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Idealizer rings and noncommutative projective geometry

Daniel Rogalski [1]

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA

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Abstract

We study noetherian graded idealizer rings which have very different behavior on the right and left sides. In particular, we construct noetherian graded algebras T over an algebraically closed field k with the following properties: T is left but not right strongly noetherian; $T \otimes_k T$ is left but not right noetherian and $T \otimes_k T^{\text{op}}$ is noetherian; the left noncommutative projective scheme T-Proj is different from the right noncommutative projective scheme Proj-T; and T satisfies left χ_d for some $d \geqslant 2$ yet fails right χ_1.

Keywords: Noetherian graded ring; Noncommutative projective geometry; Idealizer ring

1. Introduction

As a general principle, rings which are both left and right noetherian are expected to have rather symmetric properties on their left and the right sides. The theme of this paper is to show that such intuition fails quite utterly for certain properties which are important in the theory of noncommutative projective geometry. Our main result is the following theorem.

E-mail address: rogalski@math.mit.edu.

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Theorem 1.1 (Theorem 8.2). *For any integer $d \geqslant 2$, there exists a connected finitely presented graded noetherian k-algebra T, where k is an algebraically closed field, such that*

(1) *T is strongly left noetherian, but not strongly right noetherian*;
(2) *$T \otimes_k T$ is left but not right noetherian, while $T \otimes_k T^{\mathrm{op}}$ is noetherian*;
(3) *the noncommutative projective schemes T-*Proj *and* Proj-*T have equivalent underlying categories, but non-isomorphic distinguished objects*; *and*
(4) *T satisfies χ_{d-1} but not χ_d on the left, yet T fails χ_1 on the right.*

In the remainder of the introduction, we will define and briefly discuss all of the relevant terms in the statement of the theorem and indicate how the ring T is constructed. For a more detailed introduction to the theory of noncommutative geometry which motivates the study of these properties, see the survey article [16].

If R is a k-algebra, then R is called *strongly left* (*right*) *noetherian* if $R \otimes_k B$ is left (right) noetherian for every commutative noetherian k-algebra B. The study of the strong noetherian condition for graded rings in particular has recently become important because of the appearance of this property in the hypotheses of several theorems in noncommutative geometry. Most notably, Artin and Zhang showed that if A is a strongly noetherian graded k-algebra, then the set of graded A-modules with a given Hilbert function is parametrized by a projective scheme [3]. It is not a priori obvious that any noetherian finitely generated k-algebra which is not strongly noetherian should exist; in [11], Resco and Small gave the first (ungraded) such example. More recently, the author showed that there exist noncommutative noetherian graded rings which are not strongly noetherian (on either side) [12]. Theorem 1.1(1) shows that it is also possible for the strong noetherian property to fail on one side only of a noetherian graded ring.

It is natural to suspect that a ring for which the noetherian property fails after commutative base ring extension might also have strange properties when tensored with itself or its opposite ring. Theorem 1.1(2) confirms such a suspicion. The existence of a pair of finitely presented noetherian k-algebras whose tensor product is not noetherian answers [4, Appendix, Open Problem 16′]; our example shows that one can even take the algebras in question to be $\mathbb{N}$-graded.

We now explain the third part of Theorem 1.1. Let $A = \bigoplus_{n=0}^{\infty} A_n$ be an arbitrary $\mathbb{N}$-graded k-algebra, where k is an algebraically closed field. In addition, assume that A is *connected* ($A_0 = k$) and *finitely graded* ($\dim_k A_n < \infty$ for all $n \geqslant 0$). The left noncommutative projective scheme associated to A is defined to be the pair $A\text{-Proj} = (A\text{-Qgr}, \mathcal{A})$. Here A-Qgr is the quotient category of the category of $\mathbb{Z}$-graded left A-modules by the full subcategory of modules which are direct limits of modules with finite k-dimension, and $\mathcal{A}$, called the *distinguished object*, is the image of the module ${}_AA$ in A-Qgr. The right noncommutative projective scheme Proj-A of A is defined analogously. The motivation for these definitions comes from the commutative case: if A is commutative noetherian and $\operatorname{proj} A = X$ is its associated scheme, then A-Qgr and $\operatorname{Qch} X$ (the category of quasi-coherent sheaves on X) are equivalent categories, and $\mathcal{A}$ corresponds under this equivalence to the structure sheaf $\mathcal{O}_X$.

The result of Theorem 1.1(3) shows that noncommutative projective schemes associated to the two sides of a noncommutative noetherian ring may well be quite different. In fact, for the ring T of the theorem we will see that both T-Qgr and Qgr-T are equivalent to the category Qch X where $X = \mathbb{P}^d$ for some $d \geqslant 2$. However, Proj-T is isomorphic to $(\mathrm{Qch}\, X, \mathcal{O}_X)$, while T-Proj is isomorphic to $(\mathrm{Qch}\, X, \mathcal{I})$ where $\mathcal{I}$ is a non-locally-free ideal sheaf.

Next we discuss the χ conditions, which are homological properties of graded rings which arose in Artin and Zhang's work in [2] to develop the theory of noncommutative projective schemes. For each $i \geqslant 0$, the connected finitely graded k-algebra A is said to satisfy χ_i on the left (right) if $\dim_k \underline{\mathrm{Ext}}^j_A(A/A_{\geqslant 1}, M) < \infty$ for all finitely generated left (right) A-modules M and all $0 \leqslant j \leqslant i$, where $\underline{\mathrm{Ext}}$ indicates the Ext group in the ungraded module category. If A satisfies χ_i on the left for all $i \geqslant 0$, then we say that A satisfies χ on the left. The χ_1 condition is the most important of these conditions: it ensures that one can reconstruct the ring A (in large degree) from its associated scheme A-Proj [2, Theorem 4.5]. The other χ_i conditions for $i \geqslant 2$ are needed to show the finite-dimensionality of the cohomology groups associated to A-Proj [2, Theorem 7.4].

Although the χ conditions always hold for commutative rings, Stafford and Zhang constructed noetherian rings for which χ_1 fails on both sides [15]. The author studied rings in [12] which satisfy χ_1 but fail χ_2 on both sides. Theorem 1.1(3) demonstrates yet more possible behaviors of the χ conditions: first, that χ_1 may hold on one side but not the other of a noetherian ring; and second, that for any $d \geqslant 2$ there are rings which satisfy χ_{d-1} but not χ_d (on one side).

Finally, we briefly describe the construction of the rings T satisfying Theorem 1.1. Recall that if I is a left ideal in a noetherian ring S, then the *idealizer* of I, written $\mathbb{I}(I)$, is the largest subring of S which contains I as a 2-sided ideal. Explicitly, $\mathbb{I}(I) = \{s \in S \mid Is \subseteq I\}$. Now let S be a generic Zhang twist of a polynomial ring (see Section 5 for the definition), which is a noncommutative graded ring generated in degree 1. Let I be the left ideal of S generated by a generic subspace $I_1 \subseteq S_1$ with $\dim I_1 = \dim S_1 - 1$. The ring $T = \mathbb{I}(I) \subseteq S$ is then the ring of interest which will satisfy properties (1)–(4) of Theorem 1.1.

Our approach in this paper will be primarily algebraic. Since this research was completed, the article [8] has developed a geometric framework for the study of a class of algebras quite similar to the ones we study here. We remark that many of the results below can be translated into this geometric language, which would allow one to show that the properties of Theorem 1.1 hold for a wider class of idealizer rings. Specifically, one could work with idealizers inside twisted homogeneous coordinate rings over arbitrary integral projective schemes, instead of the special case of Zhang twists of polynomial rings we consider here. Since our main purpose is to construct some interesting examples, we will not attempt to be as general as possible and we will prefer the simpler algebraic constructions.

2. Idealizer rings and the left and right noetherian property

As mentioned in the introduction, the main examples of this paper will be certain idealizer rings. Idealizers have certainly proved useful in the creation of counterexamples before, but it seems that in many natural examples (for example, those in [10] or [14]), the idealizer of a left ideal is a left but not right noetherian ring. Since our intention is to create two-sided noetherian examples, in this brief section we will give some general characterizations of both the left and right noetherian properties for an idealizer ring.

Let S be a noetherian ring with left ideal I, and let $T = \mathbb{I}(I) \subseteq S = \{s \in S \mid Is \subseteq I\}$ be the idealizer of I. In [14], Stafford gives a sufficient condition for the left noetherian property of T. In the next proposition, we restate Stafford's result slightly to show that it characterizes the left noetherian property in case S is a finitely generated left T-module, which occurs in many examples of interest.

Proposition 2.1. *Let T be the idealizer of the left ideal I of a noetherian ring S, and assume in addition that ${}_TS$ is finitely generated. The following are equivalent*:

(1) *T is left noetherian.*
(2) $\mathrm{Hom}_S(S/I, S/J)$ *is a noetherian left T-module* (*or T/I-module*) *for all left ideals J of S.*

Proof. By [14, Lemma 1.2], if $\mathrm{Hom}_S(S/I, S/J)$ is a noetherian left T-module for all left ideals J of S containing I, then T is left noetherian. So if condition (2) holds, then T is certainly left noetherian.

On the other hand, if T is left noetherian, then since ${}_TS$ is finitely generated, ${}_TS$ is also noetherian. Given any left ideal J of S, we can identify the left T-module $\mathrm{Hom}_S(S/I, S/J)$ with the subfactor $\{x \in S \mid Ix \subseteq J\}/J$ of ${}_TS$, so $\mathrm{Hom}_S(S/I, S/J)$ is a noetherian T-module. □

Next, we give a characterization of the right noetherian property for idealizers of left ideals. It is formally quite similar to the characterization of Proposition 2.1, and may be of independent interest. In fact, the result applies more generally to all subrings of S inside of which I is an ideal.

Proposition 2.2. *Let S be a noetherian ring with left ideal I, and let T be a subring of S such that $I \subseteq T \subseteq \mathbb{I}(I)$. The following are equivalent*:

(1) *T is right noetherian.*
(2) *T/I is a right noetherian ring, and* $\mathrm{Tor}_1^S(S/K, S/I) = (K \cap I)/KI$ *is a noetherian right T-module* (*or T/I-module*) *for all right ideals K of S.*

Proof. The identification of $\mathrm{Tor}_1^S(S/K, S/I)$ with the subfactor $(K \cap I)/KI$ of T_T follows from [13, Corollary 11.27(iii)], and it is immediate that (1) implies (2).

Now suppose that condition (2) holds. Since S is right noetherian, T is right noetherian if and only if $(JS \cap T)/J$ is a noetherian right T-module for all finitely generated right T-ideals J—see [12, Lemma 6.10] for a proof of this in the graded case; the proof in the ungraded case is the same. Let J be an arbitrary finitely generated right ideal of T. Since T/I is right noetherian, $(JS \cap T)/(JS \cap I)$ and J/JI are noetherian right T/I-modules (the first injects into T/I, and J surjects onto the second). Then $(JS \cap T)/J$ is right noetherian over T if and only if $(JS \cap I)/JI$ is. By [13, Corollary 11.27(iii)] and the fact that $JSI = JI$, we may identify $(JS \cap I)/JI$ with $\mathrm{Tor}_1^S(S/JS, S/I)$, which is a noetherian right module over T by hypothesis. It follows that T is a right noetherian ring. $\square$

3. Noncommutative Proj of graded idealizer rings

Starting with this section, we focus our attention on idealizer rings inside connected finitely graded k-algebras in particular. Our first task is to study the properties of the left and right noncommutative schemes associated to such idealizer rings, and so we begin with a review of some of the relevant definitions.

Below, A will always be a connected finitely graded k-algebra, and we write A-Gr for the category of all $\mathbb{Z}$-graded left A-modules. A module $M \in A$-Gr is called *torsion* if for every $m \in M$ there is some $n \geqslant 0$ such that $(A_{\geqslant n})m = 0$. Let A-Tors be the full subcategory of A-Gr consisting of the torsion modules, and define A-Qgr to be the quotient category A-Gr$/A$-Tors, with quotient functor $\pi : A\text{-Gr} \to A\text{-Qgr}$. For a $\mathbb{Z}$-graded A-module M we define $M[n]$ for any $n \in \mathbb{Z}$ to be M as an ungraded module, but with a new grading given by $M[n]_m = M_{n+m}$. The shift functor $M \to M[1]$ is an autoequivalence of A-Gr which naturally descends to an autoequivalence of A-Qgr we call s, though we usually write $\mathcal{M}[n]$ instead of $s^n(\mathcal{M})$ for any $\mathcal{M} \in A$-Qgr and $n \in \mathbb{Z}$.

In general, any collection of data $(\mathcal{C}, \mathcal{F}, t)$ where $\mathcal{C}$ is an abelian category, $\mathcal{F}$ is an object of $\mathcal{C}$, and t is an autoequivalence of $\mathcal{C}$ is called an *Artin–Zhang triple*. For every connected graded ring A the data $(A\text{-Qgr}, \pi A, s)$ gives such a triple. An isomorphism of two such triples is an equivalence of categories which commutes with the autoequivalences and under which the given objects correspond; see [2, p. 237]. For example, if A is a connected graded commutative ring and $X = \operatorname{proj} A$ is the associated scheme, then by a theorem of Serre one has that $(A\text{-Qgr}, \pi A, s)$ is isomorphic to $(\operatorname{Qch} X, \mathcal{O}_X, - \otimes \mathcal{O}(1))$. Motivated by this, for any connected graded ring A one calls the pair $A\text{-Proj} = (A\text{-Qgr}, \pi A)$ the *left noncommutative projective scheme* associated to A, the object πA the *distinguished object*, and the autoequivalence s of A-Qgr the *polarization*. We define analogously the right-sided versions Qgr-A, Proj-A, etcetera of all of the notions above.

Our analysis of the noncommutative schemes for idealizer rings will be restricted to rings which satisfy the following hypotheses, which will hold for a large class of examples we study later.

Hypothesis 3.1. Let k be a field. Let S be a noetherian connected finitely $\mathbb{N}$-graded k-algebra, let I be some homogeneous left ideal of S such that $\dim_k S/I = \infty$, and put $T = \mathbb{I}(I)$. Assume in addition that ${}_T S$ is a finitely generated module, and that $\dim_k T/I < \infty$.

Under the assumptions of Hypothesis 3.1, both the left and right noncommutative schemes for the idealizer ring T are closely related to those for the ring S, as we see now.

Lemma 3.2. *Assume Hypothesis* 3.1.

(1) *There is an isomorphism of triples* $(S\text{-Qgr}, \pi I, s) \cong (T\text{-Qgr}, \pi T, s)$.
(2) *There is an isomorphism of triples* $(\text{Qgr-}S, \pi S, s) \cong (\text{Qgr-}T, \pi T, s)$.

Proof. (1) Suppose that $M \in S$-Gr. Then we claim that if ${}_T M \in T$-Tors, then ${}_S M \in S$-Tors. To prove this fact, note first that if ${}_T M$ is finitely generated, then M is finite-dimensional over k, so obviously ${}_S M \in S$-Tors. In general, ${}_T M$ is a direct limit of finite-dimensional T-modules, so $M' = S \otimes_T M$ is a direct limit of finite-dimensional S-modules and thus $M' \in S$-Tors. Since there is an S-module surjection $M' \to M$, this completes the proof of the claim.

Now we define two functors by the rules

$$F : T\text{-Gr} \to S\text{-Gr}, \qquad G : S\text{-Gr} \to T\text{-Gr},$$
$${}_T M \mapsto {}_S(I \otimes_T M), \qquad {}_S N \mapsto {}_T N$$

together with the obvious actions on morphisms. If ${}_T M \in T$-Gr, then since $\dim_k T/I < \infty$, it follows by calculating using a free resolution of M that $\underline{\operatorname{Tor}}_j^T(T/I, M)$ is a torsion left T-module for all $j \geqslant 0$. Then the natural map $I \otimes_T M \to T \otimes_T M = M$ has torsion kernel and cokernel for all $M \in T$-Gr. In particular, if $M \in T$-Tors, then $F(M) \in T$-Tors, so $F(M) \in S$-Tors by the earlier claim. It follows that $F' = \pi \circ F : T\text{-Gr} \to S\text{-Qgr}$ is an exact functor, and that $F'(M) = 0$ for all $M \in T$-Tors. Then by the universal property of the quotient category [9, Corollary 4.3.11], F' descends to a functor $\overline{F} : T\text{-Qgr} \to S\text{-Qgr}$. Similarly, it is clear that if $N \in S$-Tors then $G(N) = N \in T$-Tors. Then $G' = \pi \circ G : S\text{-Gr} \to T\text{-Qgr}$ is an exact functor with $G'(N) = 0$ for all $N \in S$-Tors, so G' descends to a functor $\overline{G} : S\text{-Qgr} \to T\text{-Qgr}$.

We conclude that $\overline{F}$ and $\overline{G}$ are inverse equivalences of categories. Moreover, obviously $\overline{F}(\pi T) \cong \pi I$, and all of the maps are compatible with the shift functors s, since F and G are compatible with the shift functors in the categories S-Gr and T-Gr.

(2) Because $SI = I \subseteq T$, we have $(S/T)I = 0$ and so since T/I is finite-dimensional we see that $(S/T)_T$ is torsion. By assumption we also know that ${}_T(S/T)$ is finitely generated. Now the proof of this triple isomorphism is entirely analogous to the proof of [15, Proposition 2.7], with the exception that the authors assume there that T is noetherian and then prove the required equivalence for the subcategories of noetherian objects. We leave it to the reader to make the obvious adjustments to the proof to show without the noetherian assumption that $(\text{Qgr-}S, \pi S, s) \cong (\text{Qgr-}T, \pi T, s)$. □

Remark 3.3. The graded idealizer rings studied by Stafford and Zhang in [15] have the special property that the ideal I is a principal ideal generated by an element of degree 1 in a graded Goldie domain S. In that case, $T = \mathbb{I}(I)$ is isomorphic to its opposite ring, and thus the differences between parts (1) and (2) of Lemma 3.2 must disappear (indeed, in this case $\pi I \cong \pi S[-1]$). In the general case, however, it is clear from Lemma 3.2 that we should expect the noncommutative schemes T-Proj and Proj-T to be non-isomorphic.

The information provided by Lemma 3.2 will allow us to prove with ease several further results about the noncommutative projective schemes of idealizer rings. First, we may show in wide generality that passing to a Veronese ring of T does not affect the associated noncommutative projective schemes. Recall that for an $\mathbb{N}$-graded ring A the nth Veronese ring of A is the graded ring $A^{(n)} = \bigoplus_{i=0}^{\infty} A_{in}$.

Proposition 3.4. *Assume Hypothesis* 3.1*, and in addition let S be generated in degree* 1*. Choose $n \geqslant 1$ and write $T' = T^{(n)}$, $S' = S^{(n)}$, and $I' = I^{(n)} = \bigoplus_{i=0}^{\infty} I_{in}$. Let $R' \subseteq S'$ be the idealizer of the left ideal I' of S'.*

(1) *T' and R' are isomorphic in large degree.*
(2) *There are isomorphisms of noncommutative projective schemes T*-Proj $\cong T'$-Proj *and* Proj-$T \cong$ Proj-T'.

Proof. (1) As ungraded rings, we may identify R', T' and S' with subrings of S. Suppose that $x \in (R'_m)$, so that $I'x \subseteq I'$. Then since the left ideal I of S is generated in some finite degree, we see that in the ring S we have $(I_{\geqslant p})x \subseteq I$ for some $p \geqslant 0$, where $x \in S_{nm}$. Since the torsion submodule of ${}_S(S/I)$ is finite-dimensional, if $m \gg 0$ then $Ix \subseteq I$ and hence $x \in T$. Then as an element of S', $x \in T'$. Since the inclusion $T' \subseteq R'$ is obvious, T' and R' must agree in large degree.

(2) Since ${}_T S$ is finitely generated and $\dim_k T/I < \infty$, we see that ${}_{T'}S'$ is finitely generated and $\dim_k T'/I' < \infty$. Then because T' and R' agree in large degree by part (1), it follows that ${}_{R'}S'$ is finitely generated and that $\dim_k R'/I' < \infty$. Also, since S is noetherian, S' must be noetherian [2, Proposition 5.10(1)].

Now we claim that we have isomorphisms of noncommutative projective schemes

$$(\text{Qgr-}T, \pi T) \cong (\text{Qgr-}S, \pi I) \cong (\text{Qgr-}S', \pi I') \cong (\text{Qgr-}R', \pi R') \cong (\text{Qgr-}T', \pi T').$$

To see this, note that since S is generated in degree 1, there is an isomorphism Proj-$S \cong$ Proj-S' [2, Proposition 5.10(3)]; the associated equivalence of categories Qgr-$S \simeq$ Qgr-S' sends πI to $\pi I'$. The second isomorphism follows, and the first and third follow from Lemma 3.2(1), applied to $T \subseteq S$ and to $R' \subseteq S'$, respectively. Last, the final isomorphism follows from part (1). Altogether this chain of isomorphisms says that Proj-$T \cong$ Proj-T'.

The argument on the left side is very similar, except using the other triple isomorphism of Lemma 3.2, and is left to the reader. □

Next we will show that under mild hypotheses the noncommutative projective schemes associated to S and T (on either side) have the same cohomological dimension; we review the definition of this property now. Cohomology groups for the noncommutative projective scheme A-Proj are defined by setting $\mathrm{H}^i(\mathcal{M}) = \mathrm{Ext}^i_{A\text{-Qgr}}(\pi A, \mathcal{M})$ for all $\mathcal{M} \in A$-Qgr. Then the *cohomological dimension* of A-Proj is

$$\mathrm{cd}(A\text{-Proj}) = \max\{i \mid \mathrm{H}^i(\mathcal{M}) \neq 0 \text{ for some } \mathcal{M} \in A\text{-Qgr}\}$$

and the *global dimension* of the category A-Qgr is

$$\mathrm{gd}(A\text{-Qgr}) = \max\bigl\{i \bigm| \mathrm{Ext}^i_{A\text{-Qgr}}(\mathcal{M}, \mathcal{N}) \neq 0 \text{ for some } \mathcal{M}, \mathcal{N} \in A\text{-Qgr}\bigr\}.$$

The right-sided versions of these notions are defined similarly.

Proposition 3.5. *Assume Hypothesis* 3.1.

(1) $\mathrm{cd}(\text{Proj-}T) = \mathrm{cd}(\text{Proj-}S)$.
(2) *Assume in addition that* S *is a domain with* $\mathrm{gd}(S\text{-Qgr}) = \mathrm{cd}(S\text{-Proj}) < \infty$. *Then* $\mathrm{cd}(T\text{-Proj}) = \mathrm{cd}(S\text{-Proj})$.

Proof. (1) This part is immediate from the triple isomorphism of Lemma 3.2(2).

(2) By Lemma 3.2(1), we have the isomorphism of triples $(T\text{-Qgr}, \pi T, s) \cong (S\text{-Qgr}, \pi I, s)$. From this it quickly follows that

$$\mathrm{cd}(T\text{-Proj}) \leqslant \mathrm{gd}(T\text{-Qgr}) = \mathrm{gd}(S\text{-Qgr}) = \mathrm{cd}(S\text{-Proj}).$$

Let $d = \mathrm{cd}(S\text{-Proj})$. To finish the proof that $\mathrm{cd}(T\text{-Proj}) = \mathrm{cd}(S\text{-Proj})$ we have only to show that there is some $\mathcal{F} \in S\text{-Qgr}$ such that $\mathrm{Ext}^d_{S\text{-Qgr}}(\pi I, \mathcal{F}) \neq 0$. Since S is a domain, we may choose some injection $S[-m] \to I$ for some $m \geqslant 0$, and passing to S-Qgr we have a short exact sequence $0 \to \pi S[-m] \to \pi I \to \mathcal{N} \to 0$ for some $\mathcal{N}$. Since S-Proj has cohomological dimension d, we may choose some $\mathcal{F} \in S\text{-Qgr}$ with $\mathrm{Ext}^d_{S\text{-Qgr}}(\pi S[-m], \mathcal{F}) \neq 0$. But $\mathrm{Ext}^{d+1}_{S\text{-Qgr}}(\mathcal{N}, \mathcal{F}) = 0$ since the global dimension of S-Qgr is d, so we conclude from the long exact sequence in Ext that $\mathrm{Ext}^d_{S\text{-Qgr}}(\pi I, \mathcal{F}) \neq 0$. □

4. The χ conditions for graded idealizers

The goal of this section is to begin an analysis of the χ conditions, which we defined in the introduction, for the case of graded idealizer rings T satisfying Hypothesis 3.1. The main result below will show that if S itself satisfies left χ, then the left χ conditions for the idealizer ring T may be characterized in terms of homological algebra over S only. We also study the right χ conditions for T; the analysis of these turns out to be a much simpler matter.

We review several definitions which we will need before proving the main result of this section. A module $M \in A$-Gr is *right bounded* if $M_n = 0$ for $n \gg 0$, *left bounded* if $M_n = 0$ for $n \ll 0$, and *bounded* if it is both left and right bounded. M is *finitely graded* if $\dim_k M_n < \infty$ for all $n \in \mathbb{Z}$. For $M, N \in A$-Gr, $\mathrm{Hom}_A(M, N)$ means the group of degree-preserving module homomorphisms, and $\mathrm{Ext}^i_A(M, -)$ is the ith right derived functor of $\mathrm{Hom}_A(M, -)$. We also set $\underline{\mathrm{Hom}}_A(M, N) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(M, N[n])$, which is the same as the group of homomorphisms in the ungraded category if M is finitely generated. More generally, we write $\underline{\mathrm{Ext}}^i_A(M, N) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Ext}^i_A(M, N[n])$. We make similar definitions in the category A-Qgr; so $\underline{\mathrm{Ext}}^i_{A\text{-Qgr}}(\mathcal{M}, \mathcal{N}) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Ext}^i_{A\text{-Qgr}}(\mathcal{M}, \mathcal{N}[n])$. Finally, let A-gr be the subcategory of all noetherian modules in A-Gr.

Note that we have defined the χ conditions for not necessarily noetherian algebras; it is easy to prove, however, that the left χ_0 condition for a connected graded ring A is equivalent to the left noetherian property for A. Recall also that if A is connected graded left noetherian with modules $M \in A$-gr and $N \in A$-Gr, then for any $j \geqslant 0$ we have $\underline{\mathrm{Ext}}^j_{A\text{-Qgr}}(\pi M, \pi N) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}^j_A(M_{\geqslant n}, N)$ [2, Proposition 7.2(1)]. In particular, in this case there is a natural map of vector spaces $\underline{\mathrm{Ext}}^j_A(M, N) \to \underline{\mathrm{Ext}}^j_{A\text{-Qgr}}(\pi M, \pi N)$. In the proof of the following proposition we will use several results of Artin and Zhang from [2] which interpret the χ conditions in terms of the properties of such maps.

Proposition 4.1. *Assume Hypothesis* 3.1*, and assume also that S satisfies χ on the left. Then T satisfies χ_i on the left for some $i \geqslant 0$ if and only if* $\dim_k \underline{\mathrm{Ext}}^j_S(S/I, M) < \infty$ *for all* $0 \leqslant j \leqslant i$ *and all* $M \in S$-gr.

Proof. Since any $M \in S$-gr has a finite filtration by cyclic S-modules, it follows from Proposition 2.1 that T is left noetherian if and only if $\underline{\mathrm{Hom}}_S(S/I, M)$ is a noetherian left T/I-module (equivalently, of finite k-dimension) for all $M \in S$-gr. Since the left noetherian property for T is equivalent to left χ_0 for T (as we remarked before the proposition), the characterization of the proposition holds when $i = 0$.

Now assume that T is left noetherian. There is an isomorphism of triples $(S\text{-Qgr}, \pi I, s) \cong (T\text{-Qgr}, \pi T, s)$ by Lemma 3.2(1). For any $M \in S$-gr we have a diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad\gamma\quad} & \underline{\mathrm{Hom}}_S(I, M) \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} \\ \underline{\mathrm{Hom}}_{T\text{-Qgr}}(\pi T, \pi M) & \xrightarrow{\ \cong\ } & \underline{\mathrm{Hom}}_{S\text{-Qgr}}(\pi I, \pi M), \end{array}$$

where the bottom arrow is an isomorphism by the triple isomorphism, α and β are the natural maps, and γ is part of the long exact sequence in $\underline{\mathrm{Ext}}$. It is straightforward to check that this diagram commutes. Now since S has χ, the map β is an isomorphism in large degree [2, Proposition 3.5(3)]. Furthermore, χ_1 holds on the left for T if and only if the map α has right bounded cokernel for all $M \in T$-gr [2, Proposition 3.14(2a)]. Note that it is equivalent to require that α have bounded cokernel for all $M \in S$-gr, as follows: if $M \in T$-gr, then $IM \in S$-gr with $\dim_k M/IM < \infty$ and thus $\pi M = \pi IM$; conversely, if $M \in S$-gr then $M \in T$-gr since ${}_T S$ is finitely generated and T is left noetherian. Thus from the diagram it follows that χ_1 holds for T on the left if and only if γ has right bounded cokernel for all $M \in S$-gr. But the cokernel of γ is $\underline{\mathrm{Ext}}^1_S(S/I, M)$, which is always finitely graded and left bounded, so is right bounded if and only if it has finite k-dimension. Thus the proposition holds for $i = 1$.

Next, assume that χ_1 holds on the left for T. Then the proof of the noncommutative version of Serre's finiteness theorem [2, Theorem 7.4] shows that χ_i holds for T for some $i \geqslant 2$ if and only if for every $M \in T$-gr, the graded cohomology group $\underline{\mathrm{H}}^j(\pi M) = \underline{\mathrm{Ext}}^j_{T\text{-Qgr}}(\pi T, \pi M)$ is finitely graded for all $0 \leqslant j < i$ and right bounded for all $1 \leqslant j < i$.

Similarly as in the last paragraph, one sees that it is equivalent to require this condition for all $M \in S$-gr. Now for every $M \in S$-gr and $j \geqslant 1$ we have a sequence of maps

$$\underline{\mathrm{Ext}}_S^{j+1}(S/I, M) \xrightarrow{\cong} \underline{\mathrm{Ext}}_S^j(I, M) \xrightarrow{\alpha} \underline{\mathrm{Ext}}_{S\text{-Qgr}}^j(\pi I, \pi M) \xrightarrow{\cong} \underline{\mathrm{Ext}}_{T\text{-Qgr}}^j(\pi T, \pi M),$$

where the first isomorphism comes from the long exact sequence in $\underline{\mathrm{Ext}}$, the natural map α is an isomorphism in large degree since S satisfies χ [2, Proposition 3.5(3)], and the final isomorphism comes from the isomorphism of triples in Lemma 3.2(1). In addition, $\underline{\mathrm{Ext}}_{S\text{-Qgr}}^j(\pi I, \pi M)$ is always finitely graded for any j, since S has χ [2, Corollary 7.3(3)]. Thus we see altogether that, assuming χ_1 holds for T, χ_i holds for T for some $i \geqslant 2$ if and only if $\underline{\mathrm{Ext}}_S^j(S/I, M)$ is right bounded (equivalently, finite-dimensional over k since it is always left bounded and finitely graded) for all $2 \leqslant j \leqslant i$ and all $M \in S$-gr. This proves the characterization of χ_i for $i \geqslant 2$, and concludes the proof of the proposition. □

In contrast to Proposition 4.1, on the right side only the χ_0 condition for T (equivalently, the right noetherian property for T) is potentially subtle to analyze. The higher χ conditions automatically must fail, as follows.

Proposition 4.2. *Assume Hypothesis* 3.1. *Then T fails χ_i on the right for all $i \geqslant 1$.*

Proof. We may assume that T is right noetherian, since otherwise right χ_0 fails for T and so by definition right χ_i fails for all $i \geqslant 0$. Also, we need only show that T fails right χ_1. For this, the same argument outlined in [15, p. 424] works here; since it is simple we briefly repeat it. By hypothesis, we have $SI = I$, $\dim_k T/I < \infty$, and $\dim_k S/I = \infty$. So the natural map

$$T \to \underline{\mathrm{Hom}}_{\mathrm{Qgr}\text{-}T}(\pi T, \pi T) = \underline{\mathrm{Hom}}_{\mathrm{Qgr}\text{-}T}(\pi I, \pi I)$$

has a cokernel which is not right bounded, since $S \subseteq \underline{\mathrm{Hom}}_{\mathrm{Qgr}\text{-}T}(\pi I, \pi I)$. Then by [2, Proposition 3.14(2a)], T must fail χ_1 on the right. □

5. Idealizers inside Zhang twists of polynomial rings

In the current section, we introduce a special class of graded idealizers on which we will focus for the remainder of the paper.

Fix a commutative polynomial ring $U = k[x_0, x_1, \ldots, x_d]$ in $d + 1$ variables, and some graded automorphism ϕ of U. Let S be the *left Zhang twist* of U by ϕ. This is a new ring which has the same underlying k-space as the ring U, but a new multiplication defined by the rule $fg = \phi^n(f) \circ g$ for $f \in S_m$, $g \in S_n$, where $\circ$ is the multiplication in U. We continue this same notational convention throughout, whereby juxtaposition means multiplication in S and the symbol $\circ$ appears when the commutative multiplication in U is intended.

It is immediate that S is a noetherian domain [17, Theorem 1.3]. One may also twist modules: given a graded U-module M, one may form a graded left S-module with the same underlying vector space as M but with S-action $fg = \phi^n(f) \circ g$ for $f \in S_m$, $g \in M_n$, where again $\circ$ indicates the U-action. In this way we get a functor U-Gr $\to$ S-Gr which is an equivalence of categories [17, Corollary 4.4(1)]. In particular, the graded left ideals of S and the graded (left) ideals of U are in one-to-one correspondence, and if J is a graded left S-ideal we use the same name J for the corresponding graded U-ideal.

Now we will idealize left ideals of S which are generated by a codimension-1 subspace of the elements of degree 1. Specifically, from now on we will consider the following hypothesis and notations.

Hypothesis 5.1. Let k be an algebraically closed base field. Choose some $d \geqslant 2$, a point $c \in \mathbb{P}^d$, and an automorphism $\varphi \in \operatorname{Aut} \mathbb{P}^d$. Let ϕ be a graded automorphism of $U = k[x_0, \dots, x_d]$ such that φ is the corresponding automorphism of $\operatorname{proj} U = \mathbb{P}^d$, and define $S = S(\varphi)$ to be the left Zhang twist of $U = k[x_0, \dots, x_d]$ by the automorphism ϕ. (Although the automorphism ϕ corresponding to φ is determined only up to scalar multiple [5, Example 7.1.1], it is easy to check that changing ϕ by a nonzero scalar does not change the ring S up to isomorphism.) Let I be the left ideal of S consisting of all homogeneous elements vanishing at the point c. Define $T = T(\varphi, c) = \mathbb{I}(I) \subseteq S$. Also write $c_n = \varphi^{-n}(c)$ for $n \in \mathbb{Z}$.

In general, the properties of the ring $T = T(\varphi, c)$ depend on the properties of the orbit $\mathcal{C} = \{c_n\}_{n \in \mathbb{Z}}$. We are most interested in the "generic" case, and so we will usually assume at least that $\mathcal{C}$ is infinite. Under such an assumption, we see next that the idealizer rings T have the following basic properties.

Lemma 5.2. *Assume Hypothesis* 5.1. *If the points* $\{c_n\}_{n \in \mathbb{Z}}$ *are all distinct, then*

(1) $T = k + I$.
(2) $T^{(n)}$ *is not generated in degree* 1 *for any* $n \geqslant 1$.
(3) $\dim_k(S/IS) < \infty$.
(4) ${}_T S$ *is finitely generated.*
(5) T *is a finitely generated* k*-algebra.*
(6) *Hypothesis* 3.1 *is satisfied.*

Proof. (1) We have $T_n = \{x \in S_n \mid Ix \subseteq I\}$. If $\phi^n(I) \neq I$, then since I is prime in U, $\phi^n(I) \circ x \subseteq I$ forces $x \in I$. Since we assume that c has infinite order under φ, $\phi^n(I) \neq I$ for all $n \neq 0$ and so $T_n = I_n$ for $n \geqslant 1$.

(2) If $T^{(n)}$ were generated in degree one for some $n \geqslant 1$, then would we have $T_n T_n = T_{2n}$, which in the commutative ring U translates to $\phi^n(I)_n \circ I_n = I_{2n}$. Since I and $\phi^n(I)$ are different homogeneous prime ideals of U which are generated in degree 1, it is easy to see that such an equation is impossible.

(3) Set $J = IS$. We have that $J = \sum_{i=0}^{\infty} IS_i = \sum_{i=0}^{\infty} \phi^i(I) \circ U_i$. Since the points $\{c_i\}$ are all distinct, it is clear that the vanishing set of the ideal J in $\mathbb{P}^d$ is empty. Thus $\dim_k U/J < \infty$ by the graded Nullstellensatz; equivalently, $\dim_k(S/IS) < \infty$.

(4) By the graded Nakayama lemma, a k-basis of $S/T_{\geqslant 1}S = S/IS$ is a minimal generating set for ${}_T S$, so (4) follows immediately from (3).

(5) T is generated as a k-algebra by some elements $t_i \in T_{\geqslant 1}$ if and only if $T_{\geqslant 1}$ is generated as left T-ideal by the t_i; so to prove (5) we just need to show that ${}_T I$ is finitely generated. Since by part (4) we know that ${}_T S$ is finitely generated, we have $TS_{\leqslant n} = S$ for some $n \geqslant 0$. Then $TS_{\leqslant n}T_1 = ST_1 = I$ is a finitely generated left T-module.

(6) Since $\dim_k I_n = \dim_k S_n - 1$ for all $n \geqslant 1$, it is clear that $\dim_k S/I = \infty$. The other necessary properties follow from (1) and (4). □

The noetherian property on the left is also straightforward to analyze.

Proposition 5.3. *Assume Hypothesis* 5.1*, and that the points* $\{c_n\}_{n\in\mathbb{Z}}$ *are all distinct. Then* T *is left noetherian.*

Proof. We have that $T = k + I$ and that ${}_T S$ is finitely generated, by Lemma 5.2. Thus the hypotheses of Proposition 2.1 are satisfied and to show that T is left noetherian we need to show that $\underline{\mathrm{Hom}}_S(S/I, S/J)$ is a left noetherian (equivalently, finite-dimensional) $T/I = k$-module for all graded left ideals J of S. Using the equivalence of categories S-Gr $\sim U$-Gr and the existence of prime filtrations in U, we see that every cyclic graded left S-module S/J has a finite graded filtration with factors of the form S/L where L is prime when considered as an ideal of U. Thus we may reduce to the case that J is a prime ideal of U. If $J = U_{\geqslant 1}$, then obviously $\underline{\mathrm{Hom}}_S(S/I, S/J)$ is finite-dimensional, so we also may assume that $J \neq U_{\geqslant 1}$.

Now we may make the identification of vector spaces

$$\underline{\mathrm{Hom}}_S(S/I, S/J)_n = \{x \in U_n \mid \phi^n(I) \circ x \subseteq J\}/J_n.$$

Since the points $\{c_i\}_{i\geqslant 0}$ are distinct, $\phi^n(I) \subseteq J$ can occur for at most one value of n; since J is prime, we see that $\{x \in U_n \mid \phi^n(I) \circ x \subseteq J\} = J_n$ for all $n \gg 0$ and so $\underline{\mathrm{Hom}}_S(S/I, S/J)_n = 0$ for $n \gg 0$. Thus $\underline{\mathrm{Hom}}_S(S/I, S/J)$ is indeed finite-dimensional over k. □

The right noetherian property and the left χ conditions for the ring T depend on a more subtle property of the set of points $\{c_n\}_{n\in\mathbb{Z}}$. Given a subset $\mathcal{C}$ of closed points of $\mathbb{P}^d$, we say that $\mathcal{C}$ is *critically dense* if every infinite subset of $\mathcal{C}$ has Zariski closure equal to all of $\mathbb{P}^d$.

Proposition 5.4. *Assume Hypothesis* 5.1*, and assume in addition that the set of points* $\{c_n\}_{n\in\mathbb{Z}}$ *is critically dense in* $\mathbb{P}^d$*. Then*

(1) T *satisfies left* χ_{d-1} *but fails left* χ_d*.*
(2) T *is right noetherian.*

Proof. (1) By [12, Lemma 8.4(2)], if J is a graded left ideal of S then we have

$$\underline{\mathrm{Ext}}^i_S(S/I, S/J)_n \cong \underline{\mathrm{Ext}}^i_U\big(U/I, U/\phi^{-n}(J)\big)_n$$

as k-spaces, for each $n \in \mathbb{Z}$. It follows that $\underline{\mathrm{Ext}}^d_S(S/I, S)_n \cong \underline{\mathrm{Ext}}^d_U(U/I, U)_n \neq 0$ for all $n \geqslant 0$, since one may calculate that $\underline{\mathrm{Ext}}^d(U/I, U) \cong (U/I)[d]$ easily from a Koszul resolution of U/I. So S fails χ_d on the left. On the other hand, [12, Proposition 8.6(1)] proves that since $\{c_n\}_{n\in\mathbb{Z}}$ is critically dense, we have $\dim_k \underline{\mathrm{Ext}}^i_S(S/I, M) < \infty$ for all $0 \leqslant i \leqslant d-1$ and all finitely generated left S-modules M. Then T satisfies χ_{d-1} on the left by Proposition 4.1.

(2) If we can show that every module of the form $(JS \cap T)/J$, for J a finitely generated right ideal of T, is finite-dimensional, then [12, Lemma 5.10] shows that T is right noetherian. Note that T is an Ore domain, since it is a domain of finite GK-dimension [7, Proposition 4.13]. Then the same proof as in [12, Lemma 5.9] shows that every module of the form $(JS \cap T)/J$ for J a finitely generated right ideal of T is filtered by subfactors of modules of the form $(fS \cap T)/fT$ and $S/(fS + T)$ for nonzero homogeneous $f \in T$. Thus we will just need to prove that modules of those forms are finite-dimensional over k.

Recall that $T = k + I$ by Lemma 5.2(1). Fix $n \geqslant 1$ and let $f \in T_n$ be arbitrary. We have for $m \geqslant n$ that $(fS + T)_m = \phi^{m-n}(f) \circ U_{m-n} + I_m$. Since $T_m = I_m$ has codimension 1 inside S_m for all $m \geqslant 1$ and I is prime in U, this implies that $(fS + T)_m = S_m$ if and only if $\phi^{m-n}(f) \notin I$.

Similarly, again assuming $m \geqslant n$, we have $(fS \cap T)_m = (\phi^{m-n}(f) \circ U_{m-n}) \cap I_m$. If $\phi^{m-n}(f) \notin I$, then as I is prime, $(\phi^{m-n}(f) \circ U_{m-n}) \cap I_m = \phi^{m-n}(f) \circ I_{m-n} = (fT)_m$. Conversely, if $\phi^{m-n}(f) \in I$, then $(fS \cap T)_m = (fS)_m \neq (fT)_m$.

Now since $\{c_n\}_{n\in\mathbb{Z}}$ is a critically dense set of points, every homogeneous $f \in S$ satisfies $f \notin \phi^n(I)$ for $n \ll 0$, which is equivalent to $\phi^n(f) \notin I$ for $n \gg 0$. We conclude that for any homogeneous $0 \neq f \in T$ the modules $(fS \cap T)/fT$ and $S/(fS + T)$ are finite-dimensional, as required. □

6. The strong noetherian property

We continue to study idealizer rings T satisfying Hypothesis 5.1, and we maintain the notation introduced in the previous section. In [12], the author showed the existence of rings which are not strongly noetherian on either side. Here we will show that the idealizer rings T are typically strongly noetherian on one side but not the other.

Let A be an arbitrary k-algebra. We call a left A-module M *strongly noetherian* if $M \otimes_k B$ is a noetherian left $A \otimes_k B$-module for every commutative noetherian k-algebra B. More generally, M is *universally noetherian* if $M \otimes_k B$ is noetherian over $A \otimes_k B$ for every noetherian k-algebra B.

Proposition 6.1. *Assume Hypothesis* 5.1*, and assume further that the set of points* $\{c_n\}_{n\in\mathbb{Z}}$ *is critically dense. Then* T *is a noetherian ring such that*

(1) T *is universally left noetherian.*
(2) T *is not strongly right noetherian.*

Proof. That T is noetherian follows from Propositions 5.3 and 5.4.

(1) We note that the ring S is universally left noetherian, as follows. For any noetherian k-algebra B, the ring $U \otimes_k B \cong B[x_0, \ldots, x_d]$ is noetherian by the Hilbert basis theorem. Then since $S \otimes_k B$ is a left Zhang twist of $U \otimes_k B$, it is also left noetherian [17,

Theorem 1.3]. Now we prove that T is universally noetherian on the left. We know that $M = {}_T(S/T)$ is finitely generated by Lemma 5.2, and since $\dim_k M_n = 1$ for all $n \geqslant 1$, we see that M must have Krull dimension 1. By [1, Theorem 4.23], M is a universally noetherian left T-module. So if B is any noetherian k-algebra, then $M \otimes_k B = (S \otimes_k B)/(T \otimes_k B)$ is a noetherian left $T \otimes_k B$-module. Then by [1, Lemma 4.2], since $S \otimes_k B$ is left noetherian, $T \otimes_k B$ is also left noetherian.

(2) The proof which we now present that T is not strongly noetherian on the right is quite analogous to the proof in [12, Section 7] that the ring R studied in that paper is not strongly noetherian. Let us first make a few comments about notation. We use subscripts to indicate extension of scalars, for example, $U_B = U \otimes_k B$. The automorphism ϕ of U naturally extends to an automorphism of U_B such that S_B is again the left Zhang twist of U_B by ϕ. We extend also our notational convention, so that juxtaposition means multiplication in S_B and $\circ$ means the commutative multiplication in U_B. Fix once and for all some particular choice of homogeneous coordinates for each of the points in $\{c_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{P}^d_k$. Then for $f \in U_B$, the expression $f(c_n)$ denotes polynomial evaluation at the fixed coordinates for c_n, giving a well-defined value in the ring B.

Because by assumption the point set $\{c_n\}_{n\in\mathbb{Z}}$ is critically dense, the same proof as in [12, Theorem 7.4] shows that there exists a noetherian commutative k-algebra B which is a unique factorization domain, constructed as an *infinite affine blowup* of affine space, and containing elements $f, g \in (U_B)_1$ with the following properties:

(1) $g(c_i) = \Omega_i f(c_i)$ for some $\Omega_i \in B$, for all $i \leqslant 0$.
(2) For all $i \ll 0$, $f(c_i)$ is not a unit in B.
(3) $\gcd(f, g) = 1$ in U_B.

Note that a homogeneous element $f \in U_B$ is in $(T_B)_{\geqslant 1} = I \otimes_k B$ if and only if $f(c_0) = 0$. Now for each $n \geqslant 1$ we may choose some element $\theta_n \in (S_B)_n \setminus (T_B)_n$ with coefficients in k. Putting $t_n = (\Omega_{-n} f - g)\theta_n$, we have in terms of the commutative multiplication in U_B that $t_n = \phi^n(\Omega_{-n} f - g) \circ \theta_n$, and since $\phi^n(\Omega_{-n} f - g)(c_0) = (\Omega_{-n} f - g)(c_{-n}) = 0$, we see that $t_n \in (T_B)_{n+1}$. Suppose for some n that $t_{n+1} = \sum_{i=1}^n t_i r_i$ with $r_i \in (T_B)_{n-i+1}$. Then

$$\phi^{n+1}(\Omega_{-n-1} f - g) \circ \theta_{n+1} = \sum_{i=1}^{n} \phi^{n+1}(\Omega_{-i} f - g) \circ \phi^{n-i+1}(\theta_i) \circ r_i.$$

Rewriting this equation in the form $h_1 \circ \phi^{n+1}(f) = h_2 \circ \phi^{n+1}(g)$, and using that $\gcd(f, g) = 1$, we may conclude that $\phi^{n+1}(g)$ divides h_1, where

$$h_1 = \Omega_{-n-1}\theta_{n+1} - \sum_{i=1}^{n} \Omega_{-i}\phi^{n-i+1}(\theta_i) \circ r_i.$$

Then $(\phi^{n+1}(g))(c_0) = g(c_{-n-1})$ divides $h_1(c_0)$. Each $r_i \in (T_B)_{\geqslant 1}$ and so $r_i(c_0) = 0$, and by assumption $\theta_{n+1} \notin T_B$ and so $\theta_{n+1}(c_0) \in k^\times$. Thus $g(c_{-n-1})$ divides Ω_{-n-1}, which implies that $f(c_{-n-1})$ is a unit in B. This contradicts property (2) above for $n \gg 0$.

Thus for $n \gg 0$ we must have $t_{n+1} \notin \sum_{i=1}^{n} t_i T_B$. We conclude that $\sum t_i T_B$ is an infinitely generated right ideal of T_B, so $T \otimes_k B$ is not right noetherian and T is not strongly right noetherian. $\square$

7. Tensor products of algebras

In Proposition 6.1 we showed explicitly that T is not strongly right noetherian by exhibiting a commutative noetherian k-algebra B such that $T \otimes_k B$ is not right noetherian. Necessarily, such a B is not a finitely generated commutative algebra. By contrast, if we allow ourselves to tensor by noncommutative rings then we may find a finitely generated noetherian k-algebra B' such that $T \otimes_k B'$ is not right noetherian. In fact, we will see in the next theorem that one may take B' to be T itself.

In order to stay within the class of $\mathbb{N}$-graded algebras, in addition to tensor products it will be useful also to consider *Segre products*, defined as follows. If A and B are two $\mathbb{N}$-graded algebras we let $A \overset{s}{\otimes}_k B$ be the $\mathbb{N}$-graded algebra $\bigoplus_{n=0}^{\infty} A_n \otimes_k B_n$. The following lemma is then elementary.

Lemma 7.1. *Let A and B be $\mathbb{N}$-graded algebras. If $A \otimes_k B$ is left* (*right*) *noetherian, then $A \overset{s}{\otimes}_k B$ is left* (*right*) *noetherian.*

Proof. Since any homogeneous left ideal I of $A \overset{s}{\otimes} B$ satisfies $(A \otimes B)I \cap (A \overset{s}{\otimes} B) = I$, a proper ascending chain of homogeneous left ideals of $A \overset{s}{\otimes} B$ induces a proper ascending chain of left ideals of $A \otimes B$. $\square$

We thank James Zhang for pointing out to us the following useful fact.

Lemma 7.2. *Let A be connected $\mathbb{N}$-graded and noetherian. Then A is finitely presented.*

Proof. Let

$$\ldots \to \bigoplus_{i=1}^{r_1} A[-d_{1i}] \to \bigoplus_{i=1}^{r_0} A[-d_{0i}] \to A \to k \to 0$$

be a graded free resolution of ${}_A k$ by free modules of finite rank. Then one may check that A has a presentation with r_0 generators and r_1 relations. $\square$

The following theorem shows that it is possible to find two connected graded noetherian rings whose tensor product is noetherian on one side only, as well a pair of connected graded noetherian rings whose tensor product is noetherian on neither side.

Theorem 7.3. *Assume Hypothesis* 5.1, *and in addition that $\{c_n\}_{n\in\mathbb{Z}}$ is critically dense. Let $T' = T \overset{s}{\otimes}_k T^{\mathrm{op}}$. Then*

(1) *T and T' are noetherian finitely presented connected graded k-algebras.*

(2) *$T \otimes_k T$ is left noetherian, but not right noetherian.*
(3) *$T' \otimes_k T' \cong T' \otimes_k (T')^{\mathrm{op}}$ is neither left nor right noetherian.*

Proof. (1) The ring T is noetherian by Propositions 5.3 and 5.4. In fact, by Proposition 6.1 T is universally left noetherian. It follows immediately that T^{op} is universally right noetherian. Thus $T \otimes_k T^{\mathrm{op}}$ is both left and right noetherian. By Lemma 7.1, T' is noetherian. Then by Lemma 7.2, both T and T' are finitely presented.

(2) As we saw in part (1), T is universally left noetherian, so that $T \otimes_k T$ is left noetherian. Now we will prove that $T \otimes T$ is not right noetherian. By Lemma 7.1, it is enough to prove that $T \overset{s}{\otimes} T$ is not right noetherian.

For a graded ring A we will use the abbreviation $A^s = A \overset{s}{\otimes} A$. Now let $X = \operatorname{proj} U^s \cong \mathbb{P}^d \times \mathbb{P}^d$. The graded ring U^s has the automorphism $\phi \otimes \phi$ with corresponding automorphism $\varphi \times \varphi$ of X. The graded ring S^s may be thought of as the left Zhang twist of U^s by $\phi \otimes \phi$, and we identify the underlying vector spaces. In particular, any homogeneous element of S^s defines a vanishing locus in X. Now let $\Delta \subset X$ be the diagonal subscheme, and let J be the left ideal of S^s consisting of those elements which vanish along Δ. Since $(\varphi \times \varphi)(\Delta) = \Delta$, it follows easily that J is a two-sided ideal of S^s. Writing $K = I \overset{s}{\otimes}_k I$, a left ideal of S^s, we have $T^s = k \oplus K$. Then to prove that T^s is not right noetherian, by Proposition 2.2 it will be enough to show that $(J \cap K)/JK$ is not finite-dimensional over k.

Let $\circ$ indicate multiplication in the commutative ring U^s. Since J is invariant under $\phi \otimes \phi$, we have $J \circ K = JK$, and so it will be equivalent to prove that $M = (J \cap K)/(J \circ K)$ is not a torsion U^s-module. To show this, we consider the corresponding sheaf $\widetilde{M}$ on X, look locally at the point $p = (c, c)$, and prove that $\widetilde{M}_p \neq 0$.

Choose local affine coordinates $u_1, \dots, u_d$ for a principal open set $\mathbb{A}^d \subseteq \mathbb{P}^d$ such that the point c corresponds to the origin. Let $v_1, \dots, v_d$ be the same coordinates for the equivalent open set $\mathbb{A}^d$ in the second copy of $\mathbb{P}^d$, so that $u_1, \dots, u_d, v_1, \dots, v_d$ are local coordinates for an affine neighborhood $\mathbb{A}^{2d}$ of p in X such that p is the origin in these coordinates. Now let $\mathfrak{p}$ be the homogeneous prime ideal of U^s corresponding to the point $p = (c, c)$. Setting $U' = (U^s)_{(\mathfrak{p})} = \mathcal{O}_{X,p}$, $J' = J_{(\mathfrak{p})}$, and $K' = K_{(\mathfrak{p})}$, we have

$$\widetilde{M}_p = M_{(\mathfrak{p})} \cong (J' \cap K')/(J'K'),$$

where we revert to the use of juxtaposition to indicate multiplication in the commutative local ring U'. Explicitly, U' is the polynomial ring $k[u_1, \dots, u_d, v_1, \dots, v_d]$ localized at the maximal ideal $\mathfrak{m} = (u_1, \dots, u_d, v_1, \dots, v_d)$, $J' = (u_1 - v_1, \dots, u_d - v_d)$, and $K' = (u_1, u_2, \dots, u_d)(v_1, v_2, \dots, v_d)$. Now it is clear that $w = u_1v_2 - u_2v_1 \in J' \cap K'$, but $w \notin J'K'$ since $w \notin \mathfrak{m}^3 \supseteq J'K'$. Thus $\widetilde{M}_p \neq 0$, as we needed to show.

(3) Note that $(T')^{\mathrm{op}} \cong T^{\mathrm{op}} \overset{s}{\otimes} T \cong T'$. The fact that $T' \otimes T'$ is neither left nor right noetherian follows immediately from part (2). □

Remark 7.4. Assuming the setup of Hypothesis 5.1, the ring $R = k\langle I_1 \rangle \subseteq S$ which is generated by the degree 1 piece of T is a graded ring of the type studied in the article [12]. In case the points $\{c_n\}_{n \in \mathbb{Z}}$ are critically dense, this ring R has similarly strange properties under tensor products. For example, a similar but slightly more complicated version of the argument in Theorem 7.3(2) above would show that $R \otimes_k R$ is neither left nor right noetherian.

8. Proof of the main theorem

In the final section, we recapitulate all of our preceding results to prove Theorem 1.1, which we restate as Theorem 8.2 below. The only thing we have left to show is that given the setup of Hypothesis 5.1, there exists a plentiful supply of choices of a point $c \in \mathbb{P}^d$ and an automorphism $\varphi \in \operatorname{Aut} \mathbb{P}^d$ such that $\mathcal{C} = \{c_n\}_{n \in \mathbb{Z}}$ is critically dense. This situation has already been studied in the paper [12]; we repeat the result for the reader's reference as the next proposition.

We call a subset of a variety X *generic* if its complement is contained in a countable union of closed subvarieties $Z \subsetneq X$. Note that as long as the base field k is uncountable, any generic subset is intuitively "almost all" of X, in particular it is nonempty. Thus the first part of the following proposition shows that if the base field k is uncountable, then any suitably general pair (φ, c) will lead to a critically dense set $\mathcal{C}$. The second part shows that in case $\operatorname{char} k = 0$ we may easily write down many explicit examples of pairs (φ, c) for which $\mathcal{C}$ is critically dense.

Proposition 8.1 [12, Theorem 12.4, Example 12.8]. *Assume Hypothesis* 5.1 *and set* $\mathcal{C} = \{c_n\}_{n \in \mathbb{Z}}$.

(1) *Let k be uncountable. For any given $c \in \mathbb{P}^d$, there is a generic subset $Y \subseteq \operatorname{Aut} \mathbb{P}^d = \mathrm{PGL}(k, d)$ such that if $\varphi \in Y$ then $\mathcal{C}$ is critically dense.*
(2) *If $\operatorname{char} k = 0$, $c = (1 : 1 : \cdots : 1)$, and φ is defined by*

$$(a_0 : a_1 : \cdots : a_d) \mapsto (a_0 : p_1 a_1 : p_2 a_2 : \cdots : p_d a_d),$$

then $\mathcal{C}$ is critically dense if and only if $p_1, \ldots, p_d$ generate a multiplicative subgroup of $k^\times$ which is isomorphic to $\mathbb{Z}^d$.

Finally, we summarize all of the properties that the ring T has in case the set of points $\mathcal{C}$ is critically dense.

Theorem 8.2. *Assume Hypothesis* 5.1. *Let k be uncountable and assume that the pair (φ, c) is chosen so that $\mathcal{C} = \{c_n\}_{n \in \mathbb{Z}}$ is critically dense. Then the idealizer ring $T = \mathbb{I}(I) = T(\varphi, c)$ is a noetherian connected finitely presented graded ring with the following properties*:

(1) *T is left universally noetherian, but not strongly right noetherian.*
(2) *$T \otimes_k T$ is left noetherian but not right noetherian. The Segre product $T' = T \overset{s}{\otimes}_k T^{\mathrm{op}}$ is also a finitely presented connected graded noetherian ring, but $T' \otimes_k T'$ is noetherian on neither side.*

(3) Proj-T *and* T-Proj *have the same underlying category but non-isomorphic distinguished objects*; *specifically,* Proj-$T \cong (\mathrm{Qch}\,\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d})$ *and* T-Proj $\cong (\mathrm{Qch}\,\mathbb{P}^d, \mathcal{I})$, *where* $\mathcal{I}$ *is the sheaf of ideals corresponding to the point* $c \in \mathbb{P}^d$.
(4) T *satisfies left* χ_{d-1} *but not left* χ_d, *and* T *fails* χ_1 *on the right.*
(5) $\mathrm{cd}(\text{Proj-}T) = \mathrm{cd}(T\text{-Proj}) = d$.
(6) *Although no Veronese ring of* T *is generated in degree* 1, *one has isomorphisms* T-Proj $\cong T^{(n)}$-Proj *and* Proj-$T \cong$ Proj-$T^{(n)}$ *for all* $n \geqslant 1$.

Proof. Note that by Proposition 8.1, we may indeed find a pair (φ, c) so that $\mathcal{C}$ is critically dense. Then T is noetherian by Propositions 5.4(2) and 5.3, and T is finitely presented by Lemma 7.2.

Now (1) follows from Proposition 6.1, and (2) from Theorem 7.3.

For (3), note that since S is a left Zhang twist of U, we have S-Gr $\simeq U$-Gr and so it easily follows that S-Proj $\cong U$-Proj. Now the opposite ring S^{op} of S is isomorphic to the left Zhang twist of U by ϕ^{-1}; this may be checked directly, or see the proof of [12, Lemma 4.2(1)]. Thus we also have an isomorphism Proj-$S \cong$ Proj-U. By Serre's theorem, we also have an equivalence of categories U-Qgr $\simeq \mathrm{Qch}\,\mathbb{P}^d$, where $\mathrm{Qch}\,\mathbb{P}^d$ is the category of quasi-coherent sheaves on $\mathbb{P}^d$.

Now using Lemma 3.2, it follows that

$$T\text{-Proj} \cong (S\text{-Qgr}, \pi I) \cong \big(\mathrm{Qch}\,\mathbb{P}^d, \mathcal{I}\big) \quad \text{and} \quad \text{Proj-}T \cong (\text{Qgr-}S, \pi S) \cong \big(\mathrm{Qch}\,\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}\big).$$

Since $d \geqslant 2$, the ideal sheaf $\mathcal{I}$ which defines the closed point c is not locally free, so in particular we have $\mathcal{I} \ncong \mathcal{O}_{\mathbb{P}^d}$ and (3) is proved.

Next, result (4) is a combination of Propositions 4.2 and 5.4(1). Since S-Proj $\cong U$-Proj and Proj-$S \cong$ Proj-U, it follows easily that $\mathrm{cd}(S\text{-Proj}) = \mathrm{gd}(S\text{-Qgr}) = \mathrm{cd}(\text{Proj-}S) = \mathrm{gd}(\text{Qgr-}S) = d$, and so (5) is a consequence of Proposition 3.5. Finally, (6) follows from Proposition 3.4 and Lemma 5.2(2). □

We close with a few remarks concerning Theorem 8.2.

Remark 8.3. Theorem 8.2(2) shows that the tensor product of two noetherian finitely presented connected graded algebras (over an algebraically closed field) can fail to be noetherian. This answers [4, Appendix, Open Question 16′].

Remark 8.4. Suppose that A is a connected graded noetherian ring satisfying left χ_1 such that A-Proj $\cong (\mathrm{Qch}\,X, \mathcal{O}_X)$ for some proper scheme X. Keeler showed that in this case A must be equal in large degree to a twisted homogeneous coordinate ring $B(X, \mathcal{L}, \sigma)$ where $\mathcal{L}$ is σ-ample [6, Theorem 7.17]. In particular, A must be universally noetherian and must satisfy χ on both sides.

Now consider instead connected graded noetherian rings A with left χ_1 such that A-Proj $\cong (\mathrm{Qch}\,X, \mathcal{F})$ for some proper scheme X, but where $\mathcal{F}$ is not assumed to be the structure sheaf. Then $A = T$, where T satisfies the conclusions of Theorem 8.2, is an example showing that rings with much more unusual behavior may occur in this case.

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