From Boolean to sign pattern matrices
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Abstract
A nonnegative sign pattern matrix is a matrix whose entries are from the set \{+, 0\}. A nonnegative sign pattern matrix can also be viewed as a Boolean matrix, by replacing each + entry with 1. In this paper, some interesting connections between nonnegative sign pattern matrices and Boolean matrices are investigated. In particular, the relations between the minimum rank and the Boolean row (or column) rank are explored; the idempotent Boolean matrices that allow idempotence are identified; and the nonnegative sign patterns that allow various types of nonnegative (or positive) generalized inverses are characterized.

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1. Introduction

In qualitative and combinatorial matrix theory, we study properties of a matrix based on combinatorial information, such as the signs of entries in the matrix. A matrix whose entries are from the set \{+, −, 0\} is called a sign pattern matrix.

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(or sign pattern, or pattern). We denote the set of all \( n \times n \) sign pattern matrices by \( Q_n \). For a real matrix \( B \), \( \text{sgn}(B) \) is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of \( B \) by + (respectively, −, 0). If \( A \in Q_n \), then the sign pattern class of \( A \) is defined by

\[
Q(A) = \{ B : \text{sgn}(B) = A \}.
\]

A subpattern of a sign pattern \( A \) is a sign pattern matrix obtained by replacing some (possibly none) of the + or − entries in \( A \) with 0. The sign pattern \( I_n \in Q_n \) is the diagonal pattern of order \( n \) with + diagonal entries.

A sign pattern matrix \( S \) is called a permutation pattern if exactly one entry in each row and column is equal to +, and all other entries are 0. A product of the form \( S^T \Lambda S \), where \( S \) is a permutation pattern, is called a permutational similarity.

Two sign pattern matrices \( A_1 \) and \( A_2 \) are said to be permutationally equivalent if there are permutation patterns \( S_1 \) and \( S_2 \) such that \( A_1 = S_1 A_2 S_2 \).

Suppose \( P \) is a property referring to a real matrix. Then a sign pattern \( A \) is said to require \( P \) if every real matrix in \( Q(A) \) has property \( P \), or to allow \( P \) if some real matrix in \( Q(A) \) has property \( P \).

A sign pattern \( A \in Q_n \) is said to be sign nonsingular if every matrix \( B \in Q(A) \) is nonsingular. It is well known that \( A \) is sign nonsingular if and only if \( \det A = + \) or \( \det A = − \), that is, in the standard expansion of \( \det A \) into \( n! \) terms, there is at least one nonzero term, and all the nonzero terms have the same sign. \( A \) is said to be sign singular if every matrix \( B \in Q(A) \) is singular, or equivalently, if \( \det A = 0 \).

A sign pattern matrix \( A \) is said to be an L-matrix (see [4]) if every real matrix \( B \in Q(A) \) has linearly independent rows. It is known that \( A \) is an L-matrix if and only if for every nonzero diagonal pattern \( D \), \( DA \) has an unsinged column (that is, a nonzero column that is nonnegative or nonpositive).

For a sign pattern matrix \( A \), the minimum rank of \( A \), denoted \( \text{mr}(A) \), is defined as

\[
\text{mr}(A) = \min_{B \in Q(A)} \{ \text{rank } B \}.
\]

Following [8], we now define some terminology concerning Boolean vectors and Boolean matrices. Let \( \mathcal{B} \) be the \((0, 1)\) Boolean algebra. A Boolean matrix (or vector) has entries (or components) in \( \mathcal{B} \). Let \( \mathcal{B}^n \) be the set of all Boolean vectors with \( n \) components. For Boolean vectors \( x_1, x_2, \ldots, x_k \in \mathcal{B}^n \), the linear manifold \( \mathcal{M}(x_1, x_2, \ldots, x_k) \) is the set of all vectors of the form \( \sum_{i=1}^{k} c_i x_i \), where \( c_i \in \mathcal{B} \). A Boolean vector \( y \in \mathcal{B}^n \) is said to be dependent on \( x_1, x_2, \ldots, x_k \) if \( y \in \mathcal{M}(x_1, x_2, \ldots, x_k) \). Otherwise, \( y \in \mathcal{B}^n \) is said to be independent of \( x_1, x_2, \ldots, x_k \). A set of Boolean vectors \( \{ x_1, x_2, \ldots, x_k \} \subseteq \mathcal{B}^n \) is said to be dependent if one vector in the set is the sum of some of the remaining vectors or the zero vector is in the set. Otherwise, the set is said to be independent.

Let \( T = \{ x_1, x_2, \ldots, x_k \} \), where \( x_i \in \mathcal{B}^n \). A set \( S \subseteq T \) is said to be a basis of \( T \) if \( S \) is independent and \( T \subseteq \mathcal{M}(S) \). It is known (see [8]) that every \( T \subseteq \mathcal{B}^n \) has a unique basis. The cardinality of the basis for \( T \) is called the rank of \( T \).
Let \( A \) be a Boolean matrix. The **Boolean row (column) rank** of \( A \) is defined to be the rank of the set of row (column) vectors of \( A \). Since a nonnegative sign pattern matrix (namely, a matrix whose entries are from the set \{+, 0\}) may be viewed as a Boolean matrix (by identifying each + entry with 1), Boolean row (column) rank is now defined for a nonnegative sign pattern matrix. Note that for a nonnegative sign pattern matrix \( A \), the Boolean row rank of \( A \) and the Boolean column rank of \( A \) may be different (see [8]). When these are indeed the same, this common value is called the **Boolean rank** of \( A \).

We now introduce two new notions, not formerly found in the literature. Let \( T = \{x_1, x_2, \ldots, x_k\} \), where \( x_i \in \mathbb{B}^n \). \( T \) is said to be **weakly dependent** if there exist two disjoint subsets \( S_1 \) and \( S_2 \) of \{1, 2, \ldots, k\}, not both empty (by convention, an empty sum is equal to 0), such that \( \sum_{i \in S_1} x_i = \sum_{j \in S_2} x_j \). Otherwise, \( T \) is said to be **strongly independent**. It can be seen that for row vectors \( x_1, x_2, \ldots, x_k \) in \( \mathbb{B}^n \), \( \{x_1, x_2, \ldots, x_k\} \) is weakly dependent if and only if the matrix

\[
\begin{bmatrix}
  x_1 \\
  \vdots \\
  x_k
\end{bmatrix}
\]

is not an L-matrix. In other words, \( \{x_1, x_2, \ldots, x_k\} \) is strongly independent if and only if the matrix

\[
\begin{bmatrix}
  x_1 \\
  \vdots \\
  x_k
\end{bmatrix}
\]

is an L-matrix. Note that for \( k \leq 3 \), \( \{x_1, \ldots, x_k\} \) is independent if and only if \( \{x_1, \ldots, x_k\} \) is strongly independent, if and only if the matrix

\[
\begin{bmatrix}
  x_1 \\
  \vdots \\
  x_k
\end{bmatrix}
\]

is an L-matrix.

Let \( B, X \) be real (or Boolean) matrices. Consider the following conditions.

1. \( BXB = B \).
2. \( XBX = X \).
3. \( BX \) is symmetric.
4. \( XB \) is symmetric.

For a real matrix \( B \), the unique matrix \( X \) satisfying all four conditions above is called the Moore–Penrose inverse of \( B \) and is denoted by \( B^\dagger \). More generally, let \( B[i, j, \ldots, l] \) denote the set of matrices \( X \) satisfying conditions \((i), (j), \ldots, (l)\) from among conditions \((1)–(4)\). A matrix \( X \in B[i, j, \ldots, l] \) is called an \((i, j, \ldots, l)\)-inverse of \( B \). For example, if \( (1) \) holds, \( X \) is called a \((1)\)-inverse of \( B \); if \( (1) \) and \( (2) \) hold, \( X \) is called a \((1, 2)\)-inverse of \( B \), and so forth. See [1] or [3] for further information on generalized inverses.

In this paper, we investigate some connections between nonnegative sign pattern matrices and Boolean matrices. In Section 2, we explore the relations between the minimum rank and the Boolean row (or column) rank. In Section 3, we study the
idempotent Boolean matrices that allow idempotence. In Sections 4 and 5, we characterize the nonnegative sign patterns that allow various types of nonnegative (or positive) generalized inverses. The results in this paper are motivated by results in [8] on Boolean matrices. However, this paper also provides new and substantial results on nonnegative sign patterns that do not follow from the results of [8]. For example, we prove that if a nonnegative sign pattern \( A \) has a nonnegative \((1, 4)\)-inverse, then \( A \) allows a nonnegative \((1, 4)\)-inverse. We also show that the same is true for \((1, 3)\)-inverse and the Moore–Penrose inverse.

2. Boolean row (column) rank and minimum rank

We start with a basic fact relating Boolean row (column) rank and minimum rank.

**Observation 2.1.** Let \( A \) be an \( m \times n \) nonnegative sign pattern matrix. Then

\[
\mr(A) \leq \min\{\text{Boolean column rank of } A, \ \text{Boolean row rank of } A\}.
\]

This observation follows from the fact that a Boolean basis for the columns (rows) of \( A \) can serve as a spanning set for the columns (rows) of some real matrix \( B \in Q(A) \). For nonnegative sign patterns that have fewer than four rows (or fewer than four columns), it can be shown (see the comment after Theorem 2.3) that we have equality in the above inequality. However, equality does not hold in general, as can be seen from the following example.

**Example 2.2.** Let

\[
A = \begin{bmatrix}
+ & + & + & 0 \\
+ & + & 0 & + \\
+ & 0 & + & + \\
0 & + & + & + \\
\end{bmatrix}.
\]

Then \( \mr(A) = 3 < 4 = \text{Boolean rank of } A \).

That \( A \) has Boolean column and row rank 4 should be clear. Now, the upper-right \( 3 \times 3 \) submatrix \( A_1 \) of \( A \) is sign nonsingular, with \( \det(A_1) = -1 \). So, \( \mr(A) \geq 3 \). However, \( A \) is not sign nonsingular as the matrix

\[
B = \begin{bmatrix}
1 & 1 & 2 & 0 \\
1 & 1 & 0 & 2 \\
2 & 0 & 1 & 1 \\
2 & 0 & 2 & 1 \\
\end{bmatrix}
\]

in \( Q(A) \) is singular. Indeed, for \( B \), row 1 + row 2 = row 3 + row 4. Hence, \( \mr(A) = 3 \). Note that the rows of \( A \) are independent, but weakly dependent.
In the following theorem, we determine exactly when we can have equality in Observation 2.1.

**Theorem 2.3.** Let $A$ be an $m \times n$ nonnegative sign pattern matrix and let $F$ be a submatrix of $A$ whose rows form a Boolean basis for the rows of $A$. Then
\[
\text{mr}(A) = \text{Boolean row rank of } A
\]
if and only if $F$ is an L-matrix.

**Proof.** We may assume that $F$ consists of say the first $r$ rows of $A$. Suppose \( \text{mr}(A) = \text{Boolean row rank of } A \), but $F$ is not an L-matrix. Then there exists $C \in Q(F)$ such that the rows of $C$ are linearly dependent. Now, the rows in $A$ but not in $F$ are Boolean linear combinations of the rows of $F$. Hence, we have a real matrix $B \in Q(A)$ whose first $r$ rows comprise the matrix $C$ and whose last $m - r$ rows are nonnegative linear combinations of the rows of $C$. But then
\[
\text{rank}(B) = \text{rank}(C) < \text{Boolean row rank of } A,
\]
which contradicts $\text{mr}(A) = \text{Boolean row rank of } A$. Thus, $F$ must be an L-matrix.

Conversely, suppose $F$ is an L-matrix. Then, for every $C \in Q(F)$, $\text{rank}(C) = r$, so that $\text{mr}(A) = r = \text{Boolean row rank of } A$. From Observation 2.1, $\text{mr}(A) \leq \text{Boolean row rank of } A$, and thus $\text{mr}(A) = \text{Boolean row rank of } A$. □

Let $A$ be an $m \times n$ nonnegative sign pattern matrix and once again let $F$ be a submatrix of $A$ whose rows form a Boolean basis for the rows of $A$ (we do not assume $F$ is an L-matrix). If $F$ has three or fewer rows, then since three or fewer Boolean vectors are independent if and only if they are strongly independent, it can be seen that $F$ is an L-matrix and thus $\text{mr}(A)$ is equal to the Boolean row rank of $A$. More generally, let $\text{mr}(F) = k$. As in the proof of Theorem 2.3, we can obtain $C \in Q(F)$ with $\text{rank}(C) = k$ and $B \in Q(A)$ such that $\text{rank}(B) = \text{rank}(C) = \text{mr}(F)$. So, $\text{mr}(A) \leq \text{mr}(F)$. But clearly, $\text{mr}(A) \geq \text{mr}(F)$ (for any submatrix $F$ of $A$), and thus $\text{mr}(A) = \text{mr}(F)$.

Next, we can take a submatrix $G$ of $F$ whose columns form a Boolean basis for the columns of $F$. Then $\text{mr}(F) = \text{mr}(G)$, and thus
\[
\text{mr}(A) = \text{mr}(F) = \text{mr}(G).
\]

We next discuss rank factorizations. In general, a nonnegative real matrix may not have a nonnegative full-rank factorization (see [9]). For nonnegative sign pattern matrices, minimum rank factorizations are crucial and we make the following definition. Let $A$ be an $m \times n$ nonnegative sign pattern matrix, with $\text{mr}(A) = r$.

We say that $A$ has a nonnegative minimum rank factorization if $A = HK$ for some $m \times r(r \times n)$ nonnegative sign pattern matrices $H(K)$ where $\text{mr}(A) = \text{mr}(H) = \text{mr}(K) = r$.

If $A$ has such a factorization, then since $r = \text{mr}(K) \leq \text{Boolean row rank of } K \leq r$, we have $\text{mr}(K) = \text{Boolean row rank of } K$; similarly, $\text{mr}(H) = \text{Boolean column
rank of $H$. Further, $H(K)$ has strongly independent columns (rows). However, non-negative minimum rank factorization is not always possible.

**Example 2.4.** As in Example 2.2, let
\[
A = \begin{bmatrix}
++&+&0 \\
++&0&+ \\
++&0&+ \\
0&+&++ \\
\end{bmatrix}.
\]
It can be shown by discussing various cases that the columns of $A$ cannot be generated (as Boolean combinations) by any three nonnegative vectors. Therefore, $A$ does not have a nonnegative minimum rank factorization.

Even when a nonnegative sign pattern matrix $A$ has a nonnegative minimum rank factorization, $\text{mr}(A)$ may not be equal to the Boolean row (column) rank of $A$.

**Example 2.5.** Let
\[
A = \begin{bmatrix}
++&+&0 &0 \\
++&0 &+ &+ \\
0 &0 &++ &+ \\
0 &+&++ &+ \\
\end{bmatrix} = \begin{bmatrix}
++&+&0 &0 \\
++&0 &0 &+ \\
0 &0 &++ &+ \\
0 &+&++ &+ \\
\end{bmatrix} = HK.
\]
Clearly, $A$ is not sign nonsingular. Since the upper-right $3 \times 3$ submatrix of $A$ is sign nonsingular, $\text{mr}(A) = 3$, and $HK$ is a nonnegative minimum rank factorization of $A$. However, both the Boolean row and column ranks of $A$ are 4.

It is worth mentioning that if $\text{mr}(A)$ (\text{mc}(A)) denotes the maximum number of strongly independent rows (columns) of a nonnegative sign pattern $A$, then clearly we have

**Proposition 2.6.** For every nonnegative sign pattern $A$, $\max\{\text{mr}(A), \text{mc}(A)\} \leq \text{mr}(A)$.

Strict inequality is possible, as the following example shows.

**Example 2.7.** Let $G$ be the $5 \times 10$ sign pattern corresponding to the matrix $\Gamma_2$ as defined on page 20 of [4]. That is,
\[
G = \begin{bmatrix}
0&0&0&0&++&++&++&++ \\
0&+&+&+&0&0&++&+ \\
+&0&+&+&0&++&0&0 \\
+&+&0&+&+&0&++&0 \\
+&+&+&0&+&0&0&0 \\
\end{bmatrix}.
\]
which is the $5 \times 10$ nonnegative sign pattern consisting of all possible columns with exactly three positive entries in each column. It is shown in [4] that $G$ is a barely L-matrix, that is to say, $G$ is an L-matrix and if one or more columns are deleted from $G$, then the resulting matrix is not an L-matrix. The fact that $G$ is an L-matrix means that the 5 rows of $G$ are strongly independent, so that $m_r(G) = 5$. However, $G$ does not have 5 strongly independent columns. In fact, if $G$ had 5 strongly independent columns, then such 5 columns would form a $5 \times 5$ sign nonsingular matrix, and thus we obtain a $5 \times 5$ submatrix of $G$ that is an L-matrix, contradicting the fact that $G$ is a barely L-matrix. On the other hand, the columns $c_1, c_2, c_3, c_5$ of $G$ can be seen to be strongly independent. Thus we have $m_r(G) = 4 < m_c(G) = 5$. Furthermore, for $A = \begin{bmatrix} 0 & G \\ G^T & 0 \end{bmatrix}$, we have $mr(A) = 2$ and $mr(G) = 10$, while $m_r(A) = m_c(A) = 9$. Thus $\max\{m_r(A), m_c(A)\} < mr(A)$.

We remark that in the characterizations in the next two sections the sign patterns $A$ have a nonnegative minimum rank factorization and also $mr(A)$ = Boolean rank of $A$. There is, however, a general open question. Suppose that $A$ has a nonnegative minimum rank factorization. Are there nontrivial necessary and sufficient conditions to describe when $mr(A)$ = Boolean row (column) rank of $A$?

3. Idempotents

Clearly, if a square nonnegative pattern $A$ allows a real idempotent, that is, there is an idempotent matrix $B \in \mathcal{Q}(A)$, then $A$ is idempotent. The converse does not hold. For example, the pattern $A = \begin{bmatrix} ++ & + \\ 0 & + \end{bmatrix}$ is idempotent, but does not allow a real idempotent. The following result from [6] determines when a nonnegative pattern allows a real idempotent.

**Proposition 3.1.** Let $A$ be a square nonnegative sign pattern matrix, with $mr(A) = r$. Then $A$ allows a real idempotent if and only if $A$ is permutationally similar to a pattern of the form

$$
\begin{bmatrix}
I_r & A_2 \\
A_3 & A_3A_2
\end{bmatrix},
$$

where $A_2A_3$ is a subpattern of $I_r$ (that is, $A_2A_3$ is a diagonal pattern).

We now obtain a theorem analogous to [8, Theorem 2.2] on Boolean matrices.

**Theorem 3.2.** Let $A$ be a square nonnegative sign pattern matrix, with $mr(A) = r$. Then $A$ is idempotent if and only if $A$ is permutationally similar to a sign pattern of the form
where $A_1$ is $r \times r$ sign nonsingular and idempotent, and $A_2A_3$ is a subpattern of $A_1$.

**Proof.** ($\Rightarrow$). Assume that $A$ is idempotent. Performing a permutational similarity on $A$, if necessary, we may assume that $A$ is in Frobenius normal form

$$
\begin{bmatrix}
A_{11} & * \\
\vdots & \ddots \\
0 & A_{mm}
\end{bmatrix},
$$

where the (square) diagonal blocks $A_{ij}$ are the irreducible components of $A$. For $q \geq 2$, a $q \times q$ nonnegative irreducible idempotent must be an all $+$ block. Further, from [5], the blocks in the strictly upper triangular part of the Frobenius form are uniformly signed $+$ or 0. Hence, any two rows (columns) of $A$ intersecting a fixed nonzero irreducible component are identical.

Let $k$ be the number of nonzero irreducible components, and let $i_j$ be the smallest row index of the $j$th nonzero irreducible component, $1 \leq j \leq k$. Let $A_1$ be the principal submatrix of $A$ with row (column) index set $\{i_1, \ldots, i_k\}$. Clearly, $A_1$ is upper triangular with $+$ diagonal entries, so that $A_1$ is sign nonsingular. Further, since $A_1$ is a principal submatrix of the nonnegative idempotent pattern $A$, we see that $A_1^2$ is a subpattern of $A_1$. Also, since $I_k$ is a subpattern of $A_1$, we have that $A_1$ is a subpattern of $A_1^2$. Hence, $A_1$ is idempotent.

Next, from $A^2 = A$, it can be seen that any row (column) of $A$ with a zero diagonal entry can be written as a Boolean combination of later (earlier) rows (columns). So, such a row (column) depends only on rows (columns) with $+$ diagonal entries, equivalently, rows or columns intersecting $A_1$.

We may thus conclude that $A$ is permutationally similar to a pattern of the form

$$
\begin{bmatrix}
A_1 & A_1A_2 \\
A_3A_1 & A_3A_1A_2
\end{bmatrix},
$$

where $A_1$ is $k \times k$ sign nonsingular and idempotent. It is then clear that $mr(A) = k$, that is, $r = k$. Finally, from $A^2 = A$, we have that $A_1^2 + A_1A_2A_3A_1 = A_1$, and so $A_1A_2A_3A_1$ is a subpattern of $A_1$. Since $I_r$ is a subpattern of $A_1$, we have that $I_rA_2A_3I_r$ is a subpattern of $A_1$, or, $A_2A_3$ is a subpattern of $A_1$.

($\Leftarrow$). The proof of the sufficiency of the conditions is straightforward.

We note that when $A$ is idempotent as in the above theorem that (a permutational similarity of)

$$
\begin{bmatrix}
A_1 \\
A_3
\end{bmatrix}
\begin{bmatrix}
A_1 & A_1A_2
\end{bmatrix}
$$

is a nonnegative minimum rank factorization of $A$. 

Theorem 3.3. Let $A$ be a nonnegative idempotent sign pattern matrix. Then

$$mr(A) = \text{Boolean rank of } A.$$ 

Proof. Let $mr(A) = r$. By Theorem 3.2, $A$ is permutationally similar to a sign pattern of the form

$$\begin{bmatrix}
A_1 & A_1A_2 \\
A_3A_1 & A_3A_1A_2
\end{bmatrix},$$

where $A_1$ is $r \times r$ sign nonsingular and idempotent. Hence,

$$mr(A) = r = mr(A_1) \leq \text{Boolean rank of } A_1.$$ 

So, Boolean rank of $A_1 = r$. Since Boolean rank of $A = \text{Boolean rank of } A_1$, the result follows. \(\square\)

Corollary 3.4. If $A$ is an $n \times n$ nonnegative idempotent sign pattern matrix with Boolean rank $n$, then $mr(A) = n$, that is, $A$ is sign nonsingular.

For symmetric patterns, the blocks in the strictly upper triangular part of the Frobenius normal form are zero. The proof of the following theorem is parallel to the proof of Theorem 3.2.

Theorem 3.5. Let $A$ be a symmetric nonnegative sign pattern matrix, with $mr(A) = r$. Then $A$ is idempotent if and only if $A$ is permutationally similar to a pattern of the form

$$\begin{bmatrix}
I_r & A_2 \\
A_2^T & A_2A_2^T
\end{bmatrix},$$

where $A_2A_2^T$ is a subpattern of $I_r$.

Proposition 3.1 and Theorem 3.5 immediately yield the following.

Theorem 3.6. Let $A$ be a symmetric nonnegative sign pattern matrix. Then $A$ is idempotent if and only if $A$ allows a real idempotent.

4. Patterns that allow nonnegative generalized inverses

Lemma 4.1. Let $A$ be an $m \times n$ nonnegative sign pattern matrix. If $A$ allows a nonnegative $(1)$-inverse, then $A$ has a nonnegative $(1)$-inverse.

The same result holds for other generalized inverses such as $(1,3)$- and Moore–Penrose inverse. For the proof just replace positive entries by $\pm$. The converse of
Lemma 4.1 does not hold. For example, the pattern \( A = \begin{bmatrix} + & + \\ 0 & + \end{bmatrix} \) is in fact a \((1,2)\)-inverse of itself, but \( A \) does not allow a nonnegative \((1)\)-inverse. This pattern \( A \) is sign nonsingular with \( B^{-1} \in Q \left( \begin{bmatrix} + & - \\ 0 & + \end{bmatrix} \right) \) for every \( B \in Q(A) \).

We first characterize the nonnegative sign patterns that have a nonnegative \((1)\)-inverse.

**Theorem 4.2.** Let \( A \) be an \( m \times n \) nonnegative sign pattern matrix, with \( mr(A) = r \). Then the following are equivalent:

(i) \( A \) has a nonnegative \((1)\)-inverse.

(ii) \( A \) is permutationally equivalent to a sign pattern of the form

\[
\begin{bmatrix}
A_1 & A_1 A_2 \\
A_3 A_1 & A_3 A_1 A_2
\end{bmatrix},
\]

where \( A_1 \) is \( r \times r \) sign nonsingular and idempotent.

(iii) \( A = HK, H = AN, K = SA \) for some \( m \times k, k \times n, n \times k, k \times m \) nonnegative patterns \( H, K, N, S \), respectively.

**Proof.** (i) \(\Rightarrow\) (ii). By [8, Theorem 2.4], we know that if \( A \) has a nonnegative \((1)\)-inverse, then \( A \) is permutationally equivalent to a sign pattern of the form

\[
\begin{bmatrix}
A_1 & A_1 A_2 \\
A_3 A_1 & A_3 A_1 A_2
\end{bmatrix},
\]

where \( A_1 \) is idempotent with full Boolean rank. In this case, if \( A_1 \) is \( k \times k \), then the Boolean rank of \( A \) is \( k \). Hence, by Corollary 3.4, \( A_1 \) is sign nonsingular. Then clearly, \( mr(A) = k \), that is, \( r = k \).

(ii) \(\Rightarrow\) (i). If

\[
A = Q \begin{bmatrix}
A_1 & A_1 A_2 \\
A_3 A_1 & A_3 A_1 A_2
\end{bmatrix} P,
\]

where \( Q \) and \( P \) are permutation patterns, and \( A_1 \) is \( r \times r \), then it is easy to check that

\[
p^T \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^T
\]

is a nonnegative \((1)\)-inverse of \( A \).

The equivalence of (i) and (iii) follows from [8, Theorem 3.1], which says that \( A \) has a nonnegative \((1)\)-inverse if and only if \( A \) has a “space decomposition”.

**Corollary 4.3.** Let \( A \) be an \( m \times n \) nonnegative sign pattern matrix that has a nonnegative \((1)\)-inverse. Then \( A \) has a nonnegative minimum rank factorization and also

\[
mr(A) = \text{Boolean rank of } A.
\]
It follows from Lemma 4.1 and Corollary 4.3 that if a nonnegative pattern allows a nonnegative (1)-inverse (in particular say a (1, 3)-inverse), then $A$ has a nonnegative minimum rank factorization and also

$$\text{mr}(A) = \text{Boolean rank of } A.$$  

We also see from Theorem 4.2 that if $A$ has a “space decomposition”, then $A$ has a nonnegative minimum rank decomposition, but not conversely.

We recall that a real nonnegative matrix $B$ is said to be monomial if and only if $B$ has exactly one nonzero entry in each row and each column, that is, $B$ can be expressed as a product of a nonsingular diagonal matrix and a permutation matrix. It is well-known that an $m \times n$ rank $r$, real, nonnegative matrix $B$ has a nonnegative (1)-inverse if and only if $B$ has a monomial submatrix of order $r$ (see [2, Theorem 4]).

**Theorem 4.4.** Let $A$ be an $m \times n$ nonnegative sign pattern matrix, with $\text{mr}(A) = r$. Then the following are equivalent:

(i) $A$ is permutationally equivalent to a sign pattern of the form

$$\begin{bmatrix} I_r & A_2 \\ A_3 & A_3 A_2 \end{bmatrix}.$$  

(ii) $A$ allows a nonnegative (1)-inverse.

(iii) $A$ allows a nonnegative (1, 2)-inverse.

(iv) $A = HK$ where $H(K)$ is an $m \times r (r \times n)$ nonnegative pattern and both $H$ and $K$ contain some row-permutation of $I_r$ as a submatrix.

**Proof.** Suppose (i) holds. (Note that since $\text{mr}(A) = r$, the identity matrix must be $r \times r$.) Replacing the $+$ entries in $A_2$ and $A_3$ by any positive real numbers, we obtain a rank $r$ nonnegative matrix $B \in Q(A)$ that has a monomial submatrix of order $r$. Hence, $B$ has a nonnegative (1)-inverse, and so (i) $\Rightarrow$ (ii).

Suppose (ii) holds, so that there exists $B \in Q(A)$ that has a nonnegative (1)-inverse. Let $\text{rank}(B) = q$. Then, $B$ has a monomial submatrix of order $q$. Hence, $B$ is permutationally equivalent to a matrix of the form

$$\begin{bmatrix} D_q \quad C \\ D & E \end{bmatrix},$$

where $D_q$ is a diagonal matrix with positive diagonal entries. Since $\text{rank}(B) = q$, we must have $E = DD_q^{-1}C$. Now, if $C \in Q(A_2)$ and $D \in Q(A_3)$, we then have that

$$\begin{bmatrix} D_q \quad C \\ D & E \end{bmatrix}$$

is in the sign pattern class of

$$\begin{bmatrix} I_q & A_2 \\ A_3 & A_3 A_2 \end{bmatrix},$$

so that $A$ is permutationally equivalent to this pattern. Since $\text{mr}(A) = r$, it must be the case that $q = r$. Hence, (ii) $\Rightarrow$ (i).

The proof of (ii) $\Rightarrow$ (iii) is easy. If $B \in Q(A)$ has a nonnegative (1)-inverse, say $X$, then $XBX$ is a nonnegative (1, 2)-inverse of $B$. Clearly, (iii) $\Rightarrow$ (ii).
Next, assume again that (i) holds, so that
\[ A = P \begin{bmatrix} I_r \\ A_3 \end{bmatrix} \begin{bmatrix} I_r & A_2 \end{bmatrix} Q \]

for some permutation patterns \( P \) and \( Q \). Letting \( H = P \begin{bmatrix} I_r \\ A_3 \end{bmatrix} \) and \( K = \begin{bmatrix} I_r & A_2 \end{bmatrix} Q \), it is then clear that (iv) holds. So, (i) \( \Rightarrow \) (iv). To show that (iv) \( \Rightarrow \) (i), simply reverse these steps. \( \square \)

Note: \( mr(H) = mr(K) = r \), so that (iv) gives a special type of minimum rank factorization.

It should be clear from the proof of Theorem 4.4 that the only matrices \( B \in Q(A) \) that can have a nonnegative (1)-inverse are of rank equal to \( mr(A) \). In fact, if \( A \) allows a nonnegative (1)-inverse, then all matrices \( B \in Q(A) \) of rank equal to \( mr(A) \) have a nonnegative (1)-inverse. Indeed, such a \( B \) is permutationally equivalent to a matrix of the form
\[ \begin{bmatrix} D_r & C \\ D & DD_r^{-1}C \end{bmatrix} \]

where \( D_r \) is a diagonal matrix with positive diagonal entries, and so \( B \) has a monomial submatrix of order \( r \). Furthermore, since these matrices \( B \) have a nonnegative (1)-inverse, they then have a nonnegative full rank factorization.

We will now show that if a nonnegative sign pattern \( A \) has a nonnegative (1,4)-inverse, then \( A \) allows a nonnegative (1,4)-inverse. The same is true for (1,3)-inverse and the Moore–Penrose inverse. As was seen earlier, this is not the case in general for (1)-inverse.

**Theorem 4.5.** Let \( A \) be an \( m \times n \) nonnegative sign pattern matrix, with \( mr(A) = r \). Then the following are equivalent:

(i) \( A \) has a nonnegative (1, 4)-inverse.

(ii) \( A \) is permutationally equivalent to a pattern of the form
\[ \begin{bmatrix} I_r & A_2 \\ A_3 & A_3 A_2 \end{bmatrix} \]

where \( A_2 A_3^T \) is a subpattern of \( I_r \).

(iii) \( A \) is permutationally equivalent to a pattern of the form \( \begin{bmatrix} F \\ G \end{bmatrix} \), where \( F \) is \( r \times n \) and has orthogonal rows, and \( G \) is nonnegative.

(iii)' \( A \) is permutationally equivalent to a pattern of the form
\[ \begin{bmatrix} J & 0 \\ GJ & 0 \end{bmatrix} \].
where \( G \) is nonnegative and
\[
J = \begin{bmatrix}
J_1 & 0 \\
& \ddots & \vphantom{J_1} \\
0 & & J_r
\end{bmatrix},
\]
with each \( J_i \) an all + row.

(iv) \( A \) allows a nonnegative \((1, 4)\)-inverse.

(v) \( A \) allows a nonnegative \((1, 2, 4)\)-inverse.

(vi) \( A = HK \), where \( H(K) \) is an \( m \times r(r \times n) \) nonnegative pattern, \( H \) contains some row-permutation of \( I_r \) as a submatrix, \( mr(K) = r \), and the rows of \( K \) are orthogonal.

**Proof.** Suppose (ii) holds. Now, since \( A_2 A_2^T \) is a subpattern of \( I_r \), \( A_2 \) has orthogonal rows. Letting \( F = \begin{bmatrix} I_r & A_2 \end{bmatrix} \) and \( G = A_3 \), we see that \( F \) has orthogonal rows and \( GF = \begin{bmatrix} A_3 & A_3 A_2 \end{bmatrix} \). So, (iii) holds and (ii) \( \Rightarrow \) (iii). (Note that since \( mr(A) = r \), \( F \) cannot have any zero rows.)

Assuming (iii) and permuting columns, we obtain (iii)', which is a special case of (iii). Likewise, assuming (iii)' and permuting columns, we can obtain (ii).

To prove the equivalences (i) \( \Leftrightarrow \) (ii) \( \Leftrightarrow \) (iv) \( \Leftrightarrow \) (v), first note that the implications (v) \( \Rightarrow \) (iv) \( \Rightarrow \) (i) are trivial. The implication (i) \( \Rightarrow \) (ii) follows from [8, Theorem 4.1]. To prove the implication (ii) \( \Rightarrow \) (v), we assume that (ii) holds. Without loss of generality, we can assume
\[
A = \begin{bmatrix} I_r & A_2 \\ A_3 & A_3 A_2 \end{bmatrix},
\]
where \( A_2 A_2^T \) is a subpattern of \( I_r \). Let \( B \in Q(I_r) \), \( C \in Q(A_2) \), and \( D \in Q(A_3) \). Since \( A_2 A_2^T \) is a subpattern of \( I_r \), we can choose \( B \) and \( C \) where \( B^2 + CC^T = I \).

Now,
\[
\begin{bmatrix} B & C \\ DB & DC \end{bmatrix} \in Q(A),
\]
and
\[
\begin{bmatrix} B & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} B & C \\ DB & DC \end{bmatrix} = \begin{bmatrix} B^2 & BC \\ C^T B & C^T C \end{bmatrix} = \begin{bmatrix} B^T B & B^T C \\ C^T B & C^T C \end{bmatrix},
\]
which is symmetric. Further,
\[
\begin{bmatrix} B & C \\ DB & DC \end{bmatrix} \begin{bmatrix} B & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} B & C \\ DB & DC \end{bmatrix} = \begin{bmatrix} B^3 + CC^T B & B^2 C + CC^T C \\ DB^3 + DCC^T B & DB^2 C + DCC^T C \end{bmatrix} = \begin{bmatrix} (B^2 + CC^T) B & (B^2 + CC^T) C \\ D(B^2 + CC^T) B & D(B^2 + CC^T) C \end{bmatrix} = \begin{bmatrix} B & C \\ DB & DC \end{bmatrix}.
\]
Hence, \( \begin{bmatrix} B & 0 \\ C^T & 0 \end{bmatrix} \) is a (1, 4)-inverse of \( \begin{bmatrix} B & C \\ DB & DC \end{bmatrix} \). Also,

\[
\begin{bmatrix} B & 0 \\ C^T & 0 \end{bmatrix} \begin{bmatrix} B & C \\ DB & DC \end{bmatrix} \begin{bmatrix} B & 0 \\ C^T & 0 \end{bmatrix} = \begin{bmatrix} B(B^2 + C C^T) & 0 \\ C^T(B^2 + C C^T) & 0 \end{bmatrix} = \begin{bmatrix} B & 0 \\ C^T & 0 \end{bmatrix}.
\]

Thus, \( \begin{bmatrix} B & 0 \\ C^T & 0 \end{bmatrix} \) is a nonnegative (1, 2, 4)-inverse of \( \begin{bmatrix} B & C \\ DB & DC \end{bmatrix} \), and (ii) \( \Rightarrow \) (v).

Finally, we show that (ii) \( \Leftrightarrow \) (vi). Suppose (ii) holds. Then, for some permutation patterns \( P \) and \( Q \),

\[
A = P \begin{bmatrix} I_r \\ A_3 \end{bmatrix} \begin{bmatrix} I_r & A_2 \\ A_3 & A_3 A_2 \end{bmatrix} Q,
\]

where \( A_2 A_3^T \) is a subpattern of \( I_r \). Letting \( H = P \begin{bmatrix} I_r \\ A_3 \end{bmatrix} \) and \( K = \begin{bmatrix} I_r & A_2 \end{bmatrix} Q \), it is then clear that (vi) holds. So, (ii) implies (vi). Conversely, suppose (vi) holds. Then \( H = P \begin{bmatrix} I_r \\ A_3 \end{bmatrix} \) for some permutation pattern \( P \) and some pattern \( A_3 \). Also, since \( mr(K) = r \), \( K \) cannot have any zero rows. Then, with the rows of \( K \) orthogonal, \( K = \begin{bmatrix} I_r & A_2 \end{bmatrix} Q \), for some permutation pattern \( Q \) and some pattern \( A_2 \), where \( AA^T \) is a subpattern of \( I_r \). With \( A = HK \), it is then clear that (ii) follows.

We point out that in Theorem 4.5, the proofs of (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (v) follow from [2], but we have given self-contained proofs.

We now present the parallel theorem for (1, 3)-inverses. This theorem follows from Theorem 4.5 and the fact that \( X \) is a (1, 4)-inverse of \( B \) if and only if \( X^T \) is a (1, 3)-inverse of \( B^T \).

**Theorem 4.6.** Let \( A \) be an \( m \times n \) nonnegative sign pattern matrix, with \( mr(A) = r \). Then the following are equivalent:

(i) \( A \) has a nonnegative (1, 3)-inverse.

(ii) \( A \) is permutationally equivalent to a pattern of the form

\[
\begin{bmatrix} I_r & A_2 \\ A_3 & A_3 A_2 \end{bmatrix},
\]

where \( A_3 A_2^T \) is a subpattern of \( I_r \).

(iii) \( A \) is permutationally equivalent to a pattern of the form \( \begin{bmatrix} F & FG \end{bmatrix} \), where \( F \) is \( m \times r \) and has orthogonal columns, and \( G \) is nonnegative.

(iii)’ \( A \) is permutationally equivalent to a pattern of the form

\[
\begin{bmatrix} J & JG \\ 0 & 0 \end{bmatrix},
\]

where \( G \) is nonnegative and
\[
J = \begin{bmatrix}
J_1 & 0 \\
\vdots & \ddots \\
0 & J_r
\end{bmatrix},
\]

with each \(J_i\) an all + column.

(iv) \(A\) allows a nonnegative (1, 3)-inverse.

(v) \(A\) allows a nonnegative (1, 2, 3)-inverse.

(vi) \(A = HK\), where \(H(K)\) is an \(m \times r(r \times n)\) nonnegative pattern, \(K\) contains some row-permutation of \(I_r\) as a submatrix, \(mr(H) = r\), and the columns of \(H\) are orthogonal.

Finally in this section, we characterize nonnegative sign pattern matrices that allow Moore–Penrose inverses. In [7], necessary and sufficient conditions are given in order that a real, nonnegative matrix have a nonnegative Moore–Penrose inverse.

**Theorem 4.7.** Let \(A\) be an \(m \times n\) nonnegative sign pattern matrix, with \(mr(A) = r\). Then the following are equivalent:

(i) \(A\) has a nonnegative Moore–Penrose inverse.

(ii) \(A\) is permutationally equivalent to a pattern of the form
\[
\begin{bmatrix}
I_r & A_2 \\
A_3 & A_3A_2
\end{bmatrix},
\]
where \(A_2A_2^T\) and \(A_3^TA_3\) are subpatterns of \(I_r\).

(iii) \(A\) is permutationally equivalent to a pattern of the form
\[
\begin{bmatrix}
J & 0 \\
0 & 0
\end{bmatrix},
\]
where
\[
J = \begin{bmatrix}
J_1 & 0 \\
\vdots & \ddots \\
0 & J_r
\end{bmatrix},
\]
with each \(J_i\) an all + (not necessarily square) block.

(iv) \(A\) allows a nonnegative Moore–Penrose inverse (in \(Q(A^T)\)).

(v) \(A = HK\), where \(H(K)\) is an \(m \times r(r \times n)\) nonnegative pattern, \(mr(H) = mr(K) = r\), and the columns (rows) of \(H(K)\) are orthogonal.

**Proof.** The implication (i) \(\Rightarrow\) (ii) follows from [8, Theorem 4.3].

Next, to show that (ii) \(\Rightarrow\) (iii), suppose (ii) holds. Now, as before,
\[
\begin{bmatrix}
I_r & A_2 \\
A_3 & A_3A_2
\end{bmatrix} = \begin{bmatrix} I_r \ni A_3 \\
A_3 \end{bmatrix} \begin{bmatrix} I_r & A_2 \\
A_3 & A_3A_2
\end{bmatrix}.\]
Since the columns of \( \begin{bmatrix} I_r \\ A_3 \end{bmatrix} \) (rows of \( \begin{bmatrix} I_r \\ A_2 \end{bmatrix} \)) are orthogonal, we can permute the rows of \( \begin{bmatrix} I_r \\ A_3 \end{bmatrix} \) (the columns of \( \begin{bmatrix} I_r \\ A_2 \end{bmatrix} \)) to \[
\begin{bmatrix}
c_1 \\
\vdots \\
c_r \\
0
\end{bmatrix}
\begin{bmatrix}
s_1 \\
\vdots \\
s_r
\end{bmatrix}
\]
where each \( c_i(s_j) \) is an all + column (row). Clearly,
\[
\begin{bmatrix}
c_1 \\
\vdots \\
c_r \\
0
\end{bmatrix}
\begin{bmatrix}
s_1 \\
\vdots \\
s_r
\end{bmatrix}
\]
is of the form described in (iii), and hence (ii) \( \Rightarrow \) (iii).

We next show that (iii) \( \Rightarrow \) (iv), and so assume (iii) holds. In (iii), say that each \( J_i \) is \( m_i \times n_i, 1 \leq i \leq r \). Replacing each + with a 1, we have \( J_i^\dagger = \frac{1}{m_i n_i} J_i^T \), and
\[
\begin{bmatrix}
J \\
0 \\
0
\end{bmatrix}^\dagger
= \begin{bmatrix}
\frac{1}{m_1 n_1} J_1^T \\
\vdots \\
\frac{1}{m_r n_r} J_r^T \\
0 \\
0
\end{bmatrix},
\]
which is in \( Q \left( \begin{bmatrix} J \\
0 \\
0 \end{bmatrix}^T \right) \). Hence, (iii) \( \Rightarrow \) (iv).

From the comment after Lemma 3.1, (iv) \( \Rightarrow \) (i). Finally, we can note that (ii) \( \Leftrightarrow \) (v) follows as in the proof of (ii) \( \Leftrightarrow \) (vi) in Theorem 4.5. \( \Box \)

5. Nonnegative patterns that allow positive generalized inverses

**Proposition 5.1.** Let \( A \) be an \( m \times n \) nonnegative sign pattern matrix. Then the following are equivalent:

(i) \( A \) allows a positive \((1)\)-inverse.
(ii) \( A \) has a positive \((1)\)-inverse.
(iii) \( A \) is permutationally equivalent to a pattern of the form
\[
\begin{bmatrix}
J \\
0 \\
0
\end{bmatrix},
\]
where \( J \) is an all + (possibly empty) pattern.

**Proof.** The implication (i) \( \Rightarrow \) (ii) is clear.

To prove (ii) \( \Rightarrow \) (iii), suppose that (ii) holds and first assume \( A \) has no zero row or column. Then \( A = AJ_{m \times m} A = J_{m \times n} A = J_{m \times n} A = J_{m \times n} \), that is, \( A \) must be the \( m \times n \)
all + pattern. More generally, suppose $A$ has a positive (1)-inverse, and (through permutational equivalence) partition $A$ as $\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$, where $A_1$ has no zero row or column. To avoid the trivial case where $A = 0$, we may assume that $A_1$ is not empty. We then have positive patterns $C, D, E, F$ (where $C$ has the same size as $A_1^T$) such that $\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C & D \\ E & F \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}.$

Hence $\begin{bmatrix} A_1 C A_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}.$

Thus, $A_1 C A_1 = A_1.$ From the above argument, $A_1$ has all + entries. Hence, $A$ is permutationally equivalent to a pattern of the form $\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}$, where $J$ is an all + (possibly empty) pattern. Thus, (ii) $\Rightarrow$ (iii).

The implication (iii) $\Rightarrow$ (i) is clear. □

In contrast, we have the following result on (2)-inverses.

**Proposition 5.2.** Let $A$ be an $m \times n$ sign pattern matrix. Then the following are equivalent:

(i) $A$ allows a positive (2)-inverse.
(ii) $A$ allows a nontrivial nonnegative (2)-inverse.
(iii) $A$ has at least one + entry.

**Proof.** The implication (i) $\Rightarrow$ (ii) is clear.

To prove (ii) $\Rightarrow$ (iii), suppose that (ii) holds and $A \preceq 0$. Then, for every $B \in Q(A)$ and every nonnegative $X$ with the same size as $A^T$, we have $X B X \preceq 0.$ Thus, $X B X \neq X$ unless $X = 0$, and $A$ does not allow a nontrivial nonnegative (2)-inverse. Hence, (ii) $\Rightarrow$ (iii).

To prove (iii) $\Rightarrow$ (i), assume that $A$ has at least one + entry. Then there is a matrix $B \in Q(A)$ with the property that the sum of all the entries of $B$ is 1. Now, $J B J = J$ where $J$ is the all ones matrix with the same size as $A^T$. Thus $J$ is a positive (2)-inverse of $B$. Hence, (iii) $\Rightarrow$ (i). □

**Corollary 5.3.** Let $A$ be an $m \times n$ nonnegative sign pattern matrix. Then the following are equivalent:

(i) $A$ allows a positive (2)-inverse.
(ii) $A$ allows a nontrivial nonnegative (2)-inverse.
(iii) $A$ has a positive (2)-inverse.
(iv) \( A \) has a nontrivial nonnegative (2)-inverse.
(v) \( A \) has at least one \( + \) entry.

We now consider nonnegative sign patterns that allow positive Moore–Penrose Inverses.

**Proposition 5.4.** Let \( A \) be an \( m \times n \) nonnegative sign pattern matrix. Then the following are equivalent:

(i) \( A \) allows a positive Moore–Penrose inverse.
(ii) \( A \) has a positive Moore–Penrose inverse.
(iii) \( A \) is the all \( + \) pattern.

**Proof.** The implication (i) \( \Rightarrow \) (ii) is clear.

To prove (ii) \( \Rightarrow \) (iii), suppose that \( A \) has a positive Moore–Penrose inverse. Clearly, \( A \neq 0 \). If \( A \) has no zero row or column, then from the proof of Proposition 5.1, we see that \( A \) is an all \( + \) sign pattern. Suppose that \( A \) has a zero row (the first row, say). Let \( J \) be the all \( + \) sign pattern with the same size as \( A^T \). Then the first row of \( AJ \) is zero while the first column of \( AJ \) is nonzero. Thus \( AJ \) is not symmetric, contradicting the fact that \( A \) has a positive Moore–Penrose inverse. Hence, \( A \) has no zero row. Similarly, \( A \) has no zero column. Therefore, it follows that \( A \) is an all \( + \) sign pattern.

If we let \( J \) denote the \( m \times n \) all 1’s matrix, then \( J^T = \frac{1}{mn} J^T \). Thus, (iii) \( \Rightarrow \) (i). \( \square \)

In a subsequent paper, we will analyze the more general sign patterns (allowing negative entries) that allow a positive (1)-inverse.

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