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ScienceDirect
J. Differential Equations 246 (2009) 164–182
Journal of Differential Equations
www.elsevier.com/locate/jde

Sharp weighted-norm inequalities for functions with compact support in $\mathbb{R}^N \setminus \{0\}$

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Received 18 January 2008

Available online 4 June 2008

Abstract

In this paper we study a class of Caffarelli–Kohn–Nirenberg inequalities without restricting the pertinent parameters. In particular, we determine the values of the corresponding optimal constants and the functions that achieve them, i.e., minimizers of a suitable functional. By studying a corresponding Euler–Lagrange equation, we also determine infinitely many sign-changing solutions at higher energy levels in addition to the found ground-state solutions.

MSC: primary: 35J20; secondary: 35J60

Keywords: Caffarelli–Kohn–Nirenberg inequalities; Best constants; Positive solutions

In this article we will discuss one particular case of the Caffarelli–Kohn–Nirenberg inequalities, introduced in [2]. It was noted by some authors (see e.g. [6] and [7]) that if one chooses to work with the smaller space $C_0^\infty(\mathbb{R}^N \setminus \{0\})$ of smooth functions in $\mathbb{R}^N$ which vanish in a neighborhood of the origin as well as outside of a compact set, then the inequality

$$C^2(N,a,b)\left(\int_{\mathbb{R}^N} \frac{u^2}{|x|^{a+b+1}}\,dx\right)^2 \leqslant \left(\int_{\mathbb{R}^N} \frac{u^2}{|x|^{2a}}\,dx\right)\left(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}}\,dx\right) \tag{1}$$

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2008.04.022

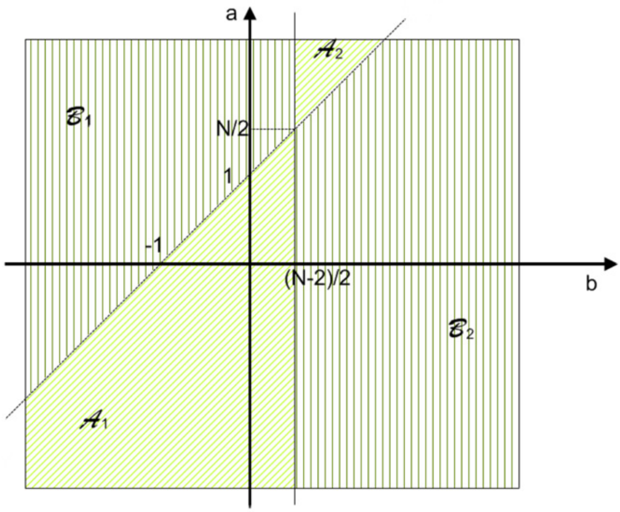

Fig. 1.

will hold for some $C(N,a,b)$ independent of u and without restriction on the parameters (a,b) in the whole plane. The inequality above is void of content if $C(N,a,b)=0$ but, as stated in Remark 1, this only happens at the point $(a,b)=(\frac{N}{2},\frac{N-2}{2})$.

In the Caffarelli–Kohn–Nirenberg article [2], since the functions u are not required to vanish at the origin, it is necessary to have a and b less than $\frac{N}{2}$, and $a+b<N-1$ for integrability reasons.

In this new setting, we point out some interesting facts regarding the exact value of the best constant $C(N,a,b)$ in (1) and the functions that achieve it. Therefore we define on $C_0^\infty(\mathbb{R}^N\setminus\{0\})$ the energy functional

$$E(u)=\frac{(\int_{\mathbb{R}^N}\frac{u^2}{|x|^{2a}}\,dx)^{\frac{1}{2}}(\int_{\mathbb{R}^N}\frac{|\nabla u|^2}{|x|^{2b}}\,dx)^{\frac{1}{2}}}{\int_{\mathbb{R}^N}\frac{u^2}{|x|^{a+b+1}}\,dx},\tag{2}$$

so that the best constant $C(N,a,b)$ in (1) is the infimum of $E(u)$.

The organization of this paper is the following. We start in Section 1 with the one-dimensional case and some preliminary lemmas needed in our approach. Then we prove Theorem 2, the one-dimensional version of Theorem 1. In Section 2 we prove Theorem 3 stating that minimizing sequences of $E(u)$ may be taken to consist entirely of radial functions. And we show how the radial case in $\mathbb{R}^N$ "folds" into the one-dimensional version of (1). In Section 3 we derive the Euler–Lagrange equation associated with the energy E, and we prove Theorem 1 below, in which we find the best constants $C(N,a,b)$ in (1) (*infima* of $E(u)$ in (2)) and possible functions that achieve them (*minimizers* of $E(u)$). In Section 4 we prove the existence of infinitely many other stationary points of $E(u)$ at discrete energy levels higher than $C(N,a,b)$. Finally, in Section 5 we provide a few concluding remarks and questions.

In order to state Theorem 1, we let the regions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$, and $\mathcal{B}_2$ be defined by the lines $b=\frac{N-2}{2}$ and $a=b+1$, as shown in Fig. 1. We define $\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2$ and $\mathcal{B}=\mathcal{B}_1\cup\mathcal{B}_2$, where we assume that the line $a = b + 1$ is disjoint of both $\mathcal{A}$ and $\mathcal{B}$, and the line $b = \frac{N-2}{2}$, with the point $(\frac{N}{2}, \frac{N-2}{2})$ removed, is common to $\mathcal{A}$ and $\mathcal{B}$.

Theorem 1. *According to the location of the point (a, b) in the plane, we have*:

(a) *In the region $\mathcal{A}$ the best constant is $C(N, a, b) = \frac{|N-(a+b+1)|}{2}$ and it is achieved by the functions $u(x) = D\exp(\frac{t|x|^{b+1-a}}{b+1-a})$, with $t < 0$ in $\mathcal{A}_1$ and $t > 0$ in $\mathcal{A}_2$, and D a nonzero constant.*
(b) *In the region $\mathcal{B}$ the best constant is $C(N, a, b) = \frac{|N-(3b-a+3)|}{2}$ and it is achieved by the functions $u(x) = D|x|^{2(b+1)-N}\exp(\frac{t|x|^{b+1-a}}{b+1-a})$, with $t > 0$ in $\mathcal{B}_1$ and $t < 0$ in $\mathcal{B}_2$.*
(c) *In addition, the only values of the parameters where the best constant is not achieved are those on the line $a = b + 1$, where $C(N, b + 1, b) = \frac{|N-2(b+1)|}{2}$.*

Remark 1. Note that along the line $b = \frac{N-2}{2}$, the best constants, as given by the formulas for regions $\mathcal{A}$ and $\mathcal{B}$, do agree. Moreover, at the point $(\frac{N}{2}, \frac{N-2}{2})$ the constant is zero.

Remark 2. The presence of the parameters D and t in the class of minimizers is due to symmetries of the inequalities (1), also inherited by the Euler–Lagrange equations (see (23)). The homogeneity evidently explains D. As for t, we have that if $u = u(x)$ is a minimizer of (1), then so is $v = v(y)$, where $y = tx$ and $u(x) = v(y)$. Indeed, if (say) we assume $t > 0$ we have

$$\int_{\mathbb{R}^N} \frac{|\nabla_x u(x)|^2}{|x|^{2b}}\,dx = t^{2b+2-N}\int_{\mathbb{R}^N} \frac{|\nabla_y v(y)|^2}{|y|^{2b}}\,dy,$$

$$\int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^{2a}}\,dx = t^{2a-N}\int_{\mathbb{R}^N} \frac{v^2(y)}{|y|^{2a}}\,dy,$$

and

$$\int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^{a+b+1}}\,dx = t^{a+b+1-N}\int_{\mathbb{R}^N} \frac{v^2(y)}{|y|^{a+b+1}}\,dy,$$

and we infer that $E(u) = E(v)$. The sign constraints on t in Theorem 1 are dictated by integrability requirements.

1. The one-dimensional case

We begin with an elementary fact involving functions u in $C_0^\infty(0,\infty)$, i.e., smooth functions $u : (0,\infty) \to \mathbb{R}$ having compact support in the open half-line $(0,\infty)$.

Lemma 1. *Given $\beta \in \mathbb{R}$ define $v(s) = s^{\beta+1}u(s^{-1})$ for each $u \in C_0^\infty(0,\infty)$. Then*

$$\int_0^\infty \frac{u_r^2}{r^\beta}\,dr = \int_0^\infty \frac{v_s^2}{s^\beta}\,ds.$$

Proof. From $u(r) = r^{\beta+1}v(r^{-1})$ and $r = s^{-1}$ we calculate

$$u_r = (\beta+1)r^{\beta}v + r^{\beta+1}\bigl(-r^{-2}\bigr)v_s = r^{\beta}\bigl[-sv_s + (\beta+1)v\bigr],$$

and

$$\frac{u_r^2}{r^{\beta}}\,dr = -s^{-(\beta+2)}\bigl[sv_s - (\beta+1)v\bigr]^2\,ds.$$

Then, noticing that the right-hand side above can be written as

$$-\frac{v_s^2}{s^{\beta}}\,ds + d\biggl(\frac{(\beta+1)v^2}{s^{\beta+1}}\biggr),$$

the result follows by integration. □

Remark 3. By letting $D^{1,2}(0,\infty; r^{-\beta}\,dr)$ denote the closure of $C_0^{\infty}(0,\infty)$ with respect to the norm $\|u\| := (\int_0^{\infty}\frac{u_r^2}{r^{\beta}}\,dr)^{1/2}$, we note that the above lemma says that the mapping $u(s) \mapsto s^{\beta+1}u(s^{-1})$ is a linear isometry from the Hilbert space $D^{1,2}(0,\infty; r^{-\beta}\,dr)$ onto itself. As such, Lemma 1 is interesting by itself.

Next, let us consider the one-dimensional version of (1) where, for simplicity, we write $C(a,b)$ instead of $C(1,a,b)$ for the best constant:

$$C^2(a,b)\Biggl(\int_0^{\infty}\frac{u(r)^2}{r^{a+b+1}}\,dr\Biggr)^2 \leqslant \Biggl(\int_0^{\infty}\frac{u(r)^2}{r^{2a}}\,dr\Biggr)\Biggl(\int_0^{\infty}\frac{u_r(r)^2}{r^{2b}}\,dr\Biggr). \tag{3}$$

A type of proof that is often found in the literature is as follows (where $u \in C_0^{\infty}(0,\infty)$ is arbitrary):

$$0 = \int_0^{\infty}\frac{d}{dr}\biggl(\frac{u^2}{r^{a+b}}\biggr)dr = -(a+b)\int_0^{\infty}\frac{u^2}{r^{a+b+1}}\,dr + 2\int_0^{\infty}\frac{uu_r}{r^{a+b}}\,dr, \tag{4}$$

hence

$$\biggl|\frac{a+b}{2}\biggr|\int_0^{\infty}\frac{u^2}{r^{a+b+1}}\,dr = \biggl|\int_0^{\infty}\frac{uu_r}{r^{a+b}}\,dr\biggr| \leqslant \int_0^{\infty}\frac{|u||u_r|}{r^{a+b}}\,dr,$$

so that the Cauchy–Schwarz inequality gives (3) with

$$C(a,b) \geqslant \biggl|\frac{a+b}{2}\biggr|. \tag{5}$$

On the other hand, if we apply Lemma 1 with $\beta = 2b$ and

$$u(r) = r^{2b+1}v\bigl(r^{-1}\bigr), \qquad s = r^{-1}, \tag{6}$$

to the last integral in (3), straightforward calculations on the other two integrals show that (3) becomes

$$C^2(a,b)\left(\int_0^\infty \frac{v(s)^2}{s^{3b-a+3}}\,ds\right)^2 \leqslant \left(\int_0^\infty \frac{v(s)^2}{s^{4b-2a+4}}\,ds\right)\left(\int_0^\infty \frac{v_s(s)^2}{s^{2b}}\,ds\right).$$

Therefore, by defining $\bar{a} = 2b - a + 2$ and $\bar{b} = b$, the above reads

$$C^2(a,b)\left(\int_0^\infty \frac{v(s)^2}{s^{\bar{a}+\bar{b}+1}}\,ds\right)^2 \leqslant \left(\int_0^\infty \frac{v(s)^2}{s^{2\bar{a}}}\,ds\right)\left(\int_0^\infty \frac{v_s(s)^2}{s^{2\bar{b}}}\,ds\right). \tag{7}$$

In other words, if $C(a,b)$ denotes the best constant in (3) for a given (a,b), and if we define the affine transformation

$$\begin{cases} \bar{a} = 2b - a + 2, \\ \bar{b} = b, \end{cases} \tag{8}$$

then the best constant $C(\bar{a},\bar{b})$ in (7) satisfies

$$C(\bar{a},\bar{b}) = C(2b - a + 2, b) = C(a,b) \quad \forall (a,b) \in \mathbb{R}^2. \tag{9}$$

For completeness we state the following lemma concerning (8), whose obvious proof we omit.

Lemma 2. *Each point (a,b) on the line $a = b + 1$ is fixed under the affine transformation* (8)*; in particular, the line $a = b + 1$ is invariant under* (8)*. In fact, the only other lines which are invariant under* (8) *are the vertical lines $b = \hat{b}$ (constant).*

From now on we denote $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, where $\mathcal{A}_1$ is the region defined by $a < b + 1$, $b \leqslant -\frac{1}{2}$ and $\mathcal{A}_2$ is the region defined by $a > b + 1$, $b \geqslant -\frac{1}{2}$. We also let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where $\mathcal{B}_1$ is defined by $a > b + 1$, $b \leqslant -\frac{1}{2}$ and $\mathcal{B}_2$ by $a < b + 1$, $b \geqslant -\frac{1}{2}$ (cf. Fig. 2).

Theorem 2 *(One-dimensional case of Theorem 1). For each $(a,b) \in \mathcal{A}$ the best constant $C(a,b)$ in* (3) *is given by*

$$C(a,b) = \frac{|a+b|}{2}$$

and is achieved by the functions

$$u(r) = D \exp\left(\frac{t}{b+1-a} r^{b+1-a}\right), \tag{10}$$

for arbitrary $D \in \mathbb{R}$ and $t \neq 0$ with $\operatorname{sgn}(t) = -\operatorname{sgn}(b+1-a)$. On the other hand, for $(a,b) \in \mathcal{B}$, the best constant $C(a,b)$ in (3) *is*

$$C(a,b) = \frac{|3b - a + 2|}{2}$$

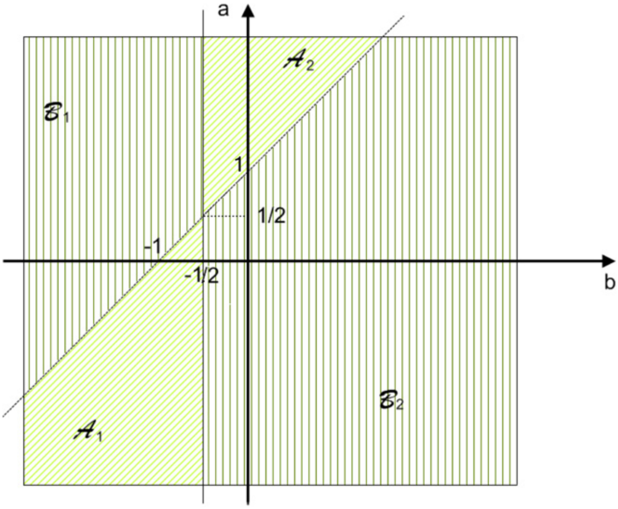

Fig. 2.

and it is achieved by the functions

$$u(r) = Dr^{2b+1} \exp\left(\frac{t}{b+1-a} r^{b+1-a}\right), \tag{11}$$

where $D \in \mathbb{R}$ *is arbitrary and* $t \neq 0$ *is such that* $\operatorname{sgn}(t) = -\operatorname{sgn}(b+1-a)$.

Proof. First note that the region $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) is the image of $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) under the affine mapping given in (8). Also note that the line given by $a = b+1$ is the complement of $\mathcal{A} \cup \mathcal{B}$.

Now, in view of (5), (8) and (9), we have that

$$C(\bar{a}, \bar{b}) = C(a, b) \geqslant \max\left\{\frac{|\bar{a}+\bar{b}|}{2}, \frac{|a+b|}{2}\right\}. \tag{12}$$

On the other hand, it was shown in [6] that, for (a, b) in the region $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, one has

$$C(a, b) = \frac{|a+b|}{2} \tag{13}$$

with $C(a, b)$ being achieved by the functions

$$u(r) = D \exp\left(\frac{t}{b+1-a} r^{b+1-a}\right),$$

for arbitrary $D \in \mathbb{R}$ and $t \neq 0$ with $\operatorname{sgn}(t) = -\operatorname{sgn}(b+1-a)$. In particular, such u satisfy

$$\int_0^\infty \frac{u_r^2}{r^{2b}}\, dr < \infty, \qquad \int_0^\infty \frac{u^2}{r^{2a}}\, dr < \infty. \tag{14}$$

Therefore, by noticing that

$$|\bar{a}+\bar{b}| > |a+b| \quad \Leftrightarrow \quad (\bar{a}+\bar{b})^2 > (a+b)^2 \quad \Leftrightarrow \quad (a-b-1)(2b+1) < 0$$

we also conclude that, for (a,b) in the region $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, one has

$$C(a,b) = \frac{|\bar{a}+\bar{b}|}{2} = \frac{|3b-a+2|}{2}, \tag{15}$$

with $C(a,b)$ being achieved (recall (6)) by the functions

$$v(s) = Ds^{2b+1}\exp\left(\frac{t}{b+1-a}s^{b+1-a}\right),$$

for arbitrary $D \in \mathbb{R}$ and t with $\operatorname{sgn}(t) = -\operatorname{sgn}(b+1-a)$.

The proof is complete. We point out that, when $a = b+1$ (i.e., (a,b) is in the complement of $\mathcal{A} \cup \mathcal{B}$), it was shown in [4] (cf. also [5]), that the best constant is $C(b+1,b) = \frac{|2b+1|}{2}$ and that it is *not* achieved. □

2. Reduction to the radial case

Let us denote by $C_{\mathrm{rad}}(N,a,b)$ the best constant in the inequality (1) when restricted to the space of radial functions

$$C^\infty_{0,\mathrm{rad}}\left(\mathbb{R}^N \setminus \{0\}\right) = \left\{u \in C_0^\infty\left(\mathbb{R}^N \setminus \{0\}\right):\ u(x) = u\left(|x|\right)\right\}.$$

That is,

$$C_{\mathrm{rad}}(N,a,b) = \inf E(u) \quad \text{when } u \in C^\infty_{0,\mathrm{rad}}\left(\mathbb{R}^N \setminus \{0\}\right) \setminus \{0\}.$$

Theorem 3. *For all pairs (a,b) we have $C_{\mathrm{rad}}(N,a,b) = C(N,a,b)$.*

The proof follows at once from the following

Lemma 3. *For any $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ there exists a function $u_{\mathrm{rad}} \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N \setminus \{0\})$ such that $E(u) \geqslant E(u_{\mathrm{rad}})$.*

Proof. For a number of other calculations (e.g. in [4], or prior to this, in [8]), the transformation of integrals over $\mathbb{R}^N$ to integrals over the cylinder $\mathcal{C} = \mathbb{S}^{N-1} \times \mathbb{R}$ has turned out to be very useful. The idea for the proof of the lemma is to convert inequality (1) into a new inequality in terms of functions on $\mathcal{C}$. Decomposition in terms of spherical harmonics then reveals that the inequality is sharp for radial functions only.

So, consider the diffeomorphism $\Upsilon : \mathbb{R}^N \setminus \{0\} \to \mathcal{C} = \mathbb{S}^{N-1} \times \mathbb{R}$ given by $\Upsilon(x) = (\frac{x}{|x|}, -\ln|x|) = (\theta, t)$. To any function $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ we associate the function $v \in C_0^\infty(\mathcal{C})$ by

$$v\left(\Upsilon(x)\right) = |x|^{\frac{N-2-2b}{2}} u(x). \tag{16}$$

It is a straightforward calculation to check that

$$\int_{\mathbb{R}^N} |x|^{-2b}|\nabla u|^2\,dx = \int_{\mathcal{C}} \left[|\nabla v|^2 + \left(\frac{N-2-2b}{2}\right)^2 v^2\right] d\mu,$$

where the gradient of v and the measure $d\mu$ correspond to the standard metric on the cylinder induced by the Euclidean metric in $\mathbb{R}^{N+1}$. Also,

$$\int_{\mathbb{R}^N} |x|^{-2a}u^2\,dx = \int_{\mathcal{C}} e^{2(a-b-1)t}v^2\,d\mu$$

and

$$\int_{\mathbb{R}^N} |x|^{-a-b-1}u^2\,dx = \int_{\mathcal{C}} e^{(a-b-1)t}v^2\,d\mu.$$

Therefore, on the cylinder $\mathcal{C}$, the inequality (1) is equivalent to

$$C^2(N,a,b) \leqslant E^2(v) = \frac{(\int_{\mathcal{C}} e^{2(a-b-1)t}v^2\,d\mu)(\int_{\mathcal{C}}[|\nabla v|^2 + (\frac{N-2-2b}{2})^2 v^2]\,d\mu)}{(\int_{\mathcal{C}} e^{(a-b-1)t}v^2\,d\mu)^2}. \tag{17}$$

We now decompose v in terms of spherical harmonics $Y_k(\theta)$, where we require

$$-\Delta_\theta Y_k = \lambda_k Y_k \quad \text{and} \quad \int_{\mathbb{S}^{N-1}} Y_k Y_l\,d\theta = \delta_{kl}.$$

Note that $\lambda_0 = 0$ is a simple eigenvalue and Y_0 is constant on $\mathbb{S}^{N-1}$. There exist functions $f_k \in C_0^\infty(\mathbb{R})$ such that

$$v(\theta,t) = \sum_{k=0}^{\infty} f_k(t)Y_k(\theta).$$

One can now check that, for any function $\rho = \rho(t)$, it holds

$$\int_{\mathcal{C}} \rho v^2\,d\mu = \int_{\mathbb{R}} \rho(t) \int_{\mathbb{S}^{N-1}} v^2(\theta,t)\,d\theta\,dt = \sum_{k=0}^{\infty} \int_{\mathbb{R}} \rho(t) f_k^2(t)\,dt.$$

Similarly, it holds that

$$\int_{\mathcal{C}} \left[|\nabla v|^2 + \left(\frac{N-2-2b}{2}\right)^2 v^2\right] d\mu$$
$$= \sum_{k=0}^{\infty} \int_{\mathbb{R}} \left\{\left(f_k'\right)^2(t) + \left[\lambda_k + \left(\frac{N-2-2b}{2}\right)^2\right] f_k^2(t)\right\} dt.$$

Therefore, if f_k is not identically zero for some $k \geqslant 1$, i.e., if v is not constant on every sphere $\mathbb{S}^{N-1} \times \{t\}$ then, since $\lambda_k > 0$, we have that

$$E^2(v) > \frac{(\sum_{k=0}^{\infty} \int_{\mathbb{R}} e^{2(a-b-1)t} f_k^2(t)\,dt)(\sum_{k=0}^{\infty} \int_{\mathbb{R}} [(f_k')^2(t) + (\frac{N-2-2b}{2})^2 f_k^2(t)]\,dt)}{(\sum_{k=0}^{\infty} \int_{\mathbb{R}} e^{(a-b-1)t} f_k^2(t)\,dt)^2}. \tag{18}$$

Next, if we let $\varphi(t) = (\sum_{k=0}^{\infty} f_k^2(t))^{\frac{1}{2}}$, then

$$\sum_{k=0}^{\infty} \left(f_k'\right)^2(t) \geqslant \frac{(\sum_{k=0}^{\infty} f_k'(t) f_k(t))^2}{\sum_{k=0}^{\infty} f_k^2(t)} = (\varphi')^2(t)$$

which, substituted into (18), gives

$$E^2(v) > \frac{(\int_{\mathbb{R}} e^{2(a-b-1)t} \varphi^2(t)\,dt)(\int_{\mathbb{R}} [(\varphi')^2(t) + (\frac{N-2-2b}{2})^2 \varphi^2(t)]\,dt)}{(\int_{\mathbb{R}} e^{(a-b-1)t} \varphi^2(t)\,dt)^2}.$$

This means that, when we view $\varphi(t)$ as a function on the cylinder which is independent of θ and we define $u_{\mathrm{rad}}(x) = |x|^{-\frac{N-2-2b}{2}} \varphi(\Upsilon(x))$, we then have

$$E(u) = E(v) \geqslant E(\varphi) = E(u_{\mathrm{rad}}),$$

with equality if and only if $u = u_{\mathrm{rad}}$. This concludes our proof. □

The inequalities (1) have a nice "folding" property. Namely, under appropriate symmetry assumptions on the functions u, they reduce to inequalities in lower dimensions of exactly the same type. We exemplify this fact by reducing the inequality for radial functions from dimension N to one dimension, i.e., from $C_{0,\mathrm{rad}}^{\infty}(\mathbb{R}^N \setminus \{0\})$ to $C_0^{\infty}(0,\infty)$.

When we work with the class of radial functions in $\mathbb{R}^N$, inequality (1) becomes

$$C^2(N,a,b)\left(\int_0^{\infty} r^{N-a-b-2} u^2\,dr\right)^2 \leqslant \left(\int_0^{\infty} r^{N-2a-1} u^2\,dr\right)\left(\int_0^{\infty} r^{N-2b-1} u_r^2\,dr\right).$$

And, when $N = 1$ this inequality reads

$$C^2(\hat{a},\hat{b})\left(\int_0^{\infty} \frac{u^2}{r^{\hat{a}+\hat{b}+1}}\,dr\right)^2 \leqslant \left(\int_0^{\infty} \frac{u^2}{r^{2\hat{a}}}\,dr\right)\left(\int_0^{\infty} \frac{u_r^2}{r^{2\hat{b}}}\,dr\right),$$

where we are using $C(\hat{a},\hat{b})$ instead of $C(1,\hat{a},\hat{b})$. Therefore, we have:

Remark 4. Regardless of the starting dimension N, the radial case (N,a,b) drops down to one dimension $(1,\hat{a},\hat{b})$, when we consider $\hat{a} = a - \frac{N-1}{2}$ and $\hat{b} = b - \frac{N-1}{2}$. Moreover, the best constant satisfies $C(N,a,b) = C(\hat{a},\hat{b})$.

3. Proof of Theorem 1 – The Euler–Lagrange equations

Proof. We have seen from Theorem 2 in Section 1 that, when $N = 1$, it holds

$$C^2(a,b)\left(\int_0^\infty \frac{u^2(r)}{r^{a+b+1}}\,dr\right)^2 \leqslant \left(\int_0^\infty \frac{u^2(r)}{r^{2a}}\,dr\right)\left(\int_0^\infty \frac{u_r^2(r)}{r^{2b}}\,dr\right), \tag{19}$$

where the best constant is given by

$$C(a,b) = \begin{cases} \frac{|a+b|}{2} & \text{if } (a,b) \in \mathcal{A}, \\ \frac{|3b-a+2|}{2} & \text{if } (a,b) \in \mathcal{B}. \end{cases} \tag{20}$$

Moreover, these optimal constants are achieved by the functions

$$u(r) = D\exp\left(\frac{t}{b+1-a} r^{b+1-a}\right) \tag{21}$$

and

$$u(r) = Dr^{2b+1}\exp\left(\frac{t}{b+1-a} r^{b+1-a}\right), \tag{22}$$

respectively, where $D \in \mathbb{R} \setminus \{0\}$ and t (of the right sign) are arbitrary. Therefore, Remark 4 above concludes the proof of Theorem 1. In other words, we have the inequality

$$C^2(N,a,b)\left(\int_{\mathbb{R}^N} \frac{u^2}{|x|^{a+b+1}}\,dx\right)^2 \leqslant \left(\int_{\mathbb{R}^N} \frac{u^2}{|x|^{2a}}\,dx\right)\left(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}}\,dx\right),$$

where the best constant is

$$C(N,a,b) = C(\hat{a},\hat{b}) = C\left(a - \frac{N-1}{2}, b - \frac{N-1}{2}\right)$$

and $C(\cdot,\cdot)$ is given in (20). In addition, $C(N,a,b)$ is achieved, respectively, by the functions in (21) and (22), with $r = |x|$. □

Remark 5. The case $a = b + 1$ was considered in [4], and it appears also in [5]. It is known that the best constant is $C(b+1,b) = \frac{|2b+1|}{2}$ and it is not achieved. The way this was shown in [4] was through the change of variables

$$u(r) = r^{-\frac{2b+1}{2}} v(-\ln r), \qquad t = -\ln r,$$

which is just the one-dimensional variant of (16). Indeed, one has

$$\int_0^\infty \frac{u_r^2(r)}{r^{2b}}\,dr = \int_{-\infty}^\infty \left[v_t^2(t) + \left(\frac{2b+1}{2}\right)^2 v^2(t)\right]dt$$

and

$$\int_0^\infty \frac{u^2(r)}{r^{2(b+1)}}\,dr = \int_{-\infty}^\infty v^2(t)\,dt.$$

Therefore, it follows that

$$C^2(b+1,b) = \inf \frac{\int_{-\infty}^\infty [v_t^2(t) + (\frac{2b+1}{2})^2 v^2(t)]\,dt}{\int_{-\infty}^\infty v^2(t)\,dt} = \left(\frac{2b+1}{2}\right)^2$$

and it is not achieved because the corresponding Euler–Lagrange equation

$$-v_{tt} + \left(\frac{2b+1}{2}\right)^2 v = \lambda v$$

has solution of exponential type, linear, or oscillatory, none of which is integrable.

Next, let us consider (cf. [6]) the space $H^1_{a,b}(\mathbb{R}^N)$ obtained from $C_0^\infty(\mathbb{R}^N \setminus \{0\})$ by completion under the weighted Sobolev norm

$$\|u\|_{H^1_{a,b}} := \left(\int_{\mathbb{R}^N} \left[\frac{|\nabla u|^2}{|x|^{2b}} + \frac{u^2}{|x|^{2a}}\right] dx\right)^{1/2}.$$

Recalling that the square of the energy functional is given by

$$E^2(u) = \frac{(\int_{\mathbb{R}^N} \frac{u^2}{|x|^{2a}}\,dx)(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}}\,dx)}{(\int_{\mathbb{R}^N} \frac{u^2}{|x|^{a+b+1}}\,dx)^2},$$

it is not hard to calculate its Euler–Lagrange equation, which is given by

$$-\mathrm{div}\Big(\frac{\nabla u}{|x|^{2b}}\Big) + K(u)\frac{u}{|x|^{2a}} - 2M(u)\frac{u}{|x|^{a+b+1}} = 0, \tag{23}$$

where

$$K(u) = \frac{\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}}\,dx}{\int_{\mathbb{R}^N} \frac{u^2}{|x|^{2a}}\,dx} \quad \text{and} \quad M(u) = \frac{\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}}\,dx}{\int_{\mathbb{R}^N} \frac{u^2}{|x|^{a+b+1}}\,dx}. \tag{24}$$

We should note that the coefficients $K(u)$ and $M(u)$ in (23) are homogeneous functionals of degree 0 and that (23) itself is homogeneous of degree 1 (so, if u is a solution so is αu for all $\alpha \in \mathbb{R}$). Also, $u \in H^1_{a,b}(\mathbb{R}^N)$ is a weak solution of (23) if and only if

$$\langle D(E^2)(u), \varphi\rangle := \frac{d}{dt}E^2(u+t\varphi)|_{t=0} = 0 \quad \forall \varphi \in H^1_{a,b}.$$

In particular, since we know from Theorem 2 that the (radial) functions u that achieve the best constants in (1) satisfy

$$\int_0^\infty \frac{u_r^2}{r^{2b}}\,dr < \infty, \qquad \int_0^\infty \frac{u^2}{r^{2a}}\,dr < \infty,$$

it follows any such function belongs to $H^1_{a,b}(\mathbb{R}^N)$ (recall Remark 4). Moreover, it is a standard fact that any such u is a weak solution of (23), i.e., one has the following lemma (for completeness we provide its easy proof):

Lemma 4. *Let $U \in H^1_{a,b}(\mathbb{R}^N)$ be a minimizer of $E(u)$. Then U is a weak solution of* (23).

Proof. Since U is a minimizer of $E(u)$, it is also a minimizer of $E^2(u)$, and so, for any given $\varphi \in H^1_{a,b}$, one has

$$E^2(U + t\varphi) \geqslant E^2(U) \quad \forall t \in \mathbb{R}.$$

Therefore, for $t > 0$ (resp. $t < 0$), it follows that

$$\frac{1}{t}\big[E^2(U + t\varphi) - E^2(U)\big] \geqslant 0 \quad (\text{resp.} \leqslant 0).$$

Letting $t \to 0$ gives

$$\big\langle D\big(E^2\big)(U), \varphi\big\rangle = 0,$$

that is, U is a weak solution of (23). □

4. Multiple solutions

In this section we construct multiple solutions for (23). Namely, we prove the following:

Theorem 4.

(a) *For any pair (a, b) with $a \neq b + 1$ Eq.* (23) *has infinitely many sign-changing solutions on discrete energy levels.*
(b) *For certain pairs (a, b), Eq.* (23) *has multiple* (*nontrivially distinct*) *solutions on the same energy level.*

Proof. For the proof we use again the transformation on the cylinder (16) that was used in Section 2.

Similarly to before, given $u \in C^\infty(\mathbb{R}^N \setminus \{0\}) \cap H^1_{a,b}(\mathbb{R}^N)$, there exist $f_k \in C^\infty(\mathbb{R}) \cap H^1(\mathbb{R})$ such that

$$v(\theta, t) = \sum_{k=0}^{\infty} f_k(t) Y_k(\theta). \tag{25}$$

Then

$$E(v)=\frac{\{\sum_{k=0}^{\infty}\int_{\mathbb{R}}e^{2(a-b-1)t}f_k^2(t)\,dt\}^{\frac{1}{2}}\{\sum_{k=0}^{\infty}\int_{\mathbb{R}}(f_k')^2(t)+[\lambda_k+(\frac{N-2-2b}{2})^2]f_k^2(t)\,dt\}^{\frac{1}{2}}}{\sum_{k=0}^{\infty}\int_{\mathbb{R}}e^{(a-b-1)t}f_k^2(t)\,dt}. \tag{26}$$

We therefore look at the equations

$$-f_k''+\left\{\left[\lambda_k+\left(\frac{N-2-2b}{2}\right)^2\right]+K(v)e^{2(a-b-1)t}-2M(v)e^{(a-b-1)t}\right\}f_k=0, \tag{27}$$

where, similarly to (24), we have

$$K(v)=\frac{\int_{\mathcal{C}}[|\nabla v|^2+(\frac{N-2-2b}{2})^2v^2]\,d\mu}{\int_{\mathcal{C}}e^{2(a-b-1)t}v^2\,d\mu} \tag{28}$$

and

$$M(v)=\frac{\int_{\mathcal{C}}[|\nabla v|^2+(\frac{N-2-2b}{2})^2v^2]\,d\mu}{\int_{\mathcal{C}}e^{(a-b-1)t}v^2\,d\mu}. \tag{29}$$

With these notations, we have

$$E(v)=M(v)/\sqrt{K(v)}.$$

Remark 6. In fact, only the ratio $E(v)=M(v)/\sqrt{K(v)}$ matters in (27), since the translation $t\to t-\frac{1}{2(a-b-1)}\ln K$ reduces (27) to

$$-f_k''+\left\{\left[\lambda_k+\left(\frac{N-2-2b}{2}\right)^2\right]+e^{2(a-b-1)t}-2E(v)e^{(a-b-1)t}\right\}f_k=0.$$

Note that K and M in (27) are non-local coefficients in that they depend on v. For the intermediate calculations we disregard this dependency and we will assume K and M to be two fixed positive constants. We then "assemble" the higher-energy solutions as in (25), sometimes using more than one Y_k. It is important to make sure that the constants K and M used to construct v are indeed related to v through (28) and (29). This will be checked (see Remark 6) in what follows.

Consider k a nonnegative integer, and K and M fixed positive constants. It is convenient to transform Eq. (27) into the Whittaker equation

$$w_{zz}+\left\{-\frac{1}{4}+\frac{\kappa}{z}+\frac{\frac{1}{4}-\mu^2}{z^2}\right\}w=0. \tag{30}$$

A change of variables realizing this transformation (using f instead of f_k) is

$$z=\frac{2\sqrt{K}}{|a-b-1|}e^{(a-b-1)t}\quad\text{and}\quad\frac{w(z)}{\sqrt{z}}=f(t), \tag{31}$$

where we denoted

$$\kappa = \frac{M/\sqrt{K}}{|a-b-1|} \quad \text{and} \quad \mu = \frac{\sqrt{\lambda_k + (\frac{N-2-2b}{2})^2}}{|a-b-1|}. \tag{32}$$

Assuming that w is a differentiable function on $(0, \infty)$ satisfying

$$\lim_{z\to 0^+} \frac{w^2(z)}{z} = \lim_{z\to\infty} \frac{w^2(z)}{z} = 0 \tag{33}$$

and that f and w are related through (31), it is not difficult to check that

$$\int_{\mathbb{R}} \left[(f')^2 + \left(\lambda_k + \left(\frac{N-2-2b}{2}\right)^2\right) f^2\right] dt = |a-b-1| \int_0^\infty \left[w_z^2 + \left(-\frac{1}{4} + \mu^2\right)\frac{1}{z^2} w^2\right] dz, \tag{34}$$

because

$$\begin{aligned} \int_{\mathbb{R}} (f')^2\, dt &= |a-b-1| \int_0^\infty \left[w_z^2 - z^{-1} w w_z + \frac{1}{4} z^{-2} w^2\right] dz \\ &= |a-b-1| \int_0^\infty \left[w_z^2 - \frac{d}{dz}\left(z^{-1}\frac{w^2}{2}\right) - \frac{1}{2} z^{-2} w^2 + \frac{1}{4} z^{-2} w^2\right] dz \\ &= |a-b-1| \int_0^\infty \left[w_z^2 - \frac{1}{4} z^{-2} w^2\right] dz, \end{aligned}$$

since the boundary terms vanish in view of (33). We also have

$$\int_{\mathbb{R}} e^{2(a-b-1)t} f^2\, dt = \frac{|a-b-1|}{4K} \int_0^\infty w^2\, dz \tag{35}$$

and

$$\int_{\mathbb{R}} e^{(a-b-1)t} f^2\, dt = \frac{1}{2\sqrt{K}} \int_0^\infty \frac{1}{z} w^2\, dz. \tag{36}$$

Note that if $v(\theta, t) = f_k(t) Y_k(\theta)$ (we call such a v a *single mode*), and f_k is obtained from w via (31), then we have from (34)–(36) that

$$E(v) = |a-b-1| \frac{(\int_0^\infty w^2\, dz)^{\frac{1}{2}} (\int_0^\infty [w_z^2 + (-\frac{1}{4} + \mu^2)\frac{1}{z^2} w^2]\, dz)^{\frac{1}{2}}}{\int_0^\infty \frac{1}{z} w^2\, dz}. \tag{37}$$

Moreover, if w is solution of (30) then, after multiplying the equation by w and integrating by parts, we obtain

$$\int_0^\infty \left[w_z^2 + \left(-\frac{1}{4} + \mu^2\right)\frac{1}{z^2}w^2\right] dz = \kappa \int_0^\infty \frac{1}{z} w^2\, dz - \frac{1}{4}\int_0^\infty w^2\, dz. \tag{38}$$

On the other hand, it is known (e.g. see [9]) that a solution of (30) is given by

$$\mathcal{M}_{\kappa,\mu}(z) = e^{-\frac{z}{2}} z^{\frac{1}{2}+\mu}\, {}_1F_1\left(\frac{1}{2} + \mu - \kappa; 1 + 2\mu; z\right),$$

where, for any β not a negative integer or zero, ${}_1F_1(\alpha; \beta; z)$ is the hypergeometric series

$${}_1F_1(\alpha; \beta; z) = 1 + \frac{\alpha}{1!\beta} z + \frac{\alpha(\alpha+1)}{2!\beta(\beta+1)} z^2 + \cdots + \frac{\alpha(\alpha+1)\cdots(\alpha+r-1)}{r!\beta(\beta+1)\cdots(\beta+r-1)} z^r + \cdots.$$

Let n be any nonnegative integer, and assume κ and μ are such that

$$\mu = \kappa - n - \frac{1}{2} > 0. \tag{39}$$

We then consider the solution of (30) given by

$$w(z) = \mathcal{M}_{\kappa,\kappa-n-\frac{1}{2}}(z) = e^{-\frac{z}{2}} z^{\kappa-n}\, {}_1F_1(-n; 2\kappa - 2n; z). \tag{40}$$

Note that, since $\alpha = -n$, the hypergeometric series ${}_1F_1(-n; 2\kappa - 2n; z)$ is a polynomial of degree n, so that w satisfies conditions (33). We now show that this w satisfies the equality

$$\int_0^\infty w^2\, dz = 2\kappa \int_0^\infty \frac{1}{z} w^2\, dz. \tag{41}$$

Indeed, we have

$$\begin{aligned}
\int_0^\infty w^2\, dz &= \int_0^\infty e^{-z} z^{2\kappa-2n}\, {}_1F_1^2(-n; 2\kappa-2n; z)\, dz \\
&= -\int_0^\infty \frac{d}{dz}\left(e^{-z} z^{2\kappa-2n}\, {}_1F_1^2(-n; 2\kappa-2n; z)\right) dz + 2\kappa \int_0^\infty \frac{1}{z} w^2\, dz \\
&\quad + \int_0^\infty e^{-z} z^{2\kappa} \frac{d}{dz}\left(z^{-n}\, {}_1F_1(-n; 2\kappa-2n; z)\right)^2 dz
\end{aligned}$$

and (41) will follow if we show that

$$\int_0^\infty e^{-z} z^{2\kappa} \frac{d}{dz}\big(z^{-n}{}_1F_1(-n; 2\kappa-2n; z)\big)^2\, dz = 0. \tag{42}$$

So, we need to show that

$$\int_0^\infty e^{-z} z^{2\kappa} \big(z^{-n}{}_1F_1(-n; 2\kappa-2n; z)\big) \frac{d}{dz}\big(z^{-n}{}_1F_1(-n; 2\kappa-2n; z)\big)\, dz = 0.$$

But it is straightforward to see that

$$\frac{d}{dz}\big(z^{-n}{}_1F_1(-n; 2\kappa-2n; z)\big) = -nz^{-(n+1)}{}_1F_1\big(-(n-1); 2\kappa-2n; z\big)$$

and, hence, (42) follows from the equality

$$\int_0^\infty e^{-z} z^{2\kappa-2n-1}{}_1F_1(-n; 2\kappa-2n; z){}_1F_1\big(-(n-1); 2\kappa-2n; z\big)\, dz = 0. \tag{43}$$

The above equality (43) is obtained from the more general fact that Laguerre polynomials $\{L_n^{\beta-1}(z)\}_n$ are orthogonal with respect to the gamma distribution $e^{-z}z^{\beta-1}\,dz$ for $\beta > 0$ (e.g. see [1], p. 282). The relevant relation is

$${}_1F_1(-n; \beta; z) = \frac{n!}{\beta(\beta+1)\cdots(\beta+n-1)} L_n^{\beta-1}(z),$$

where $\beta = 2\kappa - 2n$ in our case.

Next, having shown (41), we obtain from (38) and (41) that

$$\int_0^\infty \left[w_z^2 + \left(-\frac{1}{4} + \mu^2\right)\frac{1}{z^2}w^2\right] dz = \frac{\kappa}{2}\int_0^\infty \frac{1}{z}w^2\, dz. \tag{44}$$

And, by substituting (41) and (44) in (37), we get

$$E(v) = |a-b-1|\kappa.$$

Therefore, recalling (32), we obtain $E(v) = M/\sqrt{K}$, which verifies Remark 6.

We note that, whenever condition (39) holds, it follows by substitution from (32) that

$$E = \frac{M}{\sqrt{K}} = \sqrt{\lambda_k + \left(\frac{N-2-2b}{2}\right)^2} + |a-b-1|\left(n + \frac{1}{2}\right). \tag{45}$$

Radial solutions

For (a,b) fixed, let $n \in \{0,1,2,\ldots\}$ be arbitrary. Then, on the energy level

$$E_n = \left|\frac{N-2-2b}{2}\right| + |a-b-1|\left(n+\frac{1}{2}\right),$$

there is a radial solution $u(|x|)$ obtained from $v(\theta,t) = f(t)$ through (16) where, in turn, $f(t)$ is obtained from $w(z)$ given by (40) via (31).

Single mode nonradial solutions

More generally, for $\lambda_k > 0$ fixed, let $n \in \{0,1,2,\ldots\}$ be arbitrary. Then, on the energy level $E_{k,n}$ given by (45) there is a solution $u(x)$ obtained via (16) from the 1-mode solution $v(\theta,t) = Y_k(\theta) f_{k,n}(t)$.

Multiple mode solutions

Let a be such that, for different pairs (λ_k, n) and (λ_l, m), the energy level given by (45) satisfies $E = E_{k,n} = E_{l,m}$, that is, a is such that

$$|a-b-1| = \frac{1}{n-m}\left(\sqrt{\lambda_l + \left(\frac{N-2-2b}{2}\right)^2} - \sqrt{\lambda_k + \left(\frac{N-2-2b}{2}\right)^2}\right).$$

Then, there is a solution on the energy level E obtained from the 2-mode $v(\theta,t) = Y_k(\theta) f_{k,n}(t) + Y_l(\theta) f_{l,m}(t)$.

By the same token, if there are three pairs (λ_k, n), (λ_l, m) and (λ_j, p) such that

$$\begin{aligned}|a-b-1| &= \frac{1}{n-m}\left(\sqrt{\lambda_l + \left(\frac{N-2-2b}{2}\right)^2} - \sqrt{\lambda_k + \left(\frac{N-2-2b}{2}\right)^2}\right)\\ &= \frac{1}{p-n}\left(\sqrt{\lambda_k + \left(\frac{N-2-2b}{2}\right)^2} - \sqrt{\lambda_j + \left(\frac{N-2-2b}{2}\right)^2}\right),\end{aligned}$$

then there is a solution obtained from 3-mode solution $v(\theta,t) = Y_j(\theta) f_{j,p}(t) + Y_k(\theta) f_{k,n}(t) + Y_l(\theta) f_{l,m}(t)$.

Note that, in the cases $b = 0$ or $b = N-2$, one has $\lambda_k + (\frac{N-2-2b}{2})^2 = (k + \frac{N-2}{2})^2$, since $\lambda_k = k(k+N-2)$. Therefore, the above equality becomes

$$|a-b-1| = \frac{1}{n-m}(l-k) = \frac{1}{p-n}(k-j),$$

and if one is able to find distinct pairs $(k_1,n_1),\ldots,(k_r,n_r)$ such that $|a-b-1| = \frac{1}{n_i - n_{i+1}}(k_{i+1} - k_i)$ for all $i = 1,\ldots,r-1$, then an r-mode solution can be found at the corresponding energy level. □

In fact, we have the following (explicit) consequence of Theorem 4:

Corollary 1. *Let $b = 0$ (resp. $b = N-2$) and let $a \neq 1$ (resp. $a \neq N-1$) be an arbitrary rational number. Then, for any positive integer r, there exists a countable set of energy levels*

$$E = E_{k,n} = \left(k + \frac{N-2}{2}\right) + |a-b-1|\left(n + \frac{1}{2}\right),$$

with $k, n \to \infty$, such that, on the level $E_{k,n}$, there are r-mode solutions.

Proof. Given (a, b) as above, we write the positive rational number $|a-b-1|$ in irreducible form as

$$|a-b-1| = \frac{p}{q}.$$

In view of the preceding arguments, if we find distinct pairs $(k_1, n_1), \dots, (k_r, n_r)$ such that $|a-b-1| = \frac{1}{n_i - n_{i+1}}(k_{i+1} - k_i)$ for all $i = 1, \dots, r-1$, then an r-mode solution can be found at the corresponding energy level

$$E_{k_1,n_1} = \left(k_1 + \frac{N-2}{2}\right) + \frac{p}{q}\left(n_1 + \frac{1}{2}\right). \tag{46}$$

Then, the idea is to consider the linear Diophantine equation

$$xq - yp = 1 \tag{47}$$

and use the fact that, in view of our assumption of p and q being relatively prime, (47) has infinitely many solutions (x, y) given by

$$x = k + zp, \qquad y = n + zq,$$

with (k, n) being a fixed solution and $z \in \mathbb{Z}$ arbitrary.

For k and n fixed, the energy level $E = E_{\hat{k},\hat{n}}$ as in (46) will have at least

$$r = \left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor + 1 \tag{48}$$

solutions (as usual, $\lfloor x \rfloor$ denotes the largest integer less than, or equal to, x), with $\hat{k} = k + jp$, $\hat{n} = n - jq$, with $j \in \{-\lfloor \frac{k}{p} \rfloor, -\lfloor \frac{k}{p} \rfloor + 1, \dots, \lfloor \frac{n}{q} \rfloor\}$. Clearly, as k or n (or both) tend to infinity, $r \to \infty$ as follows from (48).

As an illustration, say p and q are fixed, and we want to construct r-mode solutions with $1 \leqslant r \leqslant 3$. For that, we can consider $k \geqslant p$ and $n \geqslant 2q$ and the energy level $E = E_{k-p,n}$ and the 3-mode solution $v(\theta, t) = Y_{k_1}(\theta) f_{k_1,n_1}(t) + Y_{k_2}(\theta) f_{k_2,n_2}(t) + Y_{k_3}(\theta) f_{k_3,n_3}(t)$, with

$$k_1 = k + p, \qquad k_2 = k, \qquad k_3 = k - p,$$
$$n_1 = n - 2q, \qquad n_2 = n - q, \qquad n_3 = n.$$

An important point to note is that the coefficients of p and q add up to the same constant.

And, for 1-mode and 2-mode solutions (at the same level $E = E_{k-p,n}$), we simply take anyone of the terms in the above $v(\theta, t)$, and the sum of any two such terms. $\square$

5. Concluding remarks and questions

Remark 7. This paper was inspired by the work in [6] after the present authors realized that, for (a, b) in $\mathcal{B}$, the value of the best constant obtained in [6] was not correct. As a result, we now obtained the best constant for all pairs (a, b) as well as the corresponding minimizers (except for $a = b + 1$, when the best constant is not achieved). In addition, we showed that all minimizers were one-signed, radial functions forming a 2-dimensional manifold parameterized by $D \in \mathbb{R} \setminus \{0\}$, $t \in (0, \infty)$ (resp. $t \in (-\infty, 0)$), and we constructed infinitely many other sign-changing, nonradial, higher-energy solutions of the Euler–Lagrange equation corresponding to the pertinent functional.

Remark 8. As we saw in Section 4, the approach we used to (explicitly) construct some of the multiple-mode solutions in the case $(a, b) \in \mathbb{Q} \times \{\hat{b}\}$ (with $\hat{b} = 0$ or $N - 2$, and $a \neq \hat{b} + 1$) involved solving a linear Diophantine equation. If we want to consider other values of $\hat{b}$, the same approach would lead to more complicated higher order Diophantine equations. Therefore, considering that the Euler–Lagrange equation (23) is "almost linear" (in the sense that it is homogeneous of degree 1), the question of describing/constructing *all* solutions of (23) (perhaps by a different method) would naturally arise. A possible difficulty may be the fact that Eq. (23) is non-local. As far as we are aware, there is not much done regarding *non-local* equations in $\mathbb{R}^N$.

In particular, it would be interesting to see whether there are solutions with infinitely many modes.

Remark 9. The general case of weighted-norm inequalities involving L^p-norms must be handled differently and constitutes work in progress [3] which will appear elsewhere.

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