J. Math. Anal. Appl. 371 (2010) 341–346

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Compactness of products of Hankel operators on the polydisk and some product domains in $\mathbb{C}^2$

Željko Čučković, Sönmez Şahutoğlu *

University of Toledo, Department of Mathematics, Toledo, OH 43606, USA

ARTICLE INFO

Article history:
Received 5 April 2010
Available online 15 May 2010
Submitted by J.H. Shapiro

Keywords:
Hankel operators
Berezin transform

ABSTRACT

Let $\mathbb{D}^n$ be the polydisk in $\mathbb{C}^n$ and the symbols $\phi, \psi \in C(\overline{\mathbb{D}^n})$ such that ϕ and ψ are pluriharmonic on any $(n-1)$-dimensional polydisk in the boundary of $\mathbb{D}^n$. Then $H^*_\psi H_\phi$ is compact on $A^2(\mathbb{D}^n)$ if and only if for every $1 \leqslant j, k \leqslant n$ such that $j \neq k$ and any $(n-1)$-dimensional polydisk D, orthogonal to the z_j-axis in the boundary of $\mathbb{D}^n$, either ϕ or ψ is holomorphic in z_k on D. Furthermore, we prove a different sufficient condition for compactness of the products of Hankel operators. In $\mathbb{C}^2$, our techniques can be used to get a necessary condition on some product domains involving annuli.

1. Introduction

In this paper we would like to understand how compactness of products of Hankel operators interacts with the behavior of the symbols on the boundary. We choose to work on the polydisk and some other product domains in $\mathbb{C}^2$. However, we believe that this approach could be useful on more general domains.

Let Ω be a domain in $\mathbb{C}^n$ and dV denote the Lebesgue volume measure on Ω. The Bergman space $A^2(\Omega)$ is the closed subspace of $L^2(\Omega, dV)$ consisting of all holomorphic functions on Ω. The Bergman projection P is the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$. For a function $\phi \in L^\infty(\Omega)$, the Toeplitz operator $T_\phi : A^2(\Omega) \to A^2(\Omega)$ is defined by $T_\phi = PM_\phi$ where M_ϕ is the multiplication operator by ϕ.

In their famous paper, Brown and Halmos [5] introduced Toeplitz operators on the Hardy space on the unit disk $\mathbb{D}$ of the complex plane and discovered the most fundamental algebraic properties of these operators. The corresponding questions for the Bergman space remained elusive for several decades. In 1991, Axler and the first author [1] characterized commuting Toeplitz operators with harmonic symbols on $\mathbb{D}$ and thus obtained an analogue of the corresponding theorem of Brown and Halmos. In 2001, Ahern and the first author [2] studied when a product of two Toeplitz operators is equal to another Toeplitz operator. They considered bounded harmonic functions ϕ and ψ, and a bounded C^2-symbol ξ with bounded invariant Laplacian. Their main result is that $T_\phi T_\psi = T_\xi$ if and only if ϕ is conjugate holomorphic or ψ is holomorphic. Later Ahern [3] removed the assumption on ξ and assumed that $\xi \in L^\infty(\mathbb{D})$ only. One of the consequences of the main result in [2] is that the semicommutator of Toeplitz operator, $T_\phi T_\psi - T_{\phi\psi} = 0$, only in trivial cases. This result was obtained earlier by Zheng [13], using different methods. In fact, Zheng characterized compact semicommutators of Toeplitz operators with harmonic symbols on the unit disk. If $\phi = \phi_1 + \overline{\phi}_2$ and $\psi = \psi_1 + \overline{\psi}_2$ are bounded and harmonic on $\mathbb{D}$, where ϕ_1, ϕ_2, ψ_1, and ψ_2 are holomorphic, then compactness of $T_\phi T_\psi - T_{\phi\psi}$ is equivalent to the condition

$$\lim_{|z|\to 1} \min\{(1-|z|^2)|\phi_1'(z)|, (1-|z|^2)|\psi_2'(z)|\} = 0.$$

* Corresponding author.
E-mail addresses: zcuckovi@math.utoledo.edu (Ž. Čučković), sonmez.sahutoglu@utoledo.edu (S. Şahutoğlu).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.05.027

Later several authors [11,6] extended this result to the Bergman space of the polydisk $\mathbb{D}^n$ and in 2007, Choe, Lee, Nam, and Zheng [8] found characterizations of compactness of $T_\phi T_\psi - T_\xi$ on the polydisk, thus extending Ahern's result.

A semicommutator of two Toeplitz operators can be expressed in terms of Hankel operators. For $\phi \in L^\infty(\Omega)$, the Hankel operator $H_\phi : A^2(\Omega) \to A^2(\Omega)^\perp$ is defined by $H_\phi = (I-P)M_\phi$. The following relation between Toeplitz operators and Hankel operators is well known:

$$T_{\bar{\phi}\psi} - T_{\bar{\phi}} T_\psi = H^*_\phi H_\psi.$$

Thus the semicommutator can be expressed as a product of an adjoint of a Hankel operator with another Hankel operator. Our approach is also motivated by our previous paper [9] in which we studied compactness of one Hankel operator on pseudoconvex domains in $\mathbb{C}^n$ in terms of the behavior of the symbol of the operator on disks in the boundary. Thus, when faced with the product of two Hankel operators, we are interested in finding how compactness of $H^*_\phi H_\psi$ interacts with the behavior of ϕ and ψ on the boundary of the domain.

We finish the introduction by listing our results. Let $\xi \in \overline{\mathbb{D}}$ and

$$D(\xi, j) = \{(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n) \in \mathbb{D}^n\colon\ z_j = \xi\}.$$

Theorem 1. *Let $\mathbb{D}^n$ be the polydisk in $\mathbb{C}^n$, $n \geqslant 2$, and the symbols $\phi, \psi \in C(\overline{\mathbb{D}^n})$ such that $\phi|_{D(\xi,j)}$ and $\psi|_{D(\xi,j)}$ are pluriharmonic for all $1 \leqslant j \leqslant n$ and all $|\xi| = 1$. Then $H^*_\psi H_\phi$ is compact on $A^2(\mathbb{D}^n)$ if and only if for any $1 \leqslant j, k \leqslant n$ such that $j \neq k$ and $|\xi| = 1$, either $\phi|_{D(\xi,j)}$ or $\psi|_{D(\xi,j)}$ is holomorphic in z_k on $D(\xi, j)$.*

In $\mathbb{C}^2$ the above theorem immediately implies the following corollary.

Corollary 1. *Let $\mathbb{D}^2$ be the bidisk in $\mathbb{C}^2$ and the symbols $\phi, \psi \in C(\overline{\mathbb{D}^2})$ such that $\phi \circ g$ and $\psi \circ g$ are harmonic for all holomorphic $g : \mathbb{D} \to \partial\mathbb{D}^2$. Then $H^*_\psi H_\phi$ is compact on $A^2(\mathbb{D}^2)$ if and only if for any holomorphic function $g : \mathbb{D} \to \partial\mathbb{D}^2$, either $\phi \circ g$ or $\psi \circ g$ is holomorphic.*

Remark 1. Let $\phi(z_1, z_2) = \chi_1(z_1, z_2) + z_1\bar{z}_2$ and $\psi(z_1, z_2) = \chi_2(z_1, z_2) + \bar{z}_1 z_2$ where $\chi_1, \chi_2 \in C^\infty_0(\mathbb{D}^2)$. Then ϕ and ψ are smooth functions but their restrictions on $\partial\mathbb{D}^2$ cannot be extended onto $\overline{\mathbb{D}^2}$ as pluriharmonic functions. So unlike the results in [11,8], Theorem 1 applies to such symbols and provides many examples of (nonzero) compact products of Hankel operators. Hence our result generalizes the previously mentioned results in the sense that our symbols do not have to be pluriharmonic on $\mathbb{D}^n$. On the other hand, we require the symbols to be continuous up to the boundary.

In fact our method can be used to remove the plurihamonicity condition on the symbols when proving the sufficiency, if we are willing to assume more about the symbols.

Theorem 2. *Let $\mathbb{D}^n$ be the polydisk in $\mathbb{C}^n$, $n \geqslant 2$, and the symbols $\phi, \psi \in C^1(\overline{\mathbb{D}^n})$. Assume that for any holomorphic function $g : \mathbb{D} \to \partial\mathbb{D}^n$, either $\phi \circ g$ or $\psi \circ g$ is holomorphic. Then $H^*_\psi H_\phi$ is compact on $A^2(\mathbb{D}^n)$.*

We also would like to note that the sufficient condition in Theorem 2 is not necessary. For example, Theorem 1 implies that $H^*_\phi H_\psi$ is compact on $A^2(\mathbb{D}^3)$ for $\phi(z_1, z_2, z_3) = \bar{z}_1 z_2$ and $\psi(z_1, z_2, z_3) = z_1 \bar{z}_2$. However, $\phi(\xi, \xi, z_3) = \psi(\xi, \xi, z_3) = |\xi|^2$ is not holomorphic.

Our technique can also be applied to some other product domains.

Theorem 3. *Let $\Omega = U \times V \subset \mathbb{C}^2$ where U and V are annuli or disks in $\mathbb{C}$, and the symbols $\phi, \psi \in C(\overline{\Omega})$. Assume that the restrictions of ϕ and ψ on any disk or annulus in the boundary of Ω are of the form $f + \overline{g}$, where f and g are holomorphic and continuous up to the boundary. If $H^*_\psi H_\phi$ is compact on $A^2(\Omega)$ then for any holomorphic function $g : \mathbb{D} \to \partial\Omega$ either $\phi \circ g$ or $\psi \circ g$ is holomorphic.*

Commutators of Toeplitz operators are connected to products of Hankel operators as follows:

$$[T_\phi, T_\psi] = H^*_{\bar{\phi}} H_\psi - H^*_{\bar{\psi}} H_\phi.$$

Hence, Theorem 2 implies the following corollary.

Corollary 2. *Let $\mathbb{D}^n$ be the polydisk in $\mathbb{C}^n$ and the symbols $\phi, \psi \in C^1(\overline{\mathbb{D}^n})$ be nonconstant. Assume that for any holomorphic function $g : \mathbb{D} \to \partial\mathbb{D}^n$, either $\phi \circ g$ and $\psi \circ g$ are holomorphic or $\overline{\phi} \circ g$ and $\overline{\psi} \circ g$ are holomorphic. Then $[T_\phi, T_\psi]$ is compact on $A^2(\mathbb{D}^n)$.*

2. Proof of Theorems 1 and 2

One of the important tools we need is the Berezin transform of an integrable function f on the polydisk in $\mathbb{C}^n$ which is defined as $B(f)(z) = \int_{\mathbb{D}^n} f(w)|k_z^n(w)|^2\,dV(w)$. Here $k_z^n(w)$ denotes the normalized Bergman kernel of $\mathbb{D}^n$. More generally, the Berezin transform of a bounded operator T is defined as $B(T)(z) = \langle Tk_z^n, k_z^n\rangle_{L^2(\mathbb{D}^n)}$.

Proof of Theorem 1. We will use the fact that if an operator T is compact then $\langle Tf_j, f_j\rangle_{L^2(\mathbb{D}^n)}$ converges to zero whenever $\{f_j\}$ converges to zero weakly. Let us assume that $H_\psi^* H_\phi$ is compact and $\phi|_{D(z_0,j)}$ and $\psi|_{D(z_0,j)}$ are pluriharmonic for all $1 \leqslant j \leqslant n$ and $|z_0| = 1$. Without loss of generality let us choose $j = n$ and let us denote $z = (z', z_n)$ where $z' = (z_1, \dots, z_{n-1})$ and define $\phi_0(z) = \phi(z) - \phi(z', z_0)$, $\psi_0(z) = \psi(z) - \psi(z', z_0)$, and denote $\psi_{z_0}(z) = \psi_1(z') = \psi(z', z_0)$ and $\phi_{z_0}(z) = \phi_1(z') = \phi(z', z_0)$. Let us fix $F \in A^2(\mathbb{D}^{n-1})$ with $\|F\|_{L^2(\mathbb{D}^{n-1})} \leqslant 1$ and choose a sequence $\{p_j\} \subset \mathbb{D}$ such that $p_j \to z_0$. Now we define $f_j(z) = F(z')k_{p_j}(z_n)$ where k_{p_j} is the normalized Bergman kernel for $\mathbb{D}$ at p_j. We note that $\phi_0(z', z_0) = \psi_0(z', z_0) = 0$ for all $z' \in \mathbb{D}^{n-1}$ and for all $\delta > 0$ the sequence $\{\|f_j\|_{L^2(\mathbb{D}^n \setminus \mathbb{D}^n_{z_0,\delta})}\}$ converges to zero, where $\mathbb{D}^n_{z_0,\delta} = \{z \in \mathbb{D}^n\colon |z_n - z_0| < \delta\}$. Then for $\delta > 0$ one can show that

$$\begin{aligned}\|\phi_0 f_j\|^2_{L^2(\mathbb{D}^n)} + \|\psi_0 f_j\|^2_{L^2(\mathbb{D}^n)} \leqslant{}& \sup\bigl\{\bigl|\phi_0(z)\bigr|^2\colon z \in \mathbb{D}^n_{z_0,\delta}\bigr\}\|f_j\|^2_{L^2(\mathbb{D}^n)} + \sup\bigl\{\bigl|\psi_0(z)\bigr|^2\colon z \in \mathbb{D}^n_{z_0,\delta}\bigr\}\|f_j\|^2_{L^2(\mathbb{D}^n)}\\ &+ \sup\bigl\{\bigl|\phi_0(z)\bigr|^2\colon z \in \overline{\mathbb{D}^n}\bigr\}\|f_j\|^2_{L^2(\mathbb{D}^n\setminus\mathbb{D}^n_{z_0,\delta})} + \sup\bigl\{\bigl|\psi_0(z)\bigr|^2\colon z \in \overline{\mathbb{D}^n}\bigr\}\|f_j\|^2_{L^2(\mathbb{D}^n\setminus\mathbb{D}^n_{z_0,\delta})}.\end{aligned}$$

For any $\varepsilon > 0$ we can choose $\delta > 0$ so that

$$\sup\bigl\{\bigl|\phi_0(z)\bigr|^2\colon z \in \mathbb{D}^n_{z_0,\delta}\bigr\} + \sup\bigl\{\bigl|\psi_0(z)\bigr|^2\colon z \in \mathbb{D}^n_{z_0,\delta}\bigr\} < \varepsilon/2.$$

Furthermore, we can choose $j_{\varepsilon,\delta}$ so that

$$\|f_j\|^2_{L^2(\mathbb{D}^n\setminus\mathbb{D}^n_{z_0,\delta})} < \frac{\varepsilon}{2\sup\{|\phi_0(z)|^2\colon z \in \overline{\mathbb{D}^n}\} + 2\sup\{|\psi_0(z)|^2\colon z \in \overline{\mathbb{D}^n}\} + 1}$$

for all $j \geqslant j_{\varepsilon,\delta}$. Combining the above inequalities with the fact that $\|f_j\|_{L^2(\mathbb{D}^n)} \leqslant 1$ we get $\|\phi_0 f_j\|^2_{L^2(\mathbb{D}^n)} + \|\psi_0 f_j\|^2_{L^2(\mathbb{D}^n)} < \varepsilon$ for $j \geqslant j_{\varepsilon,\delta}$. This implies that

$$\bigl\|H_{\phi_0}(f_j)\bigr\|_{L^2(\mathbb{D}^n)} + \bigl\|H_{\psi_0}(f_j)\bigr\|_{L^2(\mathbb{D}^n)} \to 0 \quad \text{as } j \to \infty.$$

The above statement together with the assumption that $H_\psi^* H_\phi$ is compact and $H_\phi = H_{\phi_{z_0}} + H_{\phi_0}$ and $H_\psi = H_{\psi_{z_0}} + H_{\psi_0}$ imply that $\langle H_{\phi_{z_0}}(f_j), H_{\psi_{z_0}}(f_j)\rangle_{L^2(\mathbb{D}^n)}$ converges to zero. Using the fact that $\mathbb{D}^n$ is the polydisk and the function ϕ_{z_0} depends only on z' one can show that $H_{\phi_{z_0}}(f_j)(z) = H_{\phi_1}(F)(z')k_{p_j}(z_n)$ and

$$\bigl\langle H_{\phi_{z_0}}(f_j), H_{\psi_{z_0}} f_j\bigr\rangle_{L^2(\mathbb{D}^n)} = \bigl\langle H_{\phi_1}(F), H_{\psi_1}(F)\bigr\rangle_{L^2(\mathbb{D}^{n-1})}\|k_{p_j}\|^2_{L^2(\mathbb{D})}.$$

Then compactness of $H_\psi^* H_\phi$ implies that $\langle H_{\phi_1}(F), H_{\psi_1}(F)\rangle_{L^2(\mathbb{D}^{n-1})} = 0$ for all $F \in A^2(\mathbb{D}^{n-1})$. Now Theorem 2.3 in [11] implies that for any $1 \leqslant k \leqslant n-1$ either ϕ_1 or ψ_1 is holomorphic in z_k. Therefore, for any $1 \leqslant k \leqslant n-1$ either ϕ or ψ is holomorphic in z_k.

To prove the other direction of the theorem, let q be a boundary point of $\partial\mathbb{D}^n$ and $k^n_{q_j}$ denote the normalized Bergman kernel of $\mathbb{D}^n$ centered at $q_j \in \mathbb{D}^n$ where $q_j \to q$. First, we will show that $\langle H_\phi k^n_{q_j}, H_\psi k^n_{q_j}\rangle_{L^2(\mathbb{D}^n)}$ converges to zero. Then we will use the fact ([4,12], see also [7, Theorem 2.1]) that $H_\psi^* H_\phi$ is compact if and only if $B(H_\psi^* H_\phi) \in C_0(\mathbb{D}^n)$ where $C_0(\Omega)$ denotes the class of functions that are continuous on Ω and have zero boundary limits. It is easy to see that $B(H_\psi^* H_\phi) \in C(\mathbb{D}^n)$. There exists $1 \leqslant j \leqslant n$ and $\xi \in \mathbb{C}$ such that $|\xi| = 1$ and $q \in D(\xi, j)$. We extend $\psi|_{D(\xi,j)}$ and $\phi|_{D(\xi,j)}$ trivially in z_j so that the extensions, ψ_1 and ϕ_1, are independent of z_j variable and are continuous up to the boundary of $\mathbb{D}^n$. Let us define $\phi_0 = \phi - \phi_1$ and $\psi_0 = \psi - \psi_1$. Then $\phi_0 = \psi_0 = 0$ on $D(\xi, j)$ and, as is done in the first part of this proof, one can show that both sequences $\{H_{\phi_0} k_{q_j}\}$ and $\{H_{\psi_0} k_{q_j}\}$ converge to zero. Since ϕ_1 and ψ_1 are pluriharmonic on $\mathbb{D}^n$, continuous up to the boundary, and for each variable either ϕ_1 or ψ_1 is holomorphic, Theorem 3.2 in [11] implies that $H^*_{\psi_1} H_{\phi_1} = 0$. Therefore, $H_\psi^* H_\phi$ is compact. □

In order to prove Theorem 2 we need the following lemma.

Lemma 1. *Let U be a domain in $\mathbb{C}^n$ and the functions $\phi, \psi \in C^1(U)$ are such that for any holomorphic function $g : \mathbb{D} \to U$ either $\phi \circ g$ or $\psi \circ g$ is holomorphic. Then either ϕ or ψ is holomorphic on U.*

Proof. Let $p, q \in U$ such that $\bar\partial\phi(p) \neq 0$ and $\bar\partial\psi(q) \neq 0$. Assume that $p \neq q$. Let $\varepsilon > 0$ and $\gamma : [0,1] \to U$ be a curve so that $\gamma(0) = p$, $\gamma(1) = q$, and

$$\bigl\{z \in \mathbb{C}^n\colon \operatorname{dist}(z, \gamma) < \varepsilon\bigr\} \subset U$$

where dist denotes the Euclidean distance. Using Stone–Weierstrass Theorem we choose a complex-valued (real) polynomial $P:\mathbb{R}\to\mathbb{C}^n$ so that $|P(x)-\gamma(x)|<\varepsilon/4$ for all $x\in[0,1]$. Let us define

$$f(x)=P(x)+x\big(q-P(1)\big)+(1-x)\big(p-P(0)\big).$$

The function f has a holomorphic extension to $\mathbb{C}$ and we will denote the extension by f as well. Hence, $f:\mathbb{C}\to\mathbb{C}^n$ is holomorphic such that $f(0)=p$, $f(1)=q$, and $f(z)\subset U$ for $z\in L=\{z\in\mathbb{R}\colon\ 0\leqslant z\leqslant 1\}$.

Let $\{e_1,e_2,\dots,e_n\}$ denote the standard basis in $\mathbb{C}^n$, and define $E_j=e_j$ for $1\leqslant j\leqslant n$ and $E_{n+j}=\sum_{k=1}^n k^{j-1}e_k$ for $1\leqslant j\leqslant n-1$. Using Vandermonde matrix one can show that the set $\{E_{j_1},E_{j_2},\dots,E_{j_n}\}$ is linearly independent for any $1\leqslant j_1<j_2<\dots<j_n\leqslant 2n-1$.

Let $M>0$ and define

$$g_{j,M}(z)=f(z)+\frac{z(z-1)}{M}E_j.$$

Let us fix $M>0$ large enough so that $g_{j,M}(z)\in U$ for $z\in L$. Then there exists a simply connected neighborhood V of L such that $g_{j,M}(z)\in U$ for $z\in V$. We choose a conformal mapping $h:\mathbb{D}\to V$ and define $g_j=g_{j,M}\circ h$. Then $g_j:\mathbb{D}\to U$ for $1\leqslant j\leqslant 2n-1$, and the sets $\{g'_{j_1}(z_0),g'_{j_2}(z_0),\dots,g'_{j_n}(z_0)\}$ and $\{g'_{j_1}(z_1),g'_{j_2}(z_1),\dots,g'_{j_n}(z_1)\}$ are linearly independent for $h(z_0)=0$, $h(z_1)=1$ and any $1\leqslant j_1<j_2<\dots<j_n\leqslant 2n-1$. Since for any j, either $\phi\circ g_j$ or $\psi\circ g_j$ is holomorphic, there exist $1\leqslant j_1<j_2<\dots<j_n\leqslant 2n-1$ such that either $\phi\circ g_{j_k}$ is holomorphic for $1\leqslant k\leqslant n$ or $\psi\circ g_{j_k}$ is holomorphic for $1\leqslant k\leqslant n$. Furthermore, using the chain rule together with linear independence of $\{g'_{j_k}(z_0)\colon\ 1\leqslant k\leqslant n\}$ and $\{g'_{j_k}(z_1)\colon\ 1\leqslant k\leqslant n\}$ one can show that either $\bar\partial\phi(p)=\bar\partial\phi(q)=0$ or $\bar\partial\psi(p)=\bar\partial\psi(q)=0$.

If $p=q$ then one can use affine disks along E_j's to show that either $\bar\partial\phi(p)=0$ or $\bar\partial\psi(p)=0$. Hence, we reached a contradiction completing the proof. □

Proof of Theorem 2. We will use Lemma 1 together with the ideas in the second part of the proof of Theorem 1. For any $|\xi|=1$ and $1\leqslant j\leqslant n$ we decompose the symbols as $\phi=\phi_0+\phi_1$ and $\psi=\psi_0+\psi_1$ such that

(i) $\phi_0=\psi_0=0$ on $D(\xi,j)$,
(ii) $\phi_1|_{D(\xi,j)}=\phi|_{D(\xi,j)}$, $\psi_1|_{D(\xi,j)}=\psi|_{D(\xi,j)}$,
(iii) ϕ_1 and ψ_1 are continuous on $\overline{\mathbb{D}^n}$,
(iv) either ϕ_1 or ψ_1 is holomorphic on $\mathbb{D}^n$.

Then either $H_{\phi_1}=0$ or $H_{\psi_1}=0$ and both sequences $\{H_{\phi_0}k_{q_j}\}$ and $\{H_{\psi_0}k_{q_j}\}$ converge to 0 in $L^2(\mathbb{D}^n)$ for $q_j\to q\in D(\xi,j)$. Hence, $B(H^*_\psi H_\phi)\in C_0(\mathbb{D}^n)$ and in turn this implies that $H^*_\psi H_\phi$ is compact. □

3. Proof of Theorem 3

To prove Theorem 3 one reduces the problem onto U or V as in the first part of the proof of Theorem 1. Then if the problem is reduced onto an annulus one uses the following proposition instead of Ding and Tang's Theorem.

Proposition 1. *Let $\mathcal{A}=\{z\in\mathbb{C}\colon\ 0<r<|z|<R\}$ and ϕ and ψ be holomorphic on $\mathcal{A}$ and continuous on $\overline{\Omega}$. Assume that $B(\overline{\psi}\phi)=\overline{\psi}\phi$. Then either ϕ or ψ is constant.*

Proof. Let us assume that $B(\overline{\psi}\phi)=\overline{\psi}\phi$. Then by a result of Čučković [10, Theorem 9] $B(\overline{\psi}\phi)=\overline{\psi}\phi$ implies that

$$\overline{\psi}\phi=R(\overline{\psi}\phi)+h \tag{1}$$

where h is a harmonic function and the radialization operator R is defined as $R(k)(z)=(2\pi)^{-1}\int_0^{2\pi}k(ze^{i\theta})\,d\theta$. If we apply the Laplacian to (1) we get $\overline{\psi}'\phi'=R(\Delta(\overline{\psi}\phi))$. Hence, $\overline{\psi}'\phi'$ is radial. Let $\phi'(z)=\sum_{n=-\infty}^{\infty}a_nz^n$ and $\psi'(z)=\sum_{m=-\infty}^{\infty}b_mz^m$. Then, on one hand

$$\overline{\psi}'(z)\phi'(z)=\sum_{n,m=-\infty}^{\infty}a_n\bar b_m r^{n+m}e^{i(n-m)\xi}$$

where $z=re^{i\xi}$. On the other hand, since $\overline{\psi}'\phi'$ is a radial function we get

$$\overline{\psi}'(z)\phi'(z)=\frac{1}{2\pi}\int_0^{2\pi}\overline{\psi}'\big(ze^{i\theta}\big)\phi'\big(ze^{i\theta}\big)\,d\theta$$

$$= \frac{1}{2\pi} \sum_{n,m=-\infty}^{\infty} a_n \bar{b}_m r^{n+m} e^{i(n-m)\xi} \int_0^{2\pi} e^{i(n-m)\theta}\, d\theta$$
$$= \sum_{n=-\infty}^{\infty} \frac{a_n \bar{b}_n r^{2n}}{2\pi}.$$

Hence, $\sum_{n\neq m} a_n \bar{b}_m r^{n+m} e^{i(n-m)\xi} = 0$ for all ξ. We can rewrite the last equation as

$$\sum_{k\neq 0} \left(\sum_{m=-\infty}^{\infty} a_{m+k} \bar{b}_m r^{2m+k} \right) e^{ik\xi} = 0.$$

This is a Fourier series that is equal to zero. Hence $\sum_{m=-\infty}^{\infty} a_{m+k} \bar{b}_m r^{2m+k} = 0$ for all $k \neq 0$. This is a Laurent series that is equal to zero. Therefore, $a_{m+k}\bar{b}_m = 0$ for all $k \neq 0$ and all m. In return this implies that if $b_{m_0} \neq 0$ then $a_m = 0$ for $m \neq m_0$. That is, either there exists an integer m and two nonzero constants a and b such that $\phi(z) = az^m$ and $\psi(z) = bz^m$ or either ϕ and ψ is constant. Next we will show that in the first case $m = 0$. Recall that the Bergman kernel for the annulus $\{z \in \mathbb{C}\colon\ \rho < |z| < 1\}$ is

$$K_w(z) = \frac{1}{\pi} \sum_{-\infty}^{\infty} \frac{n+1}{1-\rho^{2n+2}} (\overline{w} z)^n - \frac{1}{2\pi \ln \rho} (\overline{w} z)^{-1}.$$

Without loss of generality we may assume that $R = 1$ and $r = \rho < 1$ for a fixed m we have $B(|z|^{2m})(w) = |w|^{2m}$. Then

$$|w|^{2m} \|K_w\|^2 = \int_\rho^1 \int_0^{2\pi} r^{2m+1} \left| K_w\left(re^{i\theta}\right)\right|^2 d\theta\, dr.$$

The last equation can be expanded as

$$-\frac{|w|^{2m-2}}{2\pi \ln \rho} + \sum_{n\neq -1} \frac{(n+1)|w|^{2(m+n)}}{1-\rho^{2n+2}} = \frac{\pi(1-\rho^{2m})}{|w|^2 m (2\pi \ln \rho)^2 |w|^2} + \sum_{n\neq -1} \frac{(n+1)^2 |w|^{2n}}{(1-\rho^{2n+2})^2} \frac{(1-\rho^{2(n+m+1)})}{2(n+m+1)}.$$

Now let $k = m + n$ then $n = k - m$ and the last equation becomes

$$-\frac{|w|^{2m-2}}{2\pi \ln \rho} + \sum_{k\neq m-1} \frac{(k-m+1)|w|^{2k}}{1-\rho^{2(k-m+1)}} = \frac{\pi(1-\rho^{2m})}{|w|^2 m (2\pi \ln \rho)^2 |w|^2} + \sum_{n\neq -1} \frac{(n+1)^2 |w|^{2n}}{(1-\rho^{2n+2})^2} \frac{(1-\rho^{2(n+m+1)})}{2(n+m+1)}.$$

By equating the coefficients of each term we get

$$\frac{n-m+1}{1-\rho^{2(n-m+1)}} = \frac{(n+1)^2 (1-\rho^{2(n+m+1)})}{(n+m+1)(1-\rho^{2(n+1)})^2}$$

for $n \neq -1$ and $n \neq m-1$. Let $l = n+1$ and $\xi = \rho^2$. Then the last equation turns into

$$\frac{l^2 - m^2}{l^2} = \frac{(1-\xi^{l+m})(1-\xi^{l-m})}{(1-\xi^l)^2} \tag{2}$$

for $l \neq 0$ and $l \neq m$. From now on we will choose $l > m$. Let us define the following function

$$f_l(x) = \frac{(1-\xi^{l+x})(1-\xi^{l-x})}{l^2 - x^2}.$$

One can show that f_l is an even, nonnegative function defined on $(-l, l)$ and (2) implies that $f_l(0) = f_l(m)$. Then using the logarithmic differentiation we get

$$\left(l^2 - x^2\right) \frac{f_l'(x)}{f_l(x)} = \left(l^2 - x^2\right) \xi^l \ln \xi \left(\frac{\xi^{-x}}{1-\xi^{l-x}} - \frac{\xi^x}{1-\xi^{l+x}} \right) + 2x$$
$$= \left(l^2 - x^2\right) \xi^l \ln \xi \left(\frac{\xi^{-x} - \xi^x}{(\xi^x - \xi^l)(\xi^{-x} - \xi^l)} \right) + 2x. \tag{3}$$

Power series expansions for ξ^x and ξ^{-x} imply that

$$\xi^{-x}-\xi^{x}=-2\ln\xi\sum_{j=0}^{\infty}\frac{(\ln\xi)^{2j}}{(2j+1)!}x^{2j+1}.$$

Then there exists $0<\delta<l$ so that $|\xi^{-x}-\xi^{x}|\leqslant -3\ln\xi|x|$ for $|x|\leqslant\delta$. Now we use estimate $(\xi^{\delta}-\xi^{l})^2\leqslant(\xi^{x}-\xi^{l})(\xi^{-x}-\xi^{l})$ for $0<x\leqslant\delta$ to get

$$\left(l^2-x^2\right)\frac{f_l'(x)}{f_l(x)}\geqslant\left(\frac{-6(\ln\xi)^2(l^2-\delta^2)\xi^l}{(\xi^{\delta}-\xi^{l})^2}+2\right)x\quad\text{for } 0\leqslant x\leqslant\delta.$$

Then since $0<\xi<1$ there exists $l_0>4m$ so that $(l^2-x^2)\frac{f_l'(x)}{f_l(x)}\geqslant\frac{x}{2}$ for $l\geqslant l_0$ and $0\leqslant x\leqslant\delta$. Hence f_l are increasing functions on $[0,\delta]$ for all $l\geqslant l_0$. On the other hand for $\delta\leqslant x\leqslant m$ there exists $l_1\geqslant 4m$ such that $l\geqslant l_1$ implies that

$$\left|\left(l^2-x^2\right)\xi^l\ln\xi\left(\frac{\xi^{-x}}{1-\xi^{l-x}}-\frac{\xi^{x}}{1-\xi^{l+x}}\right)\right|\leqslant\delta.$$

Then (3) implies that $f_l'>0$ on $[\delta,m]$ for $l\geqslant l_1$. Therefore, f_l are increasing functions on $[0,m]$ and $f_l(m)>0$ for $l\geqslant\max\{l_0,l_1\}>4m$. □

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