norm \(\|\|\) is said to be stronger than the topology on \(X\) inherited from \(F\) if and only if \(x_n, x \in X\) and \(\|x_n - x\| \to 0\) as \(n \to \infty\) imply \(x_n \to x\) in \(F\).

(b) \(X\) is said to have the \(L^1\)-mapping property if and only if (a) is true and given any matrix valued function \(B \in L^1(R^+)\) with
\[
(B \ast \varphi)(t) = \int_0^t B(t - s) \varphi(s) \, ds \quad (0 \leq t < \infty),
\]
one has \(B \ast \varphi \in X\) whenever \(\varphi \in X\).

**Lemma 4.2.** If \(X\) has the \(L^1\)-mapping property, then for any \(B \in L^1(R^+)\) the function \(B \ast \varphi\) is a continuous linear map on \(X\) into \(X\).
The lemma is an immediate application of the closed graph theorem.

**Definition 4.3.** Let $X$ be a Banach subspace of $\mathcal{F}$ with a stronger norm. Fix $\tau \geq 0$. Then $X$ is said to be admissible with respect to (L) at $\tau$ if and only if given any $f \in C[0, \tau]$, if $y(t, \tau, f)$ is the solution of (L) with initial values $(\tau, f)$ then the function $Yf$ defined by

$$
(Yf)(t) = y(t + \tau, \tau, f) \quad (0 \leq t < \infty)
$$

is in $X$.

**Lemma 4.4.** Let $B(t) \in L^L(R^+)$ and let $\tau \geq 0$ be given. If $X$ is admissible w.r.t. (L) at $\tau$ then the map $Y$ defined by (4.1) is a continuous linear map of $C[0, \tau]$ into $X$.

**Proof.** It was shown in [2] that the solution $y(t, \tau, f)$ of (L) is a continuous function of $(t, \tau, f)$ for $t \geq \tau$, $\tau \geq 0$ and $f \in C(R^+)$. Therefore, the lemma follows immediately from the closed graph theorem.

The functional $h$ in system (N) will be required to be small in the following sense.

**Definition 4.5.** Let $X$ be a Banach subspace of $\mathcal{F}$ with stronger norm. Let $h: X \to X$. Then $h$ is said to be of higher order w.r.t. $X$ if and only if $h(0) = 0$ and for each $\epsilon > 0$ there exists $\delta > 0$ such that $\|h(q_1) - h(q_2)\| \leq \epsilon \|q_1 - q_2\|$ when $q_1$ and $q_2$ are in $X$ and $\|q_1\|, \|q_2\| \leq \delta$.

**Theorem 4.6.** Suppose $X$ has the $L^1$-mapping property, $X$ is admissible w.r.t. (L) at $\tau$, and $h$ is of higher order w.r.t. $X$. Assume that $B(t)$ and $L(t) \in L^L(R^+)$ and that both of the pairs $(A, B(t))$ and $(A, B(t) + L(t))$ satisfy condition (D). Then given $\epsilon > 0$ there exists a number $\delta > 0$ such that if $F \in C(R^+)$ with $\|F\|_{L^1} \leq \delta$ and if $F \in X$ with $\|F\| \leq \delta$, then equation (N) has a unique solution $x(t, \tau, f)$ such that $x(\cdot + \tau, \tau, f)$ is in $X$ and has norm less than $\epsilon$.

**Proof.** Let $y(t, \tau, f)$ be the solution of (L) with initial value $(\tau, f)$. The variation of constants formula can be employed in order to rewrite (N) in the form

$$
x(t + \tau, \tau, f) = y(t + \tau, \tau, f) + \int_0^t R(t - s) \{F(s) + h(x(s + \tau, \tau, f))(s)\} \, ds
$$

$$
+ \int_0^t R(t - s) \left\{ \int_s^t L(s - u) x(u + \tau, \tau, f) \, du \right\} \, ds.
$$
In convolution notation this equation takes the form
\[ x(t + \tau, \tau, f) = y(t + \tau, \tau, f) + R * \{ F + h(x(\cdot + \tau, \tau, f)) \} + R * L * x(\cdot + \tau, \tau, f). \] (4.2)

Let \( v = R * L \) and let \( S \) be the integral resolvent of \( v \). Since \( v \in L^1(R^+) \) and since
\[
\det(I - \delta(s)) = \det(I - R(s) L(s)) = \det(I - (s - A - B(s))^{-1} L(s)) = \det(s - A - B(s))^{-1} \det(s - A - B(s) - L(s)) \neq 0,
\]
for Re \( s \geq 0 \), then by Theorem 2.8 above it follows that \( S \in L^1(R^+) \). If \( \delta = \) identity in the convolution algebra, then (4.2) may be rewritten as
\[ x(t + \tau, \tau, f) = (\delta - S) \{ y + R * F \} + (\delta - S) * h(x(\cdot + \tau, \tau, f)) \] (4.3)

This equation has the form
\[ x = G_1 f + G_2 F + h_1(x), \] (4.4)
where \( G_1 : C[0, \tau] \to X \) and \( G_2 : X \to X \) are continuous linear maps and \( h_1 \) is of higher order w.r.t. \( X \). There exists \( \epsilon_0 > 0 \) such that if \( \varphi_1 \) and \( \varphi_2 \in X \) and \( \| \varphi_1 \|, \| \varphi_2 \| \leq \epsilon_0 \) then \( \| h_1(\varphi_1) - h_1(\varphi_2) \| \leq \| \varphi_1 - \varphi_2 \|/2 \). Given \( \epsilon \) in the interval \( 0 < \epsilon \leq \epsilon_0 \) let \( S(\epsilon) = \{ \varphi \in X : \| \varphi \| \leq \epsilon \} \). If
\[ \| G_1 f + G_2 F \| \leq \| G_1 \| \| f \|_{\tau} + \| G_2 \| \| F \| \leq \epsilon/2, \]
then the right side of (4.4) defines a contraction mapping on \( S(\epsilon) \). This proves the theorem.

The space \( X \) can be chosen in a variety of ways so that the hypotheses of Theorem 4.6 are true. All of the space considered here are easily seen to have the \( L^1 \)-mapping property. The other important consideration is admissibility w.r.t. \( L^p(R^+) \) at \( \tau \). Theorem 2.5 may be used to discuss this question. For example if \( L^p(R^+) \) is the usual Lebesgue space of \( p \)-integrable functions on \( R^+ \), then \( X = L^p(R^+) \) is admissible w.r.t. \( L^p(R^+) \) at \( \tau \) for any \( \tau \geq 0 \) and any \( P \) in the range \( 1 \leq P \leq \infty \). If \( X = \{ \varphi \in C(R^+) : \varphi' \in L^p(R^+) \} \), \( X = C(R^+) \cap L^\infty(R^+) \) or \( X = \{ \varphi \in C(R^+) \cap L^\infty(R^+) : \varphi(t) \to 0 \) as \( t \to \infty \} \), then \( X \) is admissible w.r.t. \( L^p(R^+) \) at \( \tau \) for any \( \tau \geq 0 \). One can also make up other spaces by intersection of two spaces \( X_1 \) and \( X_2 \) with norms \( \| \|_{1} \) and \( \| \|_{2} \), say \( X = X_1 \cap X_2 \) with norm \( \| \varphi \| = \| \varphi \|_{1} + \| \varphi \|_{2} \).

We conclude this section by proving a result for the case when the integral of the nonlinearity \( h \) satisfies an appropriate "smallness" condition. Consider the system (N) for some fixed \( \tau \geq 0 \). Define \( (g \varphi)(t) = \int_{0}^{t} h \varphi(s) \, ds \).
THEOREM 4.7. Suppose $X$ has the $L^1$-mapping property and is admissible w.r.t. $(L)$ at $\tau$. Suppose the functional $g$ defined above is of higher order w.r.t. $X$. Let $B(t) \in L^1(R^+)$ and let the pair $(A, B(t))$ satisfy condition (D). Then given any $\epsilon > 0$ there exists a number $\delta > 0$ such that if $f \in C[0, \tau]$ with $\|f\|_\epsilon \leq \delta$ and $F \in X$ with $\|F\| \leq \delta$, then $(N)$ has a unique solution $x(t) = x(t + \tau, \tau, f)$ in $X$ with $\|x\| \leq \epsilon$.

Proof. Let $y(t) = y(t + \tau, \tau, f)$ solve $(L)$. Then $(N)$ can be written in the equivalent form

$\quad x(t) = y(t) + (R * F)(t) + \int_0^t R(t - s) h_x(s) \, ds.$

Integration by parts yields

$\quad x(t) = y(t) + (R * F)(t) + g_x(t) - (R' * g_x)(t). \tag{4.5}$

By Theorem 2.5, $R'(t) \in L^1(R^+)$. Therefore, given any sufficiently small $\epsilon > 0$, the right side of (4.5) determines a contraction map on the sphere $S(\epsilon) = \{\varphi \in X : \|\varphi\| \leq \epsilon\}$. This proves the theorem.

V. REMARKS AND GENERALIZATIONS

Consider the integral equation

$\quad x(t) = F(t) + \int_0^t a(t - s) x(s) \, ds. \tag{5.1}$

If $a(t) \in L^1(R^+)$, then the Paley–Wiener theorem gives necessary and sufficient conditions in order that the integral resolvent $r(t)$ be in $L^1(R^+)$. Since the solution of (5.1) has the form

$\quad x(t) = F(t) - \int_0^t r(t - s) F(s) \, ds, \tag{5.2}$

this theorem gives a great deal of information about solutions of (5.1).

Theorem 3.5 will apply to (5.1) in case $F'(t)$ exists and is in $L^1(R^+)$ while

$\quad a(t) = A_0 + \int_0^t B_0(s) \, ds, \quad B_0(t) \in L^1(0, \infty).$

In this case integration by parts in (5.2) yields

$\quad x(t) = R(t) F(0) + \int_0^t R(t - s) F'(s) \, ds.$

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Theorem 3.5 gives necessary and sufficient conditions in order that the differential resolvent \( R(t) \in L^1(\mathbb{R}^+) \). If enough is known about \( F' \), then the theorem yields a great deal of information about the solution \( x(t) \). If
\[
a(t) = A_0t + A_1 + \int_0^t \int_0^{t_1} B_0(t_2) \, dt_2 \, dt_1 + \int_0^t B_1(t_1) \, dt_1, \tag{5.3}
\]
and if \( F(t) \) has two derivatives then (5.1) is equivalent to the second order system
\[
x''(t) = \{F''(t) + B_2(t)x(0)\} + A_0x(t) + A_1x'(t)
+ \int_0^t \{B_0(t - s)x(s) + B_1(t - s)x'(s)\} \, ds.
\]
We can introduce \( y(t) = x'(t) \) and then apply Theorem 3.5 to the resulting first order system. The differentiable resolvent of this resulting first order system has components \( R_{ij}(t) \) for \( i, j = 1, 2 \). Moreover, if \( r(t) \) is the integral resolvent of \( a(t) \), then
\[
R_{22}(t) = I - \int_0^t r(s) \, ds, \quad R_{12} = t - \int_0^t \int_0^s r(u) \, du \, ds
\]
and
\[
R_{21} = A_0R_{12} + B_0 \ast R_{12}, \quad R_{11} = R_{22} - A_1R_{12} - B_1 \ast R_{12}.
\]
It follows that \( \det(s^n - s(A_1 + \hat{B}_1(s)) - A_0 - \hat{B}_0(s)) \neq 0 \) for \( \text{Re } s \geq 0 \) if and only if
\[
(\frac{d^j}{dt^j}) \{t - \int_0^t \int_0^s r(u) \, du \, ds\} \in L^1(\mathbb{R}^+),
\]
for \( j = 0, 1, 2 \).

The same type of reasoning can be used to prove the following result.

**Theorem 5.1.** Suppose \( B_j \in L^1(\mathbb{R}^+) \) for \( j = 0, 1, \ldots, n \), where \( n \geq 0 \). Let
\[
a(t) = \sum_{j=0}^n \left\{ A_j t^{n-j} (n - j)! + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{j-1}} B_{n-j}(t_{j+1}) \, dt_{j+1} \cdots dt_1 \right\}
\]
and let \( r(t) \) be the integral resolvent of \( a(t) \). Then
\[
\det(s^{n+1} \{I - a(s)\}) = \det(s^{n+1} - (A_n + \hat{B}_n(s))S^n - \cdots - (A_0 + \hat{B}_0(s))) \neq 0
\]
for \( \text{Re } s \geq 0 \) if and only if
\[
\frac{d^j}{dt^j} \left\{ s^n \frac{t^n}{n!} - \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} r(t_n) \, dt_n \cdots dt_1 \right\} \in L^1(\mathbb{R}^+), \tag{5.4}
\]
for \( j = 0, 1, \ldots, n + 1 \).
If \( a(t) \) also contains a term \( B_{n+1}(t) \in L^1(\mathbb{R}^+) \), then one can still obtain part of Theorem 5.1.

**Theorem 5.2.** Suppose \( B_j \in L^1(\mathbb{R}^+) \) for \( j = 0, 1, \ldots, n + 1 \). Let

\[
a(t) = \frac{A_0 t^n}{n!} + \cdots + A_n + \int_0^t \int_0^{t_1} \cdots \int_0^{t_n} B_0(t_{n+1}) \, dt_{n+1} \cdots dt_1 + \cdots + \int_0^t \cdots \int_0^{t_1} B_{n-1}(t_2) \, dt_2 dt + \int_0^t B_n(t_1) \, dt_1 + B_{n+1}(t),
\]

and let \( r(t) \) be the integral resolvent of \( a(t) \). If

\[
\det(s^{n+1}(I - a(s))) = \det(s^n - s^{n+1}B_{n+1}(s)) - \cdots - s(A_1 + B_1(s)) - (A_0 + B_0(s)) \neq 0,
\]

for \( \text{Re} \, s \geq 0 \), then \( r(t) \in L^1(\mathbb{R}^+) \).

**Proof.** Write \( a(t) = a_I(t) + B_{n-1}(t) \). Let \( b_{n-1} \) be a \( C^1 \) function with compact support which approximates \( B_{n-1} \) in the \( L^1 \)-norm so closely that

\[
\det(s^{n+1}(I - \tilde{B}_{n+1}(s))) - s^n(A_n + \tilde{B}_n(s)) - \cdots - s(A_1 + \tilde{B}_1(s)) - (A_0 + \tilde{B}_0(s)) \neq 0,
\]

for \( \text{Re} \, s \geq 0 \). (That this is possible can be proved in the manner of Lemma 3.1.) Let \( \Delta B = B_{n+1} - b_{n+1} \) and let \( \rho(t) \) be the integral resolvent of \( a(t) - \Delta B(t) \). Since \( b_{n+1} \) has an \( L^1 \) derivative, then Theorem 5.1 applies. On using (5.4) with \( j = n + 1 \) we see that \( \rho(t) \in L^1(\mathbb{R}^+) \). Thus, (5.1) has the form

\[
x = F + \Delta B \ast x + (a - \Delta B) \ast x
\]

or

\[
x = \{f + \Delta B \ast x\} - \rho \ast \{f + \Delta B \ast x\}
\]

\[
= \{f - \rho \ast F\} + \{\Delta B - \rho \ast \Delta B\} \ast x.
\]

Since \( \Delta B - \rho \ast \Delta B \in L^1 \), then Theorem 2.8 applies. But

\[
\det(I - \Delta B(s) + \tilde{\rho}(s) \Delta B(s))
\]

\[
= \det(s^{n+1}(I - \tilde{B}_{n+1}(s))) - s^n(A_n + \tilde{B}_n(s)) - \cdots - (A_0 + \tilde{B}_0(s))^{-1}
\]

\[
\det(s^{n+1}(I - \tilde{B}_{n+1}(s))) - s^n(A_n + \tilde{B}_n(s)) - \cdots - (A_0 + \tilde{B}_0(s)) \neq 0,
\]

for \( \text{Re} \, s \geq 0 \) by the assumptions and the choice of \( b_{n+1} \). This means that the solution \( x(t) \) of (5.1) is

\[
x = \{F - \rho \ast F\} - S \ast \{F - \rho \ast F\}.
\]
where \( \rho \) and \( S \) are \( L^1 \) functions. In particular the solution \( x(t) \) of (5.1) is bounded whenever \( F \) is bounded. This can only happen if \( r(t) \in L^1(\mathbb{R}^+) \), see [6, Theorem 3].

REFERENCES