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Note

The structure of imperfect critically strongly-imperfect graphs

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Abstract

The family of all critically strongly-imperfect graphs decomposes in two nonempty classes: perfect and imperfect ones. In this paper we characterize the critically strongly-imperfect graphs which are, simultaneously, imperfect. We prove that these are precisely the holes of odd length $\geqslant 5$ or their complements.

1. Introduction

All graphs considered in this paper are finite and have neither loops nor multiple edges. We shall use the following terminology and notation: $V(G)$, $E(G)$ are the set of vertices and the set of edges of the graph G, respectively; $\bar{G}$ the graph which is complementary to G; $H \subseteq G{:}H$ an induced subgraph of G; $H \subset G{:}H$ a proper induced subgraph of G; For $A \subset V(G)$ and $K \subset E(G)$, $G - A := [V(G) - A]_G$, $G - K := (V(G), E(G) - K)$, $\mathrm{Co}(A) := \{xy\colon x \in A, y \in V(G) - A \text{ and } xy \in E(G)\}$ is the *co-circuit induced* by A in G; For $S, Q \subseteq V(G)$, S is a *stable set* in $G \Leftrightarrow E([S]) = \emptyset$, Q is a *clique* in $G \Leftrightarrow Q$ is a stable set in the complement of G, $\mathscr{S}(G) := \{S\colon S$ is a maximal stable set of $G\}$, $\mathscr{C}(G) := \{Q\colon Q$ is a maximal clique of $G\}$, $\alpha(G) := \max\{|S|\colon S \in \mathscr{S}(G)\}$ is the *stability number* of G, $\omega(G) := \max\{|C|\colon C \in \mathscr{C}(G)\}$ is the density number of G, $\chi(G)$ is the *chromatic number* of G, $\theta(G) := \chi(\bar{G})$, $\mathscr{S}_\alpha(G) := \{S\colon S \in \mathscr{S}(G)$ and $|S| = \alpha(G)\}$, $\mathscr{C}_\omega(G) := \{C\colon C \in \mathscr{C}(G)$ and $|C| = \omega(G)\}$.

Let M be a set, $\mathscr{F} = \{M_i\colon i \in I\}$ a family of subsets of M and T a subset of M. The set T is called a *transversal* of $\mathscr{F}$ iff $T \cap M_i \neq \emptyset$ for all $i \in I$. A transversal T of $\mathscr{F}$ is called *perfect* iff $|T \cap M_i| = 1$ for all $i \in I$.

Definition. A graph G is called

(1) *s-perfect* (*c-perfect*) iff for every induced subgraph H of G the family $\mathscr{S}(H)$ ($\mathscr{C}(H)$) has a perfect transversal;

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(2) *critically* (or *minimally*) *s-imperfect* (*c-imperfect*) — or, briefly, *s-critical* (*c-critical*) — iff G is *s*-imperfect (c-imperfect) and every subgraph $G - x$ ($x \in V(G)$) is s-perfect (c-perfect).

The c-perfect graphs are also known as *strongly perfect graphs*. This concept was introduced by Berge and Duchet [1].

Remark 1. It is easy to check that the following propositions hold:

(a) Let G be a graph. A transversal T of $\mathscr{S}(G)$ ($\mathscr{C}(G)$) is perfect if and only if T is a maximal clique (maximal stable set) of G (see [3])

(b) A graph G is s-perfect (s-critical) if and only if its complement is c-perfect (c-critical).

(c) Every hole C_k of length $\geqslant 5$ is critically s-imperfect; if k is odd then $\bar{C}_k$ is critically s-imperfect, too.

(d) Every s-imperfect graph contains a critically s-imperfect graph as an induced subgraph.

Definition. A graph G is called:

(1) *α-perfect* (*ω-perfect*) iff for every induced subgraph H of G, the family $\mathscr{S}_\alpha(H)$ ($\mathscr{C}_\omega(H)$) has a perfect transversal which is a clique (stable set) in H;

(2) *perfect* iff G is α-perfect as well as ω-perfect;

(3) *critically* (or *minimally*) *imperfect* if G is imperfect and every subgraph $G - x$ ($x \in V(G)$) is perfect.

Remark 2. A graph G is α-perfect (ω-perfect) if and only if, for every induced subgraph H of G, we have $\theta(H) = \alpha(H)$ ($\chi(H) = \omega(H)$).

Note that, in fact, (1) and (2) are equivalent; this is the content of the weak version of Berge's Perfect Graph Conjecture and follows from an important result of Lovász [2].

Towards a characterization of the structure of perfect graphs, the following conjecture, due to Berge, has already become famous.

Strong Perfect Graph Conjecture. *A graph G is critically imperfect if and only if G or its complement is an odd hole of length* $\geqslant 5$.

The problem is: how to characterize, in a simple way, the structure of critically s-imperfect (c-imperfect) graphs?

Notice that the class of all critically s-imperfect graphs decomposes into two nonempty classes: The graphs in class 1 are perfect (e.g., all holes of even length $\geqslant 6$) and the graphs in class 2 are imperfect (e.g., all holes of odd length $\geqslant 5$ and their complements).

Our aim is to characterize the graphs of the second class. It is easy to see that all these graphs are necessarily critically imperfect. The main result is

Theorem 1. *A graph G is critically s-imperfect (c-imperfect) and, simultaneously, imperfect if and only if G or its complement is a hole of odd length* $\geqslant 5$.

This theorem immediately implies that the following statement is equivalent to the Strong Perfect Graph Conjecture.

Every critically imperfect graph is also critically strongly imperfect.

2. Proof of Theorem 1

Definition. Let G be a graph. An edge e of G is called *α-critical* (in G) iff $\alpha(G-e)=\alpha(G)+1$.

The partial graph G_α of G with

$$E(G_\alpha)=\{e\colon e \text{ is an } \alpha\text{-critical edge of } G\}$$

is called the *α-critical skeleton* of G.

A set $A \subset V(G)$ with $\mathrm{Co(A)} \cap E(G_\alpha) \neq \emptyset$ is called an *α-critical set* of G.

A path of G_α is called an *open α-critical path* (or *α-critical*, for short) iff its extremities are not adjacent in G.

A recent result of Sebö [4, Theorem 1.2] can be reformulated as follows:

Theorem 2. *If a critically-imperfect graph contains an α-critical path, then the graph or its complement is a hole of odd length* $\geqslant 5$.

We shall show that every critically s-imperfect graph, which is also imperfect, contains at least an α-critical path. Then, the proof of Theorem 1 follows from Sebö's theorem.

Lemma. *If G is both a critically s-imperfect and an imperfect graph, then the following statements are true:*

(i) *every clique of G is α-critical;*

(ii) *G contains an α-critical path.*

Proof. (i) Let G be a critically s-imperfect graph with $\theta(G)>\alpha(G)$ and $A \subset V(G)$ a clique G. Then $G-A$ possesses a clique transversal Q (see Remark 1(a)). Clearly $\alpha(G-A)=\alpha(G)$ for, otherwise, G would be perfect. This implies the existence of a maximal stable set $S \in \mathscr{S}_\alpha(G-Q)$. The set Q being a clique transversal of $G-A$, we have

$$\alpha(G-A-Q) \leqslant \alpha(G)-1$$

and therefore $S \cap A \neq \emptyset$. From the fact that S is a stable set and A is a clique in G we infer that $A \cap S = \{x\}$, for some x, i.e., $S_x := S - \{x\}$ is a stable set in $G - A$. Since $Q \cap S_x = \emptyset$, there is a maximal stable set S' of $G - A$ such that $S_x \subset S'$ for, otherwise, Q could not be a transversal of $S(G - A)$. Let $\{y\} = S' \cap Q$. Clearly, $x \neq y$ and $xy \in E(G)$ because, if not, $S_x \cup \{x, y\}$ would be a stable set of G with $|S_x \cup \{x, y\}| = \alpha(G) + 1$ vertices, in contradiction to the definition of $\alpha(G)$.

Now, $|S_x| = \alpha(G) - 1$ and $S_x \subset S'$ entail $|S'| = \alpha(G)$, i.e., the edge xy is α-critical in G. Evidently, $xy \in \mathrm{Co}(A)$.

(ii) Let H be a (connected) component of the α-critical skeleton G_α of G. Then, $V(H)$ is not a clique in G, since otherwise (i) ensures that

$$\mathrm{Co}(V(H)) \cap E(G_\alpha) \neq \emptyset,$$

thus contradicting the fact that H is a connected component of G. Consequently, there are at least two vertices x, y in $V(H)$, which are not adjacent in G. Then, every xy-path of H is an α-critical path in G. □

Now, the proof of Theorem 1 follows directly from this lemma and Theorem 2, when G is critically s-imperfect. If G is critically c-imperfect (and imperfect, too), then $\bar{G}$ is critically s-imperfect and, consequently, $\bar{G}$ is a hole of odd length $\geqslant 5$ or its complement and therefore, G has the same structure.

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