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# Cauchon diagrams for quantized enveloping algebras

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ABSTRACT

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra, $\mathbb{K}$ a commutative field and $q$ a nonzero element of $\mathbb{K}$ which is not a root of unity. To each reduced decomposition of the longest element $w_0$ of the Weyl group $W$ corresponds a PBW basis of the quantised enveloping algebra $\mathcal{U}_q^+(\mathfrak{g})$, and one can apply the theory of deleting-derivation to this iterated Ore extension. In particular, for each decomposition of $w_0$, this theory constructs a bijection between the set of prime ideals in $\mathcal{U}_q^+(\mathfrak{g})$ that are invariant under a natural torus action and certain combinatorial objects called Cauchon diagrams. In this paper, we give an algorithmic description of these Cauchon diagrams when the chosen reduced decomposition of $w_0$ corresponds to a good ordering (in the sense of Lusztig (1990) [Lus90]) of the set of positive roots. This algorithmic description is based on the constraints that are coming from Lusztig's admissible planes Lusztig (1990) [Lus90]: each admissible plane leads to a set of constraints that a diagram has to satisfy to be Cauchon. Moreover, we explicitly describe the set of Cauchon diagrams for explicit reduced decomposition of $w_0$ in each possible type. In any case, we check that the number of Cauchon diagrams is always equal to the cardinal of $W$. In Cauchon and Mériaux (2008) [CM08], we use these results to prove that Cauchon diagrams correspond canonically to the positive subexpressions of $w_0$. So the results of this paper also give an algorithmic description of the positive subexpressions of any reduced decomposition of $w_0$ corresponding to a good ordering.


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## 1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank $n$, $\mathbb{K}$ a commutative field and $q$ a nonzero element of $\mathbb{K}$ which is not a root of unity. We follow the notation and convention of [Jan96] for the quantum group $\mathcal{U}_q(\mathfrak{g})$. In particular, to each choice of a reduced decomposition of the longest Weyl word $w_0$ of the Weyl group $W$ corresponds a generating system $(X_\beta)_{\beta\in\Phi^+}$ of the positive part $\mathcal{U}_q^+(\mathfrak{g})$ of $\mathcal{U}_q(\mathfrak{g})$ (see Section 3), where $\Phi^+$ denotes the set of positive roots associated to $\mathfrak{g}$.

The natural action of an $n$-dimensional torus on $\mathcal{U}_q^+(\mathfrak{g})$ induces a stratification of the prime spectrum $\operatorname{Spec}(\mathcal{U}_q^+(\mathfrak{g}))$ of $\mathcal{U}_q^+(\mathfrak{g})$ via the so-called Stratification Theorem (see [GL00]). In this stratification, the primitive ideals are easily identified: they are the primes of $\mathcal{U}_q^+(\mathfrak{g})$ that are maximal in their strata. This stratification was recently used in [AD08,Lau07b,Lau07a] in order to describe the automorphism group of $\mathcal{U}_q^+(\mathfrak{g})$ in the case where $\mathfrak{g}$ is of type $A_2$ and $B_2$.

As $\mathcal{U}_q^+(\mathfrak{g})$ can be presented as a skew-polynomial algebra, this stratification can also be described via the deleting-derivations theory of Cauchon [Cau03a]. In particular, in this theory, the strata are in a natural bijection with certain combinatorial objects, called Cauchon diagrams, and their geometry is completely described by the associated diagram. In fact, in the above situation, Cauchon diagrams are distinguished subsets of the set of positive roots $\Phi^+$. (For this reason, we often refer to subsets of $\Phi^+$ as diagrams.) Note that to each reduced decomposition of $w_0$ corresponds a PBW basis of $\mathcal{U}_q^+(\mathfrak{g})$ and so a notion of Cauchon diagrams.

The main aim of this paper is to give an algorithmic description of Cauchon diagrams in the case where the reduced decomposition of $w_0$ corresponds to a good order of $\Phi^+$ (in the sense of [Lus90]). Moreover, in each type, we exhibit a reduced decomposition of $w_0$ for which we are able to describe explicitly the corresponding Cauchon diagrams.

Our first ingredient in order to obtain an algorithmic description of Cauchon diagrams is the commutation relation between two generators $X_\beta$ and $X_{\beta'}$ given by Levendorskiĭ and Soibelman [LS91]. These formulas are not explicitly known, so that one cannot easily use them in order to perform the deleting-derivations algorithm. As a consequence, the description of Cauchon diagrams does not seem accessible in the general case. For this reason, we will limit ourselves to the case where the reduced decomposition of $w_0$ corresponds to a good order on $\Phi^+$ (see [Lus90]). We recall this notion in Section 2. Although we still do not know explicitly all the commutation relations between the generators of $\mathcal{U}_q^+(\mathfrak{g})$, the situation is better as we control enough commutation relations. More precisely, in this case, the commutation relation between two variables $X_\beta$, $X_{\beta'}$ is known when $\beta$ and $\beta'$ span a so-called admissible plane [Lus90] (see Section 3.4). Those relations allow the (algorithmic) construction of a set of necessary conditions, called *implications*, for a diagram $\Delta$ to be a Cauchon diagram (see Section 5.1). In Section 5.2, we prove that these conditions are necessary and sufficient (see Theorem 5.3.1), so that we get an algorithmic description of Cauchon diagrams.

In Section 6, we use this theorem to give an explicit description of these implications and these diagrams for special choices of the reduced decomposition of $w_0$. More precisely, in each type, we exhibit a reduced decomposition of $w_0$ for which we explicitly describe the corresponding Cauchon diagrams. As a corollary, we prove that in each type the number of diagrams is equal to the size $|W|$ of the Weyl group. As the strata do not depend on the choice of the reduced decomposition of $w_0$, this implies that the number of strata is always equal to $|W|$. This result was first proved by Gorelik [Gor00] by using different methods, but with the additional assumption that $q$ is transcendental.

In [CM08], we use the results of this paper in order to show that Cauchon diagrams $\Delta$ are in one-to-one correspondence with positive sub-expressions $\mathbf{w}^\Delta$ of $w_0$ as defined by Marsh and Rietsch [MR04]. More precisely, assume $w_0$ has a reduced expression of the form $w_0 = s_{\alpha_1}\circ\cdots\circ s_{\alpha_N}$ and that this decomposition corresponds to a good order on $\Phi^+$. For all $i\in\{1,\dots,N\}$, we set $\beta_i = s_{\alpha_1}\circ\cdots\circ s_{\alpha_{i-1}}(\alpha_i)$, so that $\Phi^+ = \{\beta_1 < \cdots < \beta_N\}$. For each diagram $\Delta = \{\beta_{i_1} < \cdots < \beta_{i_t}\} \subseteq \Phi^+$, we set $w^\Delta := s_{\alpha_{i_1}}\circ\cdots\circ s_{\alpha_{i_t}}$. Then we have the following results (see [CM08]).

- If $\Delta$ is a Cauchon diagram, the above decomposition of $w^\Delta$ is reduced.
- The map $\Delta \to w^\Delta$ is a bijection from the set $\mathcal{D}$ of Cauchon diagrams to $W$.

## 2. Root systems

### 2.1. Classical results on root systems

Let $\mathfrak{g}$ be a simple complex Lie algebra. Let follow the notations of [Jan96, Chapter 4]. We denote by $\Phi$ a root system and $E = \mathrm{Vect}(\Phi)$ ($\dim E = n$). When $\Pi := \{\alpha_1, \dots, \alpha_n\}$ is a basis of $\Phi$, one has a decomposition $\Phi = \Phi^+ \sqcup \Phi^-$, where $\Phi^+$ (resp. $\Phi^-$) is the set of positive (resp. negative) roots. Denote by $W$ the Weyl group associated to the root system $\Phi$; it is generated by the reflections $s_{\alpha_i}$ ($:= s_i$), $1 \leqslant i \leqslant n$. The longest Weyl word in $W$ is written $w_0$. A root system $\Phi$ is *reducible* if $\Phi = \Phi_1 \sqcup \Phi_2$ where $\Phi_1$ and $\Phi_2$ are two orthogonal root systems. Otherwise $\Phi$ is called *irreducible*.

Let us recall that there is a one-to-one correspondence between the irreducible root systems and the simple complex Lie algebras of finite dimension. We say that $\mathfrak{g}$ is of a given type if the associated root system is of the same type. The following definitions and results are taken from [Lus90].

**Definition 2.1.1.** Let $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis of $\Phi$ and $j \in [\![1, n]\!]$ ($:= \{1, 2, \dots, n\}$).

1. *The column $j$* is the set $C_j := \{\beta \in \Phi^+ \mid \beta = k_1\alpha_1 + \cdots + k_j\alpha_j,\ k_i \in \mathbb{N},\ k_j \neq 0\}$.
2. A root $\beta = k_1\alpha_1 + \cdots + k_j\alpha_j \in C_j$ is called *ordinary* if $k_j = 1$; it is called *exceptional* if $k_j = 2$.
3. A column $C_j$ is called *ordinary* if each root $\beta$ of $C_j$ is ordinary; this column is called *exceptional* if every root $\beta$ of $C_j$ is ordinary except a unique one ($\beta_{ex}$) which is exceptional.

**Definition 2.1.2.** A numbering $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is *good* if all columns $C_j$ are ordinary or exceptional.

**Example 2.1.3** *(The $G_2$ case).* The root system pf type $G_2$ has rank 2, there are two simple roots $\alpha_1$ and $\alpha_2$ such that $\|\alpha_2\| = \sqrt{3}\|\alpha_1\|$. $\Pi = \{\alpha_1, \alpha_2\}$ is a base for this roots system. The numbering $\Pi = \{\alpha_1, \alpha_2\}$ is good in this case because $C_1 = \{\alpha_1\}$ is ordinary and $C_2 = \{\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$ is exceptional.

On the contrary, the numbering $\Pi = \{\alpha_2, \alpha_1\}$ is not good. For this numbering, $C_1 = \{\alpha_2\}$ is ordinary but $C_2 = \{\alpha_1, \alpha_2 + \alpha_1, \alpha_2 + 2\alpha_1, \alpha_2 + \mathbf{3}\alpha_1, 2\alpha_2 + \mathbf{3}\alpha_1\}$ is neither ordinary nor exceptional.

**Proposition 2.1.4.** *Les $\mathfrak{g}$ be a simple Lie algebra of finite dimension. The following numberings of the associated root system $\Pi$ are examples of good numberings.*

- *If $\mathfrak{g}$ is of type $A_n$, with Dynkin diagram:* $\alpha_1 - \alpha_2 - \cdots - \alpha_{n-1} - \alpha_n$, *or*
- *if $\mathfrak{g}$ is of type $B_n$, with Dynkin diagram:* $\alpha_1 \Leftarrow \alpha_2 - \cdots - \alpha_{n-1} - \alpha_n$, *or*
- *if $\mathfrak{g}$ is of type $C_n$, with Dynkin diagram:* $\alpha_1 \Rightarrow \alpha_2 - \cdots - \alpha_{n-1} - \alpha_n$, *or*
- *if $\mathfrak{g}$ is of type $D_n$, with Dynkin diagram:*
$$\begin{array}{ccccccccc} \alpha_1 & & & & & & & & \\ & \diagdown & & & & & & & \\ & & \alpha_3 & - & \alpha_4 & - & \cdots & - & \alpha_{n-1} - \alpha_n \\ & \diagup & & & & & & & \\ \alpha_2 & & & & & & & & \end{array}$$
*, then*
$$\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n\} \text{ is a good numbering.}$$
- *If $\mathfrak{g}$ is of type $G_2$, with Dynkin diagram:* $\alpha_1 \Lleftarrow \alpha_2$, *then $\Pi = \{\alpha_1, \alpha_2\}$ is a good numbering.*
- *If $\mathfrak{g}$ is of type $F_4$, with Dynkin diagram:* $\alpha_1 - \alpha_2 \Rightarrow \alpha_3 - \alpha_4$, *then $\Pi = \{\alpha_4, \alpha_3, \alpha_2, \alpha_1\}$ is a good numbering.*
- *If $\mathfrak{g}$ is of type $E_6$, with Dynkin diagram:*
$$\begin{array}{ccccccccc} & & & & \alpha_2 & & & & \\ & & & & | & & & & \\ \alpha_1 & - & \alpha_3 & - & \alpha_4 & - & \alpha_5 & - & \alpha_6 \end{array}$$
*, then $\Pi = \{\alpha_2, \alpha_5, \alpha_4, \alpha_3, \alpha_1, \alpha_6\}$ is a good numbering.*
- *If $\mathfrak{g}$ is of type $E_7$, with Dynkin diagram:*
$$\begin{array}{ccccccccccc} & & & & \alpha_2 & & & & & & \\ & & & & | & & & & & & \\ \alpha_1 & - & \alpha_3 & - & \alpha_4 & - & \alpha_5 & - & \alpha_6 & - & \alpha_7 \end{array}$$
*, then $\Pi = \{\alpha_2, \alpha_5, \alpha_4, \alpha_3, \alpha_1, \alpha_6, \alpha_7\}$ is a good numbering.*

- *If $\mathfrak{g}$ is of type $E_8$, with Dynkin diagram:* $\begin{array}{ccccccccccccc} & & & & \alpha_2 & & & & & & & & \\ & & & & | & & & & & & & & \\ \alpha_1 & - & \alpha_3 & - & \alpha_4 & - & \alpha_5 & - & \alpha_6 & - & \alpha_7 & - & \alpha_8 \end{array}$ *, then* $\Pi = \{\alpha_2, \alpha_5, \alpha_4, \alpha_3, \alpha_1, \alpha_6, \alpha_7, \alpha_8\}$ *is a good numbering.*

The corresponding columns with these numberings are given explicitly in Section 3 and one could verify that each column is ordinary or exceptional. *In the following, the chosen numbering on $\Pi$ is always a good one. From Section* 6*, we use the numbering from the previous proposition.*

*2.2. Lusztig order*

**Definition 2.2.1.** (See [Lus90, Section 4.3].) For a root $\beta = k_1\alpha_1 + \cdots + k_j\alpha_j \in C_j$, the *height of* $\beta$ is the positive integer $h(\beta) := k_1 + \cdots + k_j$; the *Lusztig height* of $\beta$ is the rational number $h'(\beta) := \frac{1}{k_j}h(\beta)$. If $t \in h'(C_j)$, then the set $B^{j,t} := \{\beta \in C_j \mid h'(\beta) = t\}$ is called *the box of height t in the column $C_j$.*

This definition gives the following disjoint union $C_j = \bigsqcup_{t \in \mathbb{N}^*} B^{j,t}$.

**Definition 2.2.2** *(Lusztig order on $\Phi^+$).* We define a partial order on $\Phi^+$ as follows. Let $\beta_1$ and $\beta_2$ be two positive roots, if $\beta_1 \in C_{j_1}$ and $\beta_2 \in C_{j_2}$ with $j_1 < j_2$, then $\beta_1 < \beta_2$; if $\beta_1$ and $\beta_2$ are in the same column $C_j$ and if $h'(\beta_2) < h'(\beta_1)$, then $\beta_1 < \beta_2$.

One can refine the previous partial order in a total one by choosing arbitrarily an order inside the boxes. Such a total order on $\Phi^+$ is called *"a" Lusztig order*.

**Observations.** The simple root $\alpha_j$ is the greatest root in $C_j$ for any Lusztig order. The positive roots of a box are consecutive for any Lusztig order, that is, $B^{j,t} = \{\beta_p, \beta_{p+1}, \dots, \beta_{p+l}\}$. Any Lusztig order induces an order on boxes. For example, the box before $B^{j,t}$ in the column $C_j$ is $B^{j,t+1}$.

**Proposition 2.2.3.** *Let $j \in \{2, \dots, n\}$. Assume $C_j$ is an exceptional column, we denote by $\beta_{ex}$ its exceptional root. Then:*

1. $\beta_{ex} \perp (C_1 \sqcup \cdots \sqcup C_{j-1})$.
2. *If $D = \langle \beta_{ex} \rangle$ and if $s_D$ is the orthogonal against $D$, then:*
   - $s_D(C_j) = C_j$ *and for any $\beta \in C_j \setminus \{\beta_{ex}\}$, we have $\beta + s_D(\beta) = \beta_{ex}$.*
   - *Let $B^{j,t}$ be a box different from the box which contain $\beta_{ex}$. Then $s_D$ transforms $B^{j,t}$ into $B^{j,h(\beta_{ex})-t}$.*

**Proof.**

1. Let $\beta \in C_1 \cup \cdots \cup C_{j-1}$. If $\beta$ is not orthogonal to $\beta_{ex}$, then $s_\beta(\beta_{ex}) = \beta_{ex} + k\beta$ $(k \in \mathbb{Z} \setminus \{0\})$ is a root of $C_j$ whose coordinate on $\alpha_j$ is equal to 2. This is a contradiction with the unicity of the exceptional root.
2. We observe that $s_D = -s_{\beta_{ex}}$, so that $s_D(\Phi) = \Phi$.
   - Let $\beta$ be an ordinary root of $C_j$. We can decompose this root $\beta = a_1\alpha_1 + \cdots + a_{j-1}\alpha_{j-1} + \frac{1}{2}\beta_{ex}$ $(a_i \in \mathbb{Q})$. From 1. we deduce that $s_D(\beta) = -a_1\alpha_1 \ldots - a_{j-1}\alpha_{j-1} + \frac{1}{2}\beta_{ex} = \beta_{ex} - \beta$. This is a root from the previous observation. This root is clearly in $C_j$ since $\beta$ is in $C_j \setminus \{\beta_{ex}\}$.
   - By the previous assertion, $s_D$ transforms two element from $B^{j,t}$ into two roots of the same height. We deduce from this fact (and from the fact that $s_D$ is an involution) that $s_D(B^{j,t})$ is a box denoted by $B^{j,s}$. The formula $t + s = h(\beta_{ex})$ is a consequence of the previous assertion. □

**Definition 2.2.4.** The support of a root $\beta = a_1\alpha_1 + \cdots + a_j\alpha_j \in C_j$ is the set $\mathrm{Supp}(\beta) := \{\alpha_i \in \Pi \mid a_i \neq 0\}$. In particular, for $\beta \in C_j$, we have $\mathrm{Supp}(\beta) \subset \{1, \dots, j\}$.

We are now ready to prove that the box containing the exceptional root of an exceptional column is reduced to the exceptional root.

**Proposition 2.2.5.** *Let $j \in \{2, \dots, n\}$. Assume $C_j$ is an exceptional column and denote by $\beta_{ex}$ its exceptional root. Then $h'(\beta_{ex}) \notin \mathbb{N}$, so that $\beta_{ex}$ is alone in its box.*

**Proof.** Denote $\Pi_j = \{\alpha_1, \dots, \alpha_j\}$ and $\Phi_j = \Phi \cap \mathrm{Vect}(\Pi_j)$. Then $\Phi_j$ is a root system with basis $\Pi_j$ and $\Phi_j^+ = \Phi^+ \cap \mathrm{Vect}(\Pi_j)$.

Let us consider the case where $\Phi_j$ irreducible. Then we have

**Observation 1.** If $\beta$ is a root of $\Phi_j^+$ of maximal height then $\beta \in C_j$.

**Proof of Observation 1.** Assume that $\beta \in C_i$ with $i < j$. In the Dynkin diagram of $\Phi_j$ which is convex as $\Phi_j$ is irreducible, we can construct a path from $\alpha_i$ to $\alpha_j$. Denote this path by $P = (\alpha_{i_1}, \dots, \alpha_{i_t})$, where $i_1 = i$ and $i_t = j$. We know that $\alpha_i \in \mathrm{Supp}(\beta)$ and that $\alpha_j \notin \mathrm{Supp}(\beta)$. So there is a smallest index $l$ such that $\alpha_{i_l} \in \mathrm{Supp}(\beta)$ and $\alpha_{i_{l+1}} \notin \mathrm{Supp}(\beta)$. Thus, for all $\alpha \in \mathrm{Supp}(\beta)$, we have $\langle \alpha, \alpha_{i_{l+1}} \rangle \leqslant 0$ and, since $\alpha_{i_l}$ and $\alpha_{i_{l+1}}$ are two consecutive elements from $P$, we have $\langle \alpha_{i_l}, \alpha_{i_{l+1}} \rangle < 0$. We deduce that, $\langle \beta, \alpha_{i_{l+1}} \rangle < 0$ thus $\beta + \alpha_{i_{l+1}} \in \Phi_j^+$ which contradicts the maximality of the height of $\beta$. $\square$

**Observation 2.** $\beta_{ex}$ is the largest root of $\Phi_j$.

**Proof of Observation 2.** Let $\beta$ be a largest root in $\Phi_j$. We assume that $\beta \neq \beta_{ex}$. By the previous observation, we know that $\beta \in C_j$ and, by Proposition 2.2.3, $\beta_{ex} = \beta + s_D(\beta)$ is a sum of two positive roots, thus its height is greater than the height of $\beta$. So $\beta$ is equal to $\beta_{ex}$.

We note that the existence of an exceptional root implies that $\Phi_j$ is not of type $A_j$. So $\Phi_j$ is of type $B_j, C_j, D_j, E_6, E_7, E_8, F_4$ or $G_2$ and, we deduce from [Bou68] that the height of the largest root is odd. Hence it follows from Observation 2. that the height of $\beta_{ex}$ is odd, so that $h'(\beta_{ex}) \notin \mathbb{N}$.

Let us now assume that $\Phi_j$ is reducible. Denote by $\Gamma_j$ the Dynkin diagram whose vertices are $\alpha_1, \dots, \alpha_j$, and whose edges come from the Dynkin diagram of $\Phi$. We note $\Pi'$ the connected component of $\alpha_j$ in $\Gamma_j$, i.e. $\Pi' := \{\alpha_i \in \Pi_j \mid \text{there exists a path contained in } \Gamma_j \text{ connecting } \alpha_i \text{ to } \alpha_j\}$. We note $\Phi' = \Phi \cap \mathrm{Vect}(\Pi')$. It is a root system, with basis $\Pi'$, and we have $\Phi'^+ = \Phi^+ \cap \mathrm{Vect}(\Pi')$. $\square$

**Observation 3.** $C_j \subseteq \Phi'^+$.

**Proof of Observation 3.** Otherwise, there is a root in $C_j \setminus \Phi'^+$. If $\beta$ is such a root, there is a simple root in its support which is also in the set $\Pi_j \setminus \Pi'$. As the support of $\beta$ contains $\alpha_j \in \Pi'$ too, we can write $\beta = u + v$ with $u = \alpha_{i_1} + \dots + \alpha_{i_l}$ whose support is a subset of $\Pi_j \setminus \Pi'$ and $v = \alpha_{i_{l+1}} + \dots + \alpha_{i_p}$ whose support is a subset of $\Pi'$. Let us choose $\beta$ such that the integer $l$ is minimal.

- If $l = 1$, then $\beta = \alpha_{i_1} + v$. As $\alpha_{i_1} \notin \Pi'$, there is no link between $\alpha_{i_1}$ and the element of $\mathrm{Supp}(v)$. Then $s_{i_1}(\beta) = -\alpha_{i_1} + v \in \Phi$, which is impossible because the coordinates of this root in the basis $\Pi$ do not have the same sign.
- So $l \geqslant 2$. As $\langle u, u \rangle > 0$, there exists a root in $\mathrm{Supp}(u)$, for example $\alpha_{i_l}$, such that $\langle u, \alpha_{i_l} \rangle > 0$. As above:
$$\langle v, \alpha_{i_l} \rangle = 0 \quad \Rightarrow \quad \langle \beta, \alpha_{i_l} \rangle > 0 \quad \Rightarrow \quad \beta' = \beta - \alpha_{i_l} \in C_j \setminus \Phi'^+,$$
which is a contradiction with the minimality of $l$.

So we can conclude that $C_j \subseteq \Phi'^+$. Hence $C_j$ is an exceptional column of $\Phi'$ which is irreducible by construction. The proof above also shows that the exceptional root $\beta_{ex}$ satisfies $h'(\beta_{ex}) \notin \mathbb{N}$. $\square$

We can now prove that any Lusztig order is a convex order.

**Proposition 2.2.6.** "$<$" *is a convex order over $\Phi^+$.*

**Proof.** Let $\beta_1 < \beta_2$ be two positives roots such that $\beta_1 + \beta_2 \in \Phi^+$.

- If the two roots $\beta_1$ and $\beta_2$ do not belong to the same column, then $\beta_1 + \beta_2$ is in the same column as $\beta_2$. In this case, neither $\beta_2$, nor $\beta_1 + \beta_2$ are exceptional and $h'(\beta_1 + \beta_2) = h(\beta_1 + \beta_2) = h(\beta_1) + h(\beta_2) > h(\beta_2) = h'(\beta_2)$. Hence we have: $\beta_1 < \beta_1 + \beta_2 < \beta_2$.
- If the two roots $\beta_1$ and $\beta_2$ belong to the same column, then $\beta_1 + \beta_2$ is an exceptional root. We deduce from Proposition 2.2.3 that $h'(\beta_1 + \beta_2) = \frac{h'(\beta_1)+h'(\beta_2)}{2}$. Proposition 2.2.5 excludes the case where $h'(\beta_1 + \beta_2) = h'(\beta_1) = h'(\beta_2)$ because the exceptional root is alone in its box. So we get $h'(\beta_1) > h'(\beta_1 + \beta_2) > h'(\beta_2)$, so that $\beta_1 < \beta_1 + \beta_2 < \beta_2$. □

Consider a reduced decomposition of $w_0 = s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_N}$ of the longest Weyl word $w_0$. For all $j \in [\![1, N]\!]$, we set $\beta_j := s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_{j-1}}(\alpha_{i_j})$. Then it is well known (cf., for example, [BG02, I.5.1]) that $\{\beta_1, \dots, \beta_N\} = \Phi^+$. For each integer $j \in [\![1, N]\!]$, we say that $\alpha_{i_j}$ is *the simple root associated to the positive root* $\beta_j$.

We define an order on $\Phi^+$ by setting $\beta_i < \beta_j$ when $i < j$. We say that "$<$" is *the order associated to the reduced decomposition* $w_0 = s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_N}$ of $w_0$.

In [Pap94, Theorem and remark p. 662], it is shown that this is a convex order and that this leads to a one-to-one correspondence between reduced decompositions of $w_0$ and convex orders on $\Phi^+$.

*Hence, as the Lusztig order "$<$" is convex, there is a unique reduced decomposition* $w_0 = s_{i'_1} \circ s_{i'_2} \circ \cdots \circ s_{i'_N}$ of $w_0$ *whose associated order is "$<$". In this article, we always choose such a decomposition for* $w_0$.

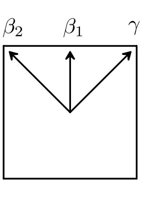
The following proposition of Lusztig [Lus90, Section 4.3] explains the behaviour of the positive roots inside (non-exceptional) boxes.

**Proposition 2.2.7.** *Inside each ordinary box (box which does not contain the exceptional root), roots are pairwise orthogonal. Moreover, simple roots associated to the positive roots of a given box are pairwise orthogonal.*

**Proof.** For the type $G_2$, explicit computations leads to the result. We now assume that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra which is not of type $G_2$. Recall that the positive roots of a box are consectutives. Let $\beta_1$ and $\beta_2$ be two consecutive roots of a box $B$ in the column $C_j$. We note $\alpha_{i_1}$ and $\alpha_{i_2}$ the associated simple roots.

Suppose that $\alpha_{i_1}$ is not orthogonal to $\alpha_{i_2}$, hence $\lambda = -\langle \alpha_{i_1}^\vee, \alpha_{i_2} \rangle = 1$ or 2 (recall that $\mathfrak{g}$ is not of type $G_2$). So we can write $\beta_2 = w \circ s_{i_1}(\alpha_{i_2}) = w(\lambda\alpha_{i_1} + \alpha_{i_2}) = \lambda\beta_1 + w(\alpha_{i_2})$.As $w(\alpha_{i_2}) \in \Phi$, we must have $\lambda = 2$, otherwise $h(w(\alpha_{i_2})) = h(\beta_2) - h(\beta_1) = 0$, which is absurd.

In this case, $\gamma = -w(\alpha_{i_2}) = 2\beta_1 - \beta_2 \in C_j$ and $h(\gamma) = 2h(\beta_1) - h(\beta_2) = h(\beta_1)$. As $\beta_1$ and $\beta_2$ are distinct roots, so they are not collinear. So the set $\Phi' = \Phi \cap \mathrm{Vect}(\beta_1, \beta_2)$ is a root system of rank 2 which contains $\beta_1$, $\beta_2$, $\gamma$ and their opposites. The equality $2\beta_1 = \gamma + \beta_2$ allows to state that $\Phi'$ is of type $B_2$ and that the situation is the following one:




So $\gamma - \beta_1 \in \Phi$, with $h(\gamma - \beta_1) = h(\gamma) - h(\beta_1) = 0$. This is impossible, and so $\alpha_{i_1} \perp \alpha_{i_2}$.

Then we get $\langle \beta_1, \beta_2 \rangle = \langle w(\alpha_{i_1}), w(s_{i_1}(\alpha_{i_2})) \rangle = \langle \alpha_{i_1}, s_{i_1}(\alpha_{i_2}) \rangle = \langle \alpha_{i_1}, \alpha_{i_2} \rangle = 0$, as desired. This finishes the case where the two roots are consecutive. One concludes using an induction on the "distance" between the two roots $\beta_1$ and $\beta_2$. □

**Convention.** For $j \in [\![1, n]\!]$, denote $\delta_j$ the smallest root of $C_j$. Let us recall that $\alpha_j$ is the largest root of $C_j$.

**Proposition 2.2.8.** *$\delta_j$ and $\alpha_j$ are alone in their boxes.*

**Proof.** The root $\alpha_j$ is alone in its box because it is the only roots of $C_j$ whose height is equal to 1.

To prove that $\delta_j$ is alone in its box, we need the following result which can be shown easily by induction on $l$.

**Lemma 2.2.9.** *Let $1 \leqslant l \leqslant N$ and $1 \leqslant m \leqslant n$. Set $\Pi_m := \{\alpha_1, \dots, \alpha_m\}$. If $\beta_l = s_{i_1} \dots s_{i_{l-1}}(\alpha_{i_l})$ is in the column $C_m$, then $\alpha_{i_j} \in \Pi_m$ for $j \in [\![1, l]\!]$.*

**Back to the proof of Proposition 2.2.8.** There is an integer $1 \leqslant l \leqslant N$ such that $\delta_j = \beta_l = s_{i_1} \circ s_{i_2} \circ \dots \circ s_{i_{l-1}}(\alpha_{i_l})$. As above, $\beta_l = \alpha_{i_l} + n_{l-1}\alpha_{i_{l-1}} + \dots + n_1\alpha_{i_1}$ ($n_t \in \mathbb{Z}$) with $\alpha_{i_1}, \dots, \alpha_{i_{l-1}}$ in $\Pi_{j-1}$ since $\beta_{l-1} \in C_{j-1}$. As $\beta_l \in C_j$, it implies that $\alpha_{i_l} = \alpha_j$.

If $\delta_j$ $(= \beta_l)$ is not alone in its box, then $\beta_{l+1}$ is also in this box and one has (Proposition 2.2.7) $\alpha_{i_l} \perp \alpha_{i_{l+1}}$. By the previous lemma, it implies that $\alpha_{i_{l+1}} \in \Pi_j \setminus \{\alpha_j\} = \Pi_{j-1}$ and $\beta_{l+1} = s_{i_1} \circ s_{i_2} \circ \dots \circ s_{i_{l-1}} \circ s_{i_l}(\alpha_{i_{l+1}}) = s_{i_1} \circ s_{i_2} \circ \dots \circ s_{i_{l-1}}(\alpha_{i_{l+1}}) = \alpha_{i_{l+1}} + n'_{l-1}\alpha_{i_{l-1}} + \dots + n'_1\alpha_{i_1}$ ($n'_t \in \mathbb{Z}$), which contradicts the hypothesis $\beta_{l+1} \in C_j$. $\square$

Let us recall the following result (see, for example, [Hum78, Lemma 9.4]).

**Lemma 2.2.10.** *Let $\beta$ and $\delta$ be two distinct roots of $\Phi^+$ such that $\langle \beta, \delta \rangle \neq 0$. If $\langle \beta, \delta \rangle > 0$, then $\beta - \delta \in \Phi$. If $\langle \beta, \delta \rangle < 0$, then $\beta + \delta \in \Phi$.*

**Proposition 2.2.11.** *Let $\beta$ be an ordinary root of a column $C_j$. Denote (as in the proof of Proposition 2.2.5) by $\Gamma_j$ the diagram whose vertices are $\alpha_1, \dots, \alpha_j$, and whose edges are the edges from the Dynkin diagram of $\Phi$. Denote by $\Omega_j$ the connected component of $\alpha_j$ in $\Gamma_j$.*

1. *If $\beta \neq \alpha_j$ then there exists $\epsilon \in \{\alpha_1, \dots, \alpha_{j-1}\}$ such that $\beta - \epsilon \in C_j$.*
2. *$\operatorname{Supp} \beta \subset \Omega_j$.*
3. *If $\beta \neq \delta_j$ then there exists $\epsilon \in \{\alpha_1, \dots, \alpha_{j-1}\}$ such that $\beta + \epsilon \in C_j$.*

The proof of this result is technical and can be found in the ArXiv version of this article [Mér08].

**Proposition 2.2.12.** *Let $j \in [\![1, n]\!]$.*

1. *If $C_j$ is ordinary, then $h'(C_j)$ is an interval of the form $[\![1, t]\!]$.*
2. *If $C_j$ is exceptional, then $h'(C_j \setminus \{\beta_{ex}\})$ is an interval of the form $[\![1, 2t]\!]$ ($t \in \mathbb{N}^\star$).*

*Moreover we have $h'(\beta_{ex}) = t + \frac{1}{2}$.*

**Proof.** The fact that $h'(C_j)$ in the ordinary case (resp. $h'(C_j \setminus \{\beta_{ex}\})$ in the exceptional case) is an interval of integers comes from Proposition 2.2.11. It contains $1 = h(\alpha_j)$, and so the first case is proved.

Let us assume that $C_j$ is exceptional. Denote by $B_1, \dots, B_t$ the boxes which contain the roots smaller than $\beta_{ex}$ for the Lusztig order. For these boxes, we have $h'(B_i) > h'(\beta_{ex})$. But the relation $h(B_i) + h(B'_i) = h(\beta_{ex})$, for the image $B'_i$ of $B_i$ by $s_D$, implies $h'(B_i) > h'(\beta_{ex}) > h'(B'_i)$. So we have exactly $t$ boxes appearing after $\beta_{ex}$ and the interval $h'(C_j \setminus \{\beta_{ex}\})$ is of the form $[\![1, 2t]\!]$ ($t \in \mathbb{N}^\star$).

Moreover $h(\beta_{ex}) = h(\alpha_j + s_D(\alpha_j)) = 1 + 2t$ and finally $h'(\beta_{ex}) = t + \frac{1}{2}$. $\square$

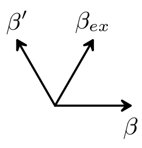
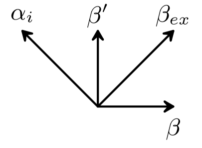
We now recall the notion of admissible planes introduced by Lusztig in [Lus90, Section 6.1].

**Definition 2.2.13.** We call *admissible plane* $P := \langle \beta, \beta' \rangle$ a plane spanned by two positive roots $\beta$ and $\beta'$ such that: $\beta$ belongs to an exceptional column $C_j$ and $\beta' = s_D(\beta)$ is such that $|h'(\beta') - h'(\beta)| = 1$. (In this case $\beta + \beta' = \beta_{ex}$ and $h'(\beta_{ex}) = t \pm \frac{1}{2}$.)

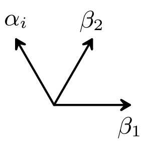
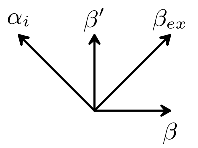
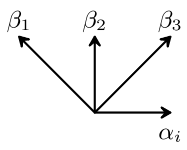
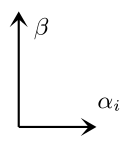
**Or** $\beta$ is an ordinary root in any column $C_j$ and $\beta' = \alpha_i$ with $i < j$. We set $\Phi_P := \Phi \cap P$ and $\Phi_P^+ := \Phi^+ \cap P$.

**Remark 2.2.14.** If $\Phi_P$ is of type $G_2$ then $\Phi = G_2$ (due to the lengths of the roots).

If $\Phi$ is not of type $G_2$ then the first condition leads to two different type of admissible planes, $\Phi_P^+$ is of one of the following types:

| Type (1.1) | Type (1.2) |
|---|---|
|   |   |
| $A_2$ | $B_2$ |
| $\beta > \beta_{ex} > \beta'$ | $\beta > \beta_{ex} > \beta' > \alpha_i$ |

The second condition leads to four types of admissible planes, $\Phi_P^+$ is of one of the following types:

| Type (2.1) | Type (2.2) | Type (2.3) | Type (2.4) |
|---|---|---|---|
|   |   |   |   |
| $A_2$ | $B_2$ with long $\alpha_i$ | $B_2$ with short $\alpha_i$ | $A_1 \times A_1$ |
| $\beta_1 > \beta_2 > \alpha_i$ | $\beta > \beta_{ex} > \beta' > \alpha_i$ | $\beta_1 > \beta_2 > \beta_3 > \alpha_i$ | $\beta > \alpha_i$ |

We note that types (1.2) and (2.2) are the same.

## 3. The quantized enveloping algebra $\mathcal{U}_q(\mathfrak{g})$

Let $\mathbb{K}$ be a field of characteristic not equal to 2 and 3, and $q$ an element $\mathbb{K}^*$ which is not a root of unity. Firstly, we recall definitions about $\mathcal{U}_q(\mathfrak{g})$ and $\mathcal{U}_q^+(\mathfrak{g})$ using notations from [Jan96, Chapter 4]. We recall then the Poincaré–Birkhoff–Witt bases of $\mathcal{U}_q(\mathfrak{g})$ construction using Lusztig automorphisms. There are several ways to construct the so called Lusztig automorphisms, we recall here three different methods. The Lusztig's one follows [Lus90, Section 3], Jantzen's one, which is the same as De Concini, Kac and Procesi, is explained in [Jan96, Section 8.14] and [DCKP95, Section 2.1] and a third one is necessary to established a link between the two others. We will explain each method and then see the links between the obtained bases.

### *3.1. Recalls on $\mathcal{U}_q(\mathfrak{g})$*

For all $a$ and $n$ integers such that $a \geqslant n \geqslant 0$, we set $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, $[n]_q^! = [n]_q[n-1]_q \dots [2]_q[1]_q$, $\left[\begin{smallmatrix} a \\ n \end{smallmatrix}\right]_q = \frac{[a]_q^!}{[n]_q^![a-n]_q^!}$. Moreover for all $\alpha \in \Pi$, we set $q_\alpha = q^{\frac{(\alpha,\alpha)}{2}}$.

**Definition 3.1.1.**

- The quantized enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{K}$-algebra with generators $E_\alpha$, $F_\alpha$, $K_\alpha$ and $K_\alpha^{-1}$ (for all $\alpha$ in $\Pi$) and relations (for all $\alpha, \beta \in \Pi$):
  - •• $K_\alpha K_\alpha^{-1} = 1 = K_\alpha^{-1} K_\alpha$, $K_\alpha K_\beta = K_\beta K_\alpha$, $K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha,\beta)} E_\beta$, $K_\alpha F_\beta K_\alpha^{-1} = q^{-(\alpha,\beta)} F_\beta$.
  - •• $E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}$, where $\delta_{\alpha\beta}$ is the Kronecker symbol.

  And (for $\alpha \neq \beta$), set $a_{\alpha\beta} = 2(\alpha,\beta)/(\alpha,\alpha)$:

$$\sum_{s=0}^{1-a_{\alpha\beta}} (-1)^s \begin{bmatrix} 1-a_{\alpha\beta} \\ i \end{bmatrix}_{q_\alpha} E_\alpha^{1-a_{\alpha\beta}-s} E_\beta E_\alpha^s = 0, \qquad \sum_{s=0}^{1-a_{\alpha\beta}} (-1)^s \begin{bmatrix} 1-a_{\alpha\beta} \\ i \end{bmatrix}_{q_\alpha} F_\alpha^{1-a_{\alpha\beta}-s} F_\beta F_\alpha^s = 0.$$

- $\mathcal{U}_q^+(\mathfrak{g})$ (resp. $\mathcal{U}_q^-(\mathfrak{g})$) is the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by all $E_\alpha$ (resp. $F_\alpha$) with $\alpha \in \Pi$.

Let us recall two important results proven for example in [BG02, Section I.6].

**Theorem 3.1.2.** *$\mathcal{U}_q^+(\mathfrak{g})$ is Noetherian domain and is graded by $\mathbb{Z}\Phi$ with $wt(E_\alpha) = \alpha$, $wt(F_\alpha) = -\alpha$ and $wt(K_\alpha^{\pm 1}) = 0$.*

*3.2. Lusztig's construction*

**Definition 3.2.1** *(Lusztig's automorphisms).* For all $i \in [\![1,n]\!]$ there is a unique automorphism $T_{\alpha_i}$ of the algebra $\mathcal{U}_q(\mathfrak{g})$ such that:

$$\big(j \in [\![1,n]\!]\big) \quad T_{\alpha_i} E_{\alpha_i} = -F_{\alpha_i} K_{\alpha_i}, \qquad T_{\alpha_i} F_{\alpha_i} = -K_{\alpha_i}^{-1} E_{\alpha_i}, \qquad T_{\alpha_i} K_{\alpha_j} = K_{\alpha_j} K_{\alpha_i}^{-a_{ij}},$$
$$(j \neq i) \quad T_{\alpha_i} E_{\alpha_j} = \sum_{r+s=-a_{ij}} (-1)^r q^{-d_i s} E_{\alpha_i}^{(r)} E_{\alpha_j} E_{\alpha_i}^{(s)}, \qquad T_{\alpha_i} F_{\alpha_j} = \sum_{r+s=-a_{ij}} (-1)^r q^{d_i s} F_{\alpha_i}^{(s)} F_{\alpha_j} F_{\alpha_i}^{(r)}$$

where $E_{\alpha_i}^{(n)} := \frac{E_{\alpha_i}^n}{[n]_{d_i}^!}$ and $d_i = \frac{(\alpha_i,\alpha_i)}{2}$.

We now a fix a Lusztig order so that we can use the notations of columns and boxes as in the Section 2. The following result is given by Lusztig in [Lus90, Section 4.3]:

**Proposition 3.2.2.** *There is a unique map from $\Phi^+$ to $[\![1,n]\!]$, sending $\beta$ to $i_\beta$ such that the following properties are satisfied*:

1. *$s_{i_{\beta_1}}$ and $s_{i_{\beta_2}}$ commute in $W$ whenever $\beta_1$ and $\beta_2$ are in the same box. Hence, for a box $B$, the product of all $s_{i_\beta}$ with $\beta \in B$ is a well-defined element $s(B)$ in $W$, independent of the order of the factors.*
2. *$i_{\alpha_j} = j$ for $j \in [\![1,n]\!]$.*
3. *If $\beta \in C_j$ and if $B_1, \dots, B_k$ are the boxes in $C_j$ whose elements are strictly greater than $\beta$ for the Lusztig order then $s(B_1)s(B_2)\dots s(B_k)(\alpha_{i_\beta}) = \beta$.*

*We then set $w_\beta := s(B_1)s(B_2)\dots s(B_k)$.*

We now recall the construction of a PBW basis of $\mathcal{U}_q^+(\mathfrak{g})$ due to Lusztig [Lus90, Theorem 3.2].

**Theorem 3.2.3.** *Let $w \in W$ and $s_{i_1} \dots s_{i_p}$ be a reduced decomposition of $w$. Then the automorphism $T_w := T_{\alpha_{i_1}} \dots T_{\alpha_{i_p}}$ depends only on $w$ and not on the choice of reduced expression for it. Hence the $T_{\alpha_i}$ define a homomorphism of the braid group of $W$ in the group of automorphisms of the algebra $\mathcal{U}_q(\mathfrak{g})$.*

**Proposition 3.2.4.** *For all positive roots $\beta$, we define $E_\beta := T_{w_\beta}(E_{i_\beta}) \in \Phi^+$. These elements form a Poincaré–Birkhoff–Witt basis of $\mathcal{U}_q^+(\mathfrak{g})$ (see* [Lus90, *Proposition* 4.2]*).*

**Notation.** If $\beta > \beta'$, then we set $[E_\beta, E_{\beta'}]_q = E_\beta E_{\beta'} - q^{(\beta,\beta')} E_{\beta'} E_\beta$.

Our aim in the remaining of this paragraph is to exhibit the form of the commutation relation between two generators $E_\gamma$ and $E_{\gamma'}$, when $\gamma$ and $\gamma'$ belong to the same admissible plane $P$.

We first consider the case where $\Phi_P(=\Phi \cap P)$ is of type $G_2$. In this case, $\Phi$ is also of type $G_2$ and the commutation relations have been computed in [Lus90, Section 5.2]. This leads us to the following result.

**Proposition 3.2.5.** *Assume that $\Phi$ is of type $G_2$. Denote by $\alpha_1$ the short simple root and by $\alpha_2$ the long simple root. This is a good numbering of the set of simple roots (see Example* 2.1.3*). The corresponding reduced decomposition of $w_0$ is $s_1 s_2 s_1 s_2 s_1 s_2$ ($s_i = s_{\alpha_i}$) and, describing the roots in the associated convex order, one has:*

$$\Phi^+ = \{\beta_1 = \alpha_1, \beta_2 = 3\alpha_1 + \alpha_2, \beta_3 = 2\alpha_1 + \alpha_2, \beta_4 = 3\alpha_1 + 2\alpha_2, \beta_5 = \alpha_1 + \alpha_2, \beta_6 = \alpha_2\}.$$

*The first column $C_1$ is reduced to $\{\beta_1\}$, the second column $C_2 = \{\beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$ is exceptional with $\beta_{ex} = \beta_4$. One has:*

*$[E_{\beta_3}, E_{\beta_1}]_q = \lambda E_{\beta_2}$ with $\lambda \neq 0$, $[E_{\beta_4}, E_{\beta_1}]_q = \lambda E_{\beta_3}^2$ with $\lambda \neq 0$, $[E_{\beta_5}, E_{\beta_1}]_q = \lambda E_{\beta_3}$ with $\lambda \neq 0$, $[E_{\beta_6}, E_{\beta_1}]_q = \lambda E_{\beta_5}$ with $\lambda \neq 0$, $[E_{\beta_3}, E_{\beta_1}]_q = \lambda E_{\beta_2}$ with $\lambda \neq 0$ and $[E_{\beta_5}, E_{\beta_3}]_q = \lambda E_{\beta_4}$ with $\lambda \neq 0$.*

If $\Phi$ is not of type $G_2$, the commutation relations between the Lusztig's generators corresponding to two roots which are in the same admissible plane are known in several cases [Lus90, Section 5.2]. In particular, we have the following relations.

**Proposition 3.2.6** *($\Phi$ not of type $G_2$).*

- *If $P = \langle \beta, \beta' \rangle$ is an admissible plane of type* (1.1)*, then $\Phi_P^+ = \{\beta, \beta_{ex} = \beta + \beta', \beta'\}$ and the relations are: $[E_\beta, E_{\beta'}]_q = \lambda E_{\beta_{ex}}$ with $\lambda \neq 0$, $[E_\beta, E_{\beta_{ex}}]_q = [E_{\beta_{ex}}, E_{\beta'}]_q = 0$.*
- *If $P = \langle \beta, \beta' \rangle$ is an admissible plane of type* (1.2)*, then $\Phi_P^+ = \{\beta, \beta_{ex} = \beta + \beta', \beta', \alpha_i\}$ and the relations are: $[E_\beta, E_{\beta'}]_q = \lambda E_{\beta_{ex}}$ with $\lambda \neq 0$, $[E_{\beta_{ex}}, E_{\alpha_i}]_q = \lambda' E_{\beta'}^2$ with $\lambda' \neq 0$, $[E_\beta, E_{\alpha_i}]_q = \lambda'' E_{\beta'}$ with $\lambda'' \neq 0$, $[E_\beta, E_{\beta_{ex}}]_q = [E_{\beta_{ex}}, E_{\beta'}]_q = [E_{\beta'}, E_{\alpha_i}]_q = 0$.*
- *If $P = \langle \beta, \alpha_i \rangle$ is an admissible plane of type* (2.1)*, then $\Phi_P^+ = \{\beta_1, \beta_2 = \beta_1 + \alpha_i, \alpha_i\}$ ($\beta = \beta_1$ or $\beta_2$) and the relations are: $[E_{\beta_1}, E_{\alpha_i}]_q = \lambda E_{\beta_2}$ with $\lambda \neq 0$, $[E_{\beta_1}, E_{\beta_2}]_q = [E_{\beta_2}, E_{\alpha_i}]_q = 0$.*
- *If $P = \langle \beta, \alpha_i \rangle$ is an admissible plane of type* (2.2)*, then we have the same relations as in type* (1.2)*.*
- *If $P = \mathrm{Vect}(\beta, \alpha_i)$ is an admissible plane of type* (2.3)*, then $\Phi_P^+ = \{\beta_1, \beta_2 = \beta_1 + \alpha_i, \beta_3 = \beta_1 + 2\alpha_i, \alpha_i\}$ ($\beta = \beta_1, \beta_2$ or $\beta_3$) and the relations are: $[E_{\beta_2}, E_{\alpha_i}]_q = \lambda E_{\beta_3}$ with $\lambda \neq 0$, $[E_{\beta_1}, E_{\beta_3}]_q = \lambda' E_{\beta_2}^2$ with $\lambda' \neq 0$, $[E_{\beta_1}, E_{\alpha_i}]_q = \lambda'' E_{\beta_2}$ with $\lambda'' \neq 0$, $[E_{\beta_1}, E_{\beta_2}]_q = [E_{\beta_2}, E_{\beta_3}]_q = [E_{\beta_3}, E_{\alpha_i}]_q = 0$.*
- *If $P = \langle \beta, \alpha_i \rangle$ is an admissible plane of type* (2.4)*, then $\Phi_P^+ = \{\beta, \alpha_i\}$ with $\beta \perp \alpha_i$ and, if $\beta$ is ordinary, then $[E_\beta, E_{\alpha_i}]_q = 0$.*

**Corollary 3.2.7.** *Assume $\Phi$ is not of type $G_2$. Let $i, l$ be two integers such that $1 \leqslant i < l \leqslant n$ and $\eta \in C_l$:*

1. *If $(\eta, \alpha_i) > 0$, then $[E_\eta, E_{\alpha_i}]_q = 0$.*
2. *If $\eta + \alpha_i = m\gamma$ with $\gamma \in \Phi^+$ and $m \in \mathbb{N}^\star$, then $[E_\eta, E_{\alpha_i}]_q = \lambda E_\gamma^m$, with $\lambda \in \mathbb{K}^\star$.*
3. *If $\eta = \eta_1 + \eta_2$ with $\eta_1$ and $\eta_2$ in $C_l$ such that $h(\eta_1) + 1 = h(\eta_2)$ then $[E_{\eta_1}, E_{\eta_2}]_q = \lambda E_\eta$, with $\lambda \in \mathbb{K}^\star$.*

**Proof.** $P = \mathrm{Vect}(\eta, \alpha_i)$ is an admissible plane of type (2.1), (2.2), (2.3) or (2.4) by definition.

1. $P$ is not of type (2.4) because $(\eta, \alpha_i) \neq 0$. We distinguish between three remaining cases.

- If $P$ is of type (2.1), then with the notations of Remark 2.2.14, we have $\eta = \beta_2$, and so the result follows from Proposition 3.2.6.
- If $P$ is of type (2.2), then we have $\eta = \beta'$, and so the result follows from Proposition 3.2.6.
- If $P$ is of type (2.3), then we have $\eta = \beta_3$, and so the result follows from Proposition 3.2.6.

2. Since $m \neq 0$, we have $\gamma \in P \cap \Phi^+ = \Phi_P^+$, so $P$ is not of type (2.4). We distinguish between three remaining cases.
   - If $P$ is of type (2.1), then we deduce that $m = 1$, $\eta = \beta_1$ and $\gamma = \beta_2$, and so the result follows from Proposition 3.2.6.
   - If $P$ is of type (2.2), then there are two possibilities: ($m = 1$, $\eta = \beta$ and $\gamma = \beta'$) or ($m = 2$, $\eta = \beta_{ex}$ and $\gamma = \beta'$). In both cases, the result follows from Proposition 3.2.6.
   - If $P$ is of type (2.3), then we have $m = 1$, $\eta = \beta_1$ (resp. $\eta = \beta_2$) and $\gamma = \beta_2$ (resp. $\gamma = \beta_3$), and so the result follows from Proposition 3.2.6.
3. Let us consider the plane $P := \langle \eta_1, \eta_2 \rangle$. It is an admissible plane (see Definition 2.2.13) and $\Phi_P^+ = \{\eta_1, \eta, \eta_2\}$ (if $P$ is of type 1.1) or $\{\eta_1, \eta, \eta_2, \alpha_i\}$ with $i < l$ (if $P$ is of type 1.2). The previous proposition implies that $[E_{\eta_1}, E_{\eta_2}]_q = \lambda E_\eta$, with $\lambda \in \mathbb{K}^\star$ in both cases. □

### 3.3. *Jantzen's construction*

In [Jan96, Section 8.14], a different construction of a PBW basis is explained which also uses the automorphisms $T_\alpha, (\alpha \in \Pi)$. For a given reduced decomposition of $w_0 = s_{i_1} \dots s_{i_N}$, we know that, for all $\beta \in \Phi^+$, there exists $i_\beta \in [\![1, N]\!]$ such that $\beta = s_{i_1} \dots s_{i_\beta - 1}(\alpha_{i_\beta})$.

**Definition 3.3.1.** Let $\beta \in \Phi^+$, we set $w'_\beta := s_{i_1} \dots s_{i_\beta - 1}$ and define $X_\beta := T_{w'_\beta}(E_{\alpha_{i_\beta}})$, $Y_\beta := T_{w'_\beta}(F_{\alpha_{i_\beta}})$.

The following result follows from [Jan96, Theorems 4.21 and 8.24].

**Theorem 3.3.2.**

- *If $\alpha \in \Pi$, then $X_\alpha = E_\alpha$ (see* [Jan96, *Proposition* 8.20]*).*
- *The products $X_{\beta_1}^{k_1} \dots X_{\beta_N}^{k_N}$ ($k_i \in \mathbb{N}$) form a basis of $\mathcal{U}_q^+(\mathfrak{g})$.*
- *The products $X_{\beta_1}^{k_1} \dots X_{\beta_N}^{k_N} K_{\alpha_1}^{m_1} \dots K_{\alpha_n}^{m_n} Y_{\beta_1}^{l_1} \dots Y_{\beta_N}^{l_N}$ (resp. $K_{\alpha_1}^{m_1} \dots K_{\alpha_n}^{m_n} Y_{\beta_1}^{l_1} \dots Y_{\beta_N}^{l_N} X_{\beta_1}^{k_1} \dots X_{\beta_N}^{k_N}$, resp. $Y_{\beta_1}^{l_1} \dots Y_{\beta_N}^{l_N} K_{\alpha_1}^{m_1} \dots K_{\alpha_n}^{m_n} X_{\beta_1}^{k_1} \dots X_{\beta_N}^{k_N}$), ($k_i, l_i \in \mathbb{N}$, $m_i \in \mathbb{Z}$) form a basis of $\mathcal{U}_q(\mathfrak{g})$.*

The following theorem was proved by Levendorskiĭ and Soibelman [LS91, Proposition 5.5.2] in a slightly different case. One can find other formulations in the literature (several containing small mistakes). That is why we give a proof of this result in [Mér08, Section 3.3]. We make this proof essentially by rewriting the one from [LS91, Proposition 5.5.2].

**Theorem 3.3.3** *(of Levendorskiĭ and Soibelman). If $i$ and $j$ are two integers such that $1 \leqslant i < j \leqslant N$, then we have*

$$X_{\beta_i} X_{\beta_j} - q^{(\beta_i, \beta_j)} X_{\beta_j} X_{\beta_i} = \sum_{\substack{\beta_i < \gamma_1 < \dots < \gamma_p < \beta_j \\ p \geqslant 1,\ k_i \in \mathbb{N}}} c_{\bar{\mathbf{k}}, \bar{\boldsymbol{\gamma}}} X_{\gamma_1}^{k_1} \dots X_{\gamma_p}^{k_p},$$

*where $c_{\bar{\mathbf{k}}, \bar{\boldsymbol{\gamma}}} \in \mathbb{K}$ and $c_{\bar{\mathbf{k}}, \bar{\boldsymbol{\gamma}}} \neq 0$ only if $wt(X_{\gamma_1}^{k_1} \dots X_{\gamma_p}^{k_p}) := k_1 \times \gamma_1 + \dots + k_p \times \gamma_p = \beta_i + \beta_j$.*

### 3.4. *Commutation relations between $X_\gamma$ in admissible planes*

The goal of this section is to show that the $X_\gamma$ satisfy analogous relations to the $E_\gamma$ (see Section 3.2). In order to achieve this aim, we start by introducing an intermediate generating system.

*3.4.1. Construction of a third generating system*

Let us recall the following well-known result:

**Lemma 3.4.1.** *(See [Jan96, Section 4.6].)*

1. *There is a unique automorphism $\omega$ of $\mathcal{U}_q(\mathfrak{g})$ such that $\omega(E_\alpha) = F_\alpha$, $\omega(F_\alpha) = E_\alpha$ and $\omega(K_\alpha) = K_\alpha^{-1}$. One has $\omega^2 = 1$.*
2. *There is a unique anti-automorphism $\tau$ of $\mathcal{U}_q(\mathfrak{g})$ such that $\tau(E_\alpha) = E_\alpha$, $\tau(F_\alpha) = F_\alpha$ and $\tau(K_\alpha) = K_\alpha^{-1}$. One has $\tau^2 = 1$.*

**Convention.**

- Let $i$ be an integer of $[\![1, n]\!]$. And set $T'_{\alpha_i} := \tau \circ T_{\alpha_i} \circ \tau$. This is an automorphism of $\mathcal{U}_q(\mathfrak{g})$ which satisfies the following conditions:

$$T'_{\alpha_i} E_{\alpha_i} = -K_{\alpha_i}^{-1} F_{\alpha_i}, \quad T'_{\alpha_i} F_{\alpha_i} = -E_{\alpha_i} K_{\alpha_i}, \quad T'_{\alpha_i} K_{\alpha_j} = K_{\alpha_j} K_{\alpha_i}^{-a_{ij}} \quad \bigl(j \in [\![1, n]\!]\bigr)$$

  and for $j \neq i$:

$$T'_{\alpha_i} E_{\alpha_j} = \sum_{r+s=-a_{ij}} (-1)^r q^{d_i s} E_{\alpha_i}^{(s)} E_{\alpha_j} E_{\alpha_i}^{(r)} \quad \text{and} \quad T'_{\alpha_i} F_{\alpha_j} = \sum_{r+s=-a_{ij}} (-1)^r q^{-d_i s} F_{\alpha_i}^{(r)} F_{\alpha_j} F_{\alpha_i}^{(s)}.$$

- If $w_p \in W$ has a reduced decomposition given by $w_p = s_{i_1} \dots s_{i_p}$, then we set $T'_{w_p} := \tau \circ T_{w_p} \circ \tau$. We have $T'_{w_p} = T'_{\alpha_{i_1}} \dots T'_{\alpha_{i_p}}$.
- If $\beta \in \Phi^+$, then we set $w_\beta := s_{i_1} \dots s_{i_\beta - 1}$ and we define $X'_\beta := T'_{w_\beta}(E_{\alpha_{i_\beta}})$ and $Y'_\beta := T_{w_\beta}(F_{\alpha_{i_\beta}})$. One has $X'_\alpha = E_\alpha$ and $Y'_\alpha = F_\alpha$ for $\alpha \in \Pi$.

The theorem of Levendorskiĭ and Soibelman can be rewritten as below. The proof can be found in [Mér08, Section 3.4]:

**Proposition 3.4.2.** *If $i$ and $j$ are two integers such that $1 \leqslant i < j \leqslant N$ then we have*

$$X'_{\beta_i} X'_{\beta_j} - q^{-(\beta_i, \beta_j)} X'_{\beta_j} X'_{\beta_i} = \sum_{\substack{\beta_i < \gamma_1 < \dots < \gamma_p < \beta_j \\ p \geqslant 1,\ k_i \in \mathbb{N}}} c_{\bar{\mathbf{k}}, \bar{\boldsymbol{\gamma}}} X'^{k_1}_{\gamma_1} \dots X'^{k_p}_{\gamma_p}$$

*with $c_{\bar{\mathbf{k}}, \bar{\boldsymbol{\gamma}}} \in \mathbb{K}$ and $c_{\bar{\mathbf{k}}, \bar{\boldsymbol{\gamma}}} \neq 0$ only if $wt(X'^{k_1}_{\gamma_1} \dots X^{k_p}_{'\gamma_p}) := k_1 \times \gamma_1 + \dots + k_p \times \gamma_p = \beta_i + \beta_j$.*

*3.4.2. Relations between $E_\beta$ and $X'_\beta$*

As in previous sections, $\Phi^+$ is provided with a given Lusztig order associated to a reduced decomposition of $w_0 = s_{i_1} \dots s_{i_N}$. In this case, we can improve the theorem of Levendorskiĭ and Soibelman.

**Theorem 3.4.3.** *If $i$ and $j$ are two integers such that $1 \leqslant i < j \leqslant N$, then one has*:

$$X'_{\beta_i} X'_{\beta_j} - q^{-(\beta_i, \beta_j)} X'_{\beta_j} X'_{\beta_i} = \sum_{\beta_i < \gamma_1 < \dots < \gamma_p < \beta_j} C_{\bar{\mathbf{k}}, \bar{\boldsymbol{\gamma}}} X'^{k_1}_{\gamma_1} \dots X'^{k_p}_{\gamma_p}.$$

*The monomials on the left-hand side whose coefficient $C_{\bar{\mathbf{k}}, \bar{\boldsymbol{\gamma}}}$ is not equal to zero satisfies*: $wt(X'^{k_1}_{\gamma_1} \dots X'^{k_p}_{\gamma_p}) =$ *$\beta_i + \beta_j$; $\gamma_1$ is not in the same box as $\beta_i$ and $\gamma_p$ is not in the same box as $\beta_j$.*

The proof of this theorem is essentially based on the following result:

**Lemma 3.4.4.** *Let $B = \{\beta_p, \dots, \beta_{p+l}\}$ be a box and $\alpha_{i_p}, \dots, \alpha_{i_{p+l}}$ be the corresponding simple roots. Then $\forall k \in [\![0, l]\!]$, we have*:

$$T'_{\alpha_{i_p}} \dots T'_{\alpha_{i_{p+k-1}}}(E_{\alpha_{i_{p+k}}}) = E_{\alpha_{i_{p+k}}} = T'_{\alpha_{i_p}} \dots T'_{i_{\alpha_{p+k-1}}} T'_{i_{\alpha_{p+k+1}}} \dots T'_{\alpha_{i_{p+l}}}(E_{\alpha_{i_{p+k}}}).$$

**Proof.** We already know that if $\alpha_1$ and $\alpha_2$ are two simple orthogonal roots, then $T_{\alpha_1}(E_{\alpha_2}) = E_{\alpha_2} = \tau(E_{\alpha_2})$, hence $T'_{\alpha_1}(E_{\alpha_2}) = E_{\alpha_2}$. As $\alpha_{i_p}, \dots, \alpha_{i_{p+l}}$ are orthogonal to each others by Proposition 2.2.7, the formulas above are proved. $\square$

**Proof of Theorem 3.4.3.** The first point is provided by Proposition 3.4.2. If in the reduced decomposition of $w_0$, we change the order of the reflexions associated to the simple roots coming from a single box $B$, we find a new reduced decomposition of $w_0$. The positive roots of $B$ constructed with this new decomposition of $w_0$ are permuted as the simple roots are but the other roots are not moved. By the previous lemma, the $X'_\beta$, $\beta \in B$, are also permuted in the same way but are not modified, and the $X'_\gamma$, $\gamma \notin B$, are not modified. Thus, without lost of generality, we can assume that $\beta_i$ is maximal in its box and that $\beta_j$ is minimal in its box. As a result, if $\beta_i < \gamma_1 < \dots < \gamma_p < \beta_j$, then $\gamma_1$ is not in the same box as $\beta_i$ and $\gamma_p$ is not in the same box as $\beta_j$. $\square$

**Remark 3.4.5.** The proof of the previous theorem can be rewritten with the elements $X_\beta (\beta \in \Phi_+)$ so that we also apply Theorem 3.4.3 to those elements.

We can now establish a link between the $X'_\beta$'s and the $E_\beta$'s.

**Theorem 3.4.6.**

$$\forall \beta \in \Phi^+,\ \exists \lambda_\beta \in \mathbb{K} \setminus \{0\} \quad \textit{such that} \quad X'_\beta = \lambda_\beta E_\beta.$$

**Proof.** Let $\beta$ and $\beta'$ be two positive roots such that $\beta > \beta'$. Set $[X'_\beta, X'_{\beta'}]_q = X'_\beta X'_{\beta'} - q^{(\beta,\beta')} X'_{\beta'} X'_\beta$.

Let us deal first with the case where $\Phi$ is of type $G_2$. We keep the conventions of Proposition 3.2.5. It is known (Conventions 3.4.1) that, since $\beta_1$ and $\beta_6$ are simple, one has $X'_{\beta_1} = E_{\beta_1}$ and $X'_{\beta_6} = E_{\beta_6}$.

Thus

$$\left[X'_{\beta_6}, X'_{\beta_1}\right]_q = [E_{\beta_6}, E_{\beta_1}]_q = \lambda E_{\beta_5} \quad \text{with } \lambda \in \mathbb{K}^\star.$$

By Theorem 3.4.3, one also has $[X'_{\beta_6}, X'_{\beta_1}]_q = \mu X'_{\beta_5}$ with $\mu \in \mathbb{K}$ and, then, $X'_{\beta_5} = \lambda_{\beta_5} E_{\beta_5}$ with $\lambda_{\beta_5} \in \mathbb{K}^\star$.

It implies that $[X'_{\beta_5}, X'_{\beta_1}]_q = \lambda_{\beta_5}[E_{\beta_5}, E_{\beta_1}]_q = \nu E_{\beta_3}$ with $\nu \in \mathbb{K}^\star$. We deduce as above that $X'_{\beta_3} = \lambda_{\beta_3} E_{\beta_3}$ with $\lambda_{\beta_3} \in \mathbb{K}^\star$.

Using the same method and considering $[X'_{\beta_3}, X'_{\beta_1}]_q = \lambda_{\beta_3}[E_{\beta_3}, E_{\beta_1}]_q$, one proves that $X'_{\beta_2} = \lambda_{\beta_2} E_{\beta_2}$ with $\lambda_{\beta_2} \in \mathbb{K}^\star$.

At last, one has $[X'_{\beta_5}, X'_{\beta_3}]_q = \lambda_{\beta_5}\lambda_{\beta_3}[E_{\beta_5}, E_{\beta_3}]_q = \nu E_{\beta_4}$ with $\nu \in \mathbb{K}^\star$, so it implies that $X'_{\beta_4} = \lambda_{\beta_4} E_{\beta_4}$ with $\lambda_{\beta_4} \in \mathbb{K}^\star$.

Suppose now that $\Phi$ is of type $G_2$, and consider a column $C_t$ ($t \in [\![1, n]\!]$). We just prove the theorem for all the roots of $C_t$.

We first study the case of ordinary roots.

Let $\beta \in C_t$ be an ordinary root. Let us prove the result by induction on $h(\beta)$.

If $h(\beta) = 1$, then $\beta = \alpha_t$ and as above $X'_{\alpha_t} = E_{\alpha_t}$.

Assume $h(\beta) > 1$ and the result proved for all $\delta \in C_t$ an ordinary root such that $h(\delta) < h(\beta)$. By Proposition 2.2.11, there is a simple root $\alpha_i$ ($i < t$) such that $\beta - \alpha_i = \gamma \in C_t$. Moreover, $\gamma$ is ordinary

$$C_t : \begin{array}{|c|}\hline \beta_k \\ \hline \vdots \\ \hline \gamma \\ \hline \beta \\ \hline \delta \\ \hline \vdots \\ \hline \beta_N \\ \hline \end{array}$$

**Fig. 1.**

because, if it is not the case $\beta = \gamma + \alpha_i$ would be exceptional which contradicts the uniqueness of an exceptional root in a column. So $P := \langle \alpha_i, \beta \rangle$ is an admissible plane of type (2.1), (2.2) or (2.3) and then $[E_\gamma, E_{\alpha_i}]_q = cE_\beta$ $(c \in \mathbb{K} \setminus \{0\})$ (see Section 3.2).

As $h(\gamma) = h(\beta) - 1 < h(\beta)$, one has $X'_\gamma = \lambda_\gamma E_\gamma$ $(\lambda_\gamma \in \mathbb{K} \setminus \{0\})$, and as $E_{\alpha_i} = X'_{\alpha_i}$, one has:

$$\left[X'_\gamma, X'_{\alpha_i}\right]_q = \lambda_\gamma [E_\gamma, E_{\alpha_i}]_q = \lambda_\gamma c E_\beta.$$

By Theorem 3.4.3, $E_\beta$ is a linear combination of monomials $X'_{\delta_1} \dots X'_{\delta_s}$ with $\alpha_i < \delta_1 \leqslant \dots \leqslant \delta_s < \gamma$, $\delta_s$ not in the same box as $\gamma$, $\delta_1$ not in the same box as $\alpha_i$ and

$$\delta_1 + \dots + \delta_s = \alpha_i + \gamma = \beta. \tag{$\star$}$$

For all monomials, $\delta_s \in C_t$ and $\delta_s$ is ordinary (because $\beta \in C_t$ and $\beta$ is ordinary). As $\delta_s < \gamma$ and $\delta_s$ does not belong to the same box as $\gamma$, one has $h(\delta_s) > h(\gamma)$. Hence $h(\delta_s) \geqslant h(\beta)$, so that $s = 1$ and $\delta_1 = \beta$. This implies $E_\beta = aX'_\beta$ with $a \in \mathbb{K} \setminus \{0\}$, and the result is proved.

Let us now assume that $\beta$ is the exceptional root of $C_t$. Let $\gamma$ be the root of $C_t$ which precedes $\beta$ in the Lusztig order and let $\delta = s_D(\gamma)$, so that $\delta + \gamma = \beta$ (see Fig. 1). By Proposition 2.2.12, one has $h'(\beta) = m + \frac{1}{2}$ with $m \in \mathbb{N}^\star$ and $h'(C_t) = [\![1, 2m]\!]$. If $B$ is the box in $C_t$ which precedes $\beta$, then $h'(B) = h(B) = t + 1$. As $\beta$ is alone in its box, we have $\gamma \in B$, so that $h(\gamma) = m + 1$. Hence $h(\delta) = m$. Thus $P = \mathrm{Vect}(\gamma, \delta)$ is an admissible plane of type (1.1) or (1.2), and $[E_\delta, E_\gamma]_q = cE_\beta (c \neq 0)$ (see Section 3.2).

As $\gamma$ and $\delta$ are ordinary roots, we already know that $X'_\gamma = \lambda_\gamma E_\gamma$ and $X'_\delta = \lambda_\delta E_\delta$ with $\lambda_\gamma \neq 0$ and $\lambda_\delta \neq 0$. Thus, one has:

$$\left[X'_\delta, X'_\gamma\right]_q = \lambda_\gamma \lambda_\delta [E_\delta, E_\gamma]_q = \lambda_\gamma \lambda_\delta c E_\beta \ (\lambda_\gamma \neq 0, \lambda_\delta \neq 0).$$

As above, $E_\beta$ is a linear combination of monomials $X'_{\delta_1} \dots X'_{\delta_s}$ with $\gamma < \delta_1 \leqslant \dots \leqslant \delta_s < \delta$, $\delta_s$ not in the same box as $\delta$ and $\delta_1$ not in the same box as $\gamma$. As $\beta$ is the only root of $C_t$ which satisfies $\gamma < \beta < \delta$, $\beta$ is not in the same box as $\delta$ and $\beta$ is not in the same box as $\gamma$. Hence $s = 1$ and $\delta_1 = \beta$. So that $E_\beta = aX'_\beta$ with $a \in \mathbb{K} \setminus \{0\}$. □

From Theorems 3.4.3 and 3.4.6, we deduce the following result.

**Corollary 3.4.7.** *If $i$ and $j$ are two integers such that $1 \leqslant i < j \leqslant N$, one has:*

$$E_{\beta_i} E_{\beta_j} - q^{-(\beta_i, \beta_j)} E_{\beta_j} E_{\beta_i} = \sum_{\substack{\beta_i < \gamma_1 < \dots < \gamma_p < \beta_j \\ p \geqslant 1,\ k_i \in \mathbb{N}}} C'_{\overline{\mathbf{k}}, \overline{\boldsymbol{\gamma}}} E_{\gamma_1}^{k_1} \dots E_{\gamma_p}^{k_p}.$$

*The monomials on the left-hand side whose coefficient $C'_{\overline{\mathbf{k}}, \overline{\boldsymbol{\gamma}}}$ is not equal to zero satisfies: $wt(X'^{k_1}_{\gamma_1} \dots X'^{k_p}_{\gamma_p}) = \beta_i + \beta_j$; $\gamma_1$ is not in the same box as $\beta_i$ and $\gamma_p$ is not in the same box as $\beta_j$.*

*3.4.3. Link with Jantzen's construction*

**Proposition 3.4.8.** *Let $\beta_1 < \beta_2$ be two positive roots.*

1. *If $E_{\beta_1}E_{\beta_2} - q^{-(\beta_1,\beta_2)}E_{\beta_2}E_{\beta_1} = kE_\gamma^m$ ($k \neq 0, m \geqslant 1$ and $\gamma \in \Phi^+$), then $X_{\beta_1}X_{\beta_2} - q^{+(\beta_1,\beta_2)}X_{\beta_2}X_{\beta_1} = k'X_\gamma^m$ ($k' \neq 0$).*
2. *If $E_{\beta_1}E_{\beta_2} - q^{-(\beta_1,\beta_2)}E_{\beta_2}E_{\beta_1} = kE_\gamma E_\delta$ ($k \neq 0, \gamma, \delta \in \Phi^+, \gamma$ and $\delta$ belonging to the same box), then $X_{\beta_1}X_{\beta_2} - q^{+(\beta_1,\beta_2)}X_{\beta_2}X_{\beta_1} = k'X_\gamma X_\delta$ ($k' \neq 0$).*

**Proof.** Let $\beta \in \Phi^+$. Let us recall (see Section 3.4.1) that $X_\beta := T_{w'_\beta}(E_{\alpha_{i_\beta}})$, $X'_\beta := T'_{w'_\beta}(E_{\alpha_{i_\beta}})$, and that $T_{w'_\beta} = \tau \circ T'_{w'_\beta} \circ \tau$. So we have $X_\beta = \tau \circ T'_{w'_\beta} \circ \tau(E_{\alpha_{i_\beta}}) = \tau(X'_\beta)$. Let us also recall (see Theorem 3.4.6) that $X'_\beta = \lambda_\beta E_\beta$ with $\lambda_\beta \in \mathbb{K}^\star$.

Let $\beta_1 < \beta_2$ be two positive roots.

1. If $E_{\beta_1}E_{\beta_2} - q^{-(\beta_1,\beta_2)}E_{\beta_2}E_{\beta_1} = kE_\gamma^m$ ($k \neq 0, \gamma \in \Phi^+$), then:

$$\begin{aligned}
X_{\beta_1}X_{\beta_2} - q^{+(\beta_1,\beta_2)}X_{\beta_2}X_{\beta_1} &= \tau\left(X'_{\beta_1}\right)\tau\left(X'_{\beta_2}\right) - q^{+(\beta_1,\beta_2)}\tau(X_{\beta_2})\tau(X_{\beta_1}) \\
&= \tau\left(X'_{\beta_2}X'_{\beta_1} - q^{+(\beta_1,\beta_2)}X'_{\beta_1}X'_{\beta_2}\right) \\
&= -q^{+(\beta_1,\beta_2)}\tau\left(X'_{\beta_1}X'_{\beta_2} - q^{-(\beta_1,\beta_2)}X'_{\beta_2}X'_{\beta_1}\right) \\
&= -q^{+(\beta_1,\beta_2)}\lambda_{\beta_1}\lambda_{\beta_2}\tau\left(E_{\beta_1}E_{\beta_2} - q^{-(\beta_1,\beta_2)}E_{\beta_2}E_{\beta_1}\right) \\
&= -q^{+(\beta_1,\beta_2)}\lambda_{\beta_1}\lambda_{\beta_2}\tau\left(kE_\gamma^m\right) \\
&= \frac{-q^{+(\beta_1,\beta_2)}\lambda_{\beta_1}\lambda_{\beta_2}k}{\lambda_\gamma}\tau\left(\left(X'_\gamma\right)^m\right) = k'X_\gamma^m \quad \text{with } k' \in \mathbb{K}^\star.
\end{aligned}$$

2. If $E_{\beta_1}E_{\beta_2} - q^{-(\beta_1,\beta_2)}E_{\beta_2}E_{\beta_1} = kE_\gamma E_\delta$ ($k \neq 0, \gamma, \delta \in \Phi^+, \gamma$ and $\delta$ belonging two the same box) so, by doing the same computations as in 1., we obtain:

$$X_{\beta_1}X_{\beta_2} - q^{(\beta_1,\beta_2)}X_{\beta_2}X_{\beta_1} = k'\tau\left(X'_\gamma X'_\delta\right) = k'X_\delta X_\gamma \quad \left(k' \neq 0\right).$$

As $\gamma$ and $\delta$ are in the same box, we know (see Proposition 2.2.7) that $(\delta, \gamma) = 0$, so that, by Theorem 3.3.3, we get $X_\gamma X_\delta = X_\gamma X_\delta$, as desired. □

## 4. Deleting derivations in $\mathcal{U}_q^+(\mathfrak{g})$

*4.1. $\mathcal{U}_q^+(\mathfrak{g})$ is a CGL extension*

In this section, we set $A := \mathcal{U}_q^+(\mathfrak{g})$, $X_i := X_{\beta_i}$ for $1 \leqslant i \leqslant N$, and $\lambda_{i,j} := q^{-(\beta_j,\beta_i)}$ for $1 \leqslant i, j \leqslant N$. We know from Proposition 3.3.3 that, if $1 \leqslant i < j \leqslant N$, then one has:

$$X_jX_i - \lambda_{j,i}X_iX_j = P_{j,i} \tag{1}$$

with

$$P_{j,i} = \sum_{\bar{k}=(k_{i+1},\dots,k_{j-1})} c_{\bar{k}}X_{i+1}^{k_{i+1}}\dots X_{j-1}^{k_{j-1}}, \tag{2}$$

where $c_{\bar{k}} \in \mathbb{K}$. Moreover, as $\mathcal{U}_q^+(\mathfrak{g})$ is $\Phi$-gradued, one has

$$c_{\bar{k}} \neq 0 \quad \Rightarrow \quad \lambda_{l,i+1}^{k_{i+1}} \dots \lambda_{l,j-1}^{k_{j-1}} = \lambda_{l,j}\lambda_{l,i} \quad \text{for all } 1 \leqslant l \leqslant N. \tag{3}$$

Thus, $A$ satisfies [Cau03a, Hypothesis 6.1.1]. From Theorem 3.3.2, ordered monomials in $X_i$ are a basis of $A$, so that we deduce from [Cau03a, Proposition 6.1.1]:

**Proposition 4.1.1.**

1. *A is skew polynomial ring which could be expressed as*:

$$A = \mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \dots [X_N; \sigma_N, \delta_N],$$

*where the $\sigma_j$'s are $\mathbb{K}$-linear automorphisms and the $\delta_j$'s are $\mathbb{K}$-linear $\sigma_j$-derivations such that, for $1 \leqslant i < j \leqslant N$, $\sigma_j(X_i) = \lambda_{j,i} X_i$ and $\delta_j(X_i) = P_{j,i}$.*

2. *If $1 \leqslant m \leqslant N$, then there is a (unique) automorphism $h_m$ of the algebra A which satisfies $h_m(X_i) = \lambda_{m,i} X_i$ for $1 \leqslant i \leqslant N$.*

Moreover, we deduce from [Cau03a, Proposition 6.1.2] the following result.

**Proposition 4.1.2.**

1. *A satisfies conventions from* [Cau03a, *Section* 3.1]*, that is to say*:
   - *For all $j \in [\![2, N]\!]$, $\sigma_j$ is a $\mathbb{K}$-linear automorphism and $\delta_j$ is a $\mathbb{K}$-linear (left sided) $\sigma_j$-derivation and locally nilpotent.*
   - *For all $j \in [\![2, N]\!]$, one has $\sigma_j \circ \delta_j = q_j \delta_j \circ \sigma_j$ with $q_j = \lambda_{j,j} = q^{-\|\beta_j\|^2}$, and for all $i \in [\![1, j-1]\!]$, $\sigma_j(X_i) = \lambda_{j,i} X_i$.*
   - *None of the $q_j$ $(2 \leqslant j \leqslant N)$ is a root of unity.*
2. *A satisfies* [Cau03a, *Hypothesis* 4.1.2]*, that is to say*:
   *The subgroup H of the automorphisms group of A generated by the elements $h_l$ satisfies*:
   - *For all h in H, the indeterminates $X_1, \dots, X_N$ are h-eigenvectors.*
   - *The set $\{\lambda \in \mathbb{K}^* \mid (\exists h \in H)\ h(X_1) = \lambda X_1\}$ is infinite.*
   - *If $m \in [\![2, N]\!]$, there is $h_m \in H$ such that $h_m(X_i) = \lambda_{m,i} X_i$ if $1 \leqslant i < m$ and $h_m(X_m) = q_m X_m$.*

The previous proposition shows that $\mathcal{U}_q^+(\mathfrak{g})$ is a CGL extension in the sens of [LLR06] and so allows us to apply the deleting derivation theory [Cau03a]. We describe this theory in the following section.

### *4.2. The deleting derivation algorithm*

It follows from Propositions 4.1.1 and 4.1.2, that $A$ is an integral domain which is Noetherian. Denote by $F$ its fields of fraction. We define, by induction, the families $X^{(l)} = (X_i^{(l)})_{1 \leqslant i \leqslant N}$ of elements of $F^\star := F \setminus \{0\}$, and the algebras $A^{(l)} := \mathbb{K}\langle X_1^{(l)}, \dots, X_N^{(l)}\rangle$ when $l$ decreases from $N+1$ to 2 as in [Cau03a, Section 3.2]. So we have for all $l \in [\![2, N+1]\!]$:

**Lemma 4.2.1.** *If $1 \leqslant i < j \leqslant N$, one has*:

$$X_j^{(l)} X_i^{(l)} - \lambda_{j,i} X_i^{(l)} X_j^{(l)} = P_{j,i}^{(l)} \tag{4}$$

*with*

$$P_{j,i}^{(l)} = \begin{cases} 0 & \text{if } j \geqslant l, \\ \sum_{\bar{k}=(k_{i+1},\dots,k_{j-1})} c_{\bar{k}} (X_{i+1}^{(l)})^{k_{i+1}} \dots (X_{j-1}^{(l)})^{k_{j-1}} & \text{if } j < l, \end{cases} \tag{5}$$

*where* $c_{\bar{k}}$ *are* the same *as in the formula* (2)*, so that we also have the implication* (3)*.*

**Proof.** See [Cau03a, Théorème 3.2.1]. □

**Lemma 4.2.2.** *The ordered monomials on* $X_1^{(l)}, \dots, X_N^{(l)}$ *form a basis* $A^{(l)}$ *as a* $\mathbb{K}$*-vectorial space.*

**Proof.** See [Cau03a, Théorème 3.2.1]. □

From Lemmas 4.2.1 and 4.2.2 above and from [Cau03a, Proposition 6.1.1], we deduce that:

**Lemma 4.2.3.**

1. $A^{(l)}$ *is an iterated ore extension which can be written*:

$$A^{(l)} = \mathbb{K}\big[X_1^{(l)}\big]\big[X_2^{(l)}; \sigma_2^{(l)}, \delta_2^{(l)}\big] \dots \big[X_N^{(l)}; \sigma_N^{(l)}, \delta_N^{(l)}\big]$$

*where* $\sigma_j^{(l)}$ *are* $\mathbb{K}$*-linear automorphisms and* $\delta_j^{(l)}$ *are* $\mathbb{K}$*-linear* (*left sided*) $\sigma_j^{(l)}$*-derivations such that, for* $1 \leqslant i < j \leqslant N$, $\sigma_j^{(l)}(X_i^{(l)}) = \lambda_{j,i} X_i^{(l)}$ *and* $\delta_j^{(l)}(X_i^{(l)}) = P_{j,i}^{(l)}$*.*

2. $A^{(l)}$ *is the* $\mathbb{K}$ *algebra generated by the elements* $X_1^{(l)}, \dots, X_N^{(l)}$ *with relations* (4)*.*

Let us recall that the automorphisms $h_m$ $(1 \leqslant m \leqslant N)$ of the algebra $A$ defined in Proposition 4.1.1 can be extended (uniquely) in automorphisms, also denoted by $h_m$, of the field $F$.

**Lemma 4.2.4.** *If* $1 \leqslant m, i \leqslant N$*, one has* $h_m(X_i^{(l)}) = \lambda_{m,i} X_i^{(l)}$ *so that* $h_m$ *induces* (*by restriction*) *an automorphism of the algebra* $A^{(l)}$*, denoted by* $h_m^{(l)}$*.*

**Proof.** See [Cau03a, Lemme 4.2.1]. □

**Convention.** Denote by $H^{(l)}$ the subgroup of the automorphism group of $A^{(l)}$ generated by $h_m^{(l)}$ $(1 \leqslant m \leqslant N)$.

By [Cau03a, Proposition 6.1.2], one has:

**Lemma 4.2.5.** *The iterated Ore extension* $A^{(l)} = \mathbb{K}[X_1^{(l)}][X_2^{(l)}; \sigma_2^{(l)}, \delta_2^{(l)}] \dots [X_N^{(l)}; \sigma_N^{(l)}, \delta_N^{(l)}]$ *satisfies the conventions of* [Cau03a, *Section* 3.1] *with, as above,* $\lambda_{i,j} = q^{-(\beta_i,\beta_j)}$ *and* $q_i = \lambda_{i,i} = q^{-\|\beta_i\|^2}$ *for* $1 \leqslant i, j \leqslant N$*. It also satisfies the Hypothesis* 4.1.2 *of* [Cau03a] *with* $H^{(l)}$ *replacing* $H$*.*

**Corollary 4.2.6.** *If* $J$ *is an* $H^{(l)}$*-prime ideal of* $A^{(l)}$ *in the sense of* [BG02, II.1.9]*, then* $J$ *is completely prime.*

**Proof.** One has:

- $A^{(l)} = \mathbb{K}[X_1^{(l)}][X_2^{(l)}; \sigma_2^{(l)}, \delta_2^{(l)}] \dots [X_N^{(l)}; \sigma_N^{(l)}, \delta_N^{(l)}]$ is an iterated Ore extension by Lemma 4.2.3.
- $X_1^{(l)}, X_2^{(l)}, \dots, X_N^{(l)}$ are $H^{(l)}$-eigenvectors by Lemma 4.2.4.
- If $1 \leqslant i < j \leqslant N$, then one has $h_i^{(l)}(X_j^{(l)}) = \lambda_{j,i} X_i^{(l)} = \sigma_j^{(l)}(X_i^{(l)})$ and $h_j^{(l)}(X_j^{(l)}) = q_j X_j^{(l)}$ with $q_j = \lambda_{j,j} \in \mathbb{K}^\star$ is not a root of unity by Lemmas 4.2.3 and 4.2.4.

Hence we deduce from [BG02, Theorem II.5.12] that $J$ is completely prime. □

From the construction of the deleting algorithm (see [Cau03a, Section 3.2]), one has:

**Lemma 4.2.7.**

1. $X_i^{(N+1)} = X_i$ *for all* $i \in [\![1, N]\!]$.
2. *If* $2 \leqslant l \leqslant N$ *and if* $i \in [\![1, N]\!]$, *one has*

$$X_i^{(l)} = \begin{cases} X_i^{(l+1)} & \text{if } i \geqslant l, \\ \sum_{n=0}^{+\infty}[\frac{(1-q_l)^{-n}}{[n]!_{q_l}}(\delta_l^{(l+1)})^n \circ (\sigma_l^{(l+1)})^{-n}(X_i^{(l+1)})](X_l^{(l+1)})^{-n} & \text{if } i < l. \end{cases} \tag{6}$$

**Lemma 4.2.8.** *Let $J$ be an $H^{(l)}$-invariant (two sided) ideal of $A^{(l)}$. Let us consider an integer $j \in [\![2, N]\!]$ and denote by $B = \mathbb{K}[X_1^{(l)}][X_2^{(l)}; \sigma_2^{(l)}, \delta_2^{(l)}]\dots[X_{j-1}^{(l)}; \sigma_{j-1}^{(l)}, \delta_{j-1}^{(l)}]$ the subalgebra of $A^{(l)}$ generated by $X_1^{(l)}, \dots, X_{j-1}^{(l)}$. Then $\sigma_j^{(l)}(B \cap J) = B \cap J$ and $\delta_j^{(l)}(B \cap J) \subset B \cap J$.*

**Proof.** By Lemmas 4.2.3 and 4.2.4, one has for $1 \leqslant i < j$,

$$\sigma_j^{(l)}\big(X_i^{(l)}\big) = \lambda_{j,i} X_i^{(l)} = h_j^{(l)}\big(X_i^{(l)}\big). \tag{7}$$

As a result, for all $b \in B$, $\sigma_j^{(l)}(b) = h_j^{(l)}(b)$. As $J$ is $H^{(l)}$-invariant, and as $B$ is $\sigma_j^{(l)}$-invariant, we deduce that, for all $b \in B \cap J$, we have $\sigma_j^{(l)}(b) \in B \cap J$. So, $\sigma_j^{(l)}(B \cap J) \subset B \cap J$. From the equality (7), we get that:

$$\big(\sigma_j^{(l)}\big)^{-1}\big(X_i^{(l)}\big) = \lambda_{j,i}^{-1} X_i^{(l)} = \big(h_i^{(l)}\big)^{-1}\big(X_i^{(l)}\big). \tag{8}$$

As above, we deduce that $(\sigma_j^{(l)})^{-1}(B \cap J) \subset B \cap J$, so that $\sigma_j^{(l)}(B \cap J) = B \cap J$.

Finally, if $b \in B \cap J$, then we have $\delta_j^{(l)}(b) = X_j^{(l)} b - \sigma_j^{(l)}(b) X_j^{(l)} \in B \cap J$. $\square$

If $l \in [\![2, N]\!]$, then it follows from (6) that $X_l^{(l)} = X_l^{(l+1)}$. This element is a nonzero element which belongs to the two algebras $A^{(l)}$ and $A^{(l+1)}$ (recall that none of the $X_i^{(l)}$ is null). So, the set $S_l := \{(X_l^{(l)})^p \mid p \in \mathbb{N}\}$ is a multiplicative system of regular elements of $A^{(l)}$ and $A^{(l+1)}$. From [Cau03a, Theorem 3.2.1], we deduce:

**Lemma 4.2.9.** *Let $l \in [\![2, N]\!]$. Then $S_l$ is an Ore set in $A^{(l)}$ and also in $A^{(l+1)}$. Moreover, one has:*

$$A^{(l)} S_l^{-1} = A^{(l+1)} S_l^{-1}.$$

*4.3. Prime spectrum and diagrams*

Let us recall that the convention are $X_i = X_{\beta_i}$ for $1 \leqslant i \leqslant N$. Denote $\overline{A} := A^{(2)} = \mathbb{K}\langle T_{\beta_1}, \dots, T_{\beta_N}\rangle$ with $T_{\beta_i} = X_i^{(2)}$ for all $i$. By Lemmas 4.2.1 and 4.2.3, $\overline{A}$ is the quantum affine space generated by $T_{\beta_i}$ ($1 \leqslant i \leqslant N$) with relations $T_{\beta_j} T_{\beta_i} = \lambda_{j,i} T_{\beta_i} T_{\beta_j}$ for $1 \leqslant i < j \leqslant N$.

Let us consider an integer $l \in [\![2, N]\!]$ and a prime ideal $P \in \mathrm{Spec}(A^{(l+1)})$.

- Assume $X_l^{(l+1)} \notin P$. Then, by [Cau03a, Lemmas 4.2.2 and 4.3.1], we have $S_l \cap P = \emptyset$ and $Q := A^{(l)} \cap P S_l^{-1} \in \mathrm{Spec}(A^{(l)})$.

- Assume $X_l^{(l+1)} \in P$. Then, by [Cau03a, Lemma 4.3.2], there is a (unique) surjective algebra homomorphism

$$g : A^{(l)} \to \frac{A^{(l+1)}}{(\mathcal{P}^{(l+1)})}$$

which satisfies, for all $i$, $g(X_i^{(l)}) = \overline{X_i^{(l+1)}}$ $(:= X_i^{(l+1)} + (\mathcal{P}^{(l+1)}))$, so that $Q = g^{-1}(\frac{P}{(X_l^{(l+1)})}) \in \mathrm{Spec}(A^{(l)})$.

We define this way a map $\phi_l : \mathrm{Spec}(A^{(l+1)}) \to \mathrm{Spec}(A^{(l)})$ that maps $P$ to $Q$ and, by composing these maps, we obtain a map $\phi = \phi_2 \circ \cdots \circ \phi_N$: $\mathrm{Spec}(A) \to \mathrm{Spec}(\bar{A})$. By [Cau03a, Proposition 4.3.1], one has:

**Lemma 4.3.1.** *Each $\phi_l$ $(2 \leqslant l \leqslant N)$ is injective, so that $\phi$ is injective.*

We can now define the notion of diagrams and Cauchon diagrams.

**Definition 4.3.2.**

1. We call *diagram* a subset $\Delta$ of the set of positive roots $\Phi^+$, and we note:

$$\mathrm{Spec}_\Delta(\overline{A}) := \left\{ Q \in \mathrm{Spec}(\overline{A}) \;\middle|\; Q \cap \{T_{\beta_1}, \ldots, T_{\beta_N}\} = \{T_\beta \mid \beta \in \Delta\} \right\}.$$

2. A diagram $\Delta$ is a *Cauchon diagram* if there is $P \in \mathrm{Spec}(A)$ such that $\phi(P) \in \mathrm{Spec}_\Delta(\overline{A})$, that is to say, if $\phi(P) \cap \{T_{\beta_1}, \ldots, T_{\beta_N}\} = \{T_\beta \mid \beta \in \Delta\}$. In this case, we set

$$\mathrm{Spec}_\Delta(A) = \left\{ P \in \mathrm{Spec}(A) \;\middle|\; \phi(P) \in \mathrm{Spec}_\Delta(\overline{A}) \right\}.$$

By [Cau03a, Theorems 5.1.1, 5.5.1 and 5.5.2], we have:

**Proposition 4.3.3.**

1. *If $\Delta$ is a Cauchon diagram, then $\phi(\mathrm{Spec}_\Delta(A)) = \mathrm{Spec}_\Delta(\overline{A})$ and $\phi$ induced a bi-increasing homeomorphism from $\mathrm{Spec}_\Delta(A)$ onto $\mathrm{Spec}_\Delta(\overline{A})$.*
2. *The family $\mathrm{Spec}_\Delta(A)$ (with $\Delta$ Cauchon diagram) coincide with the Goodearl–Letzter $H$-stratification of $\mathrm{Spec}(A)$ [BG02].*

In the following section, we describe more precisely Cauchon Diagrams. In order to do this, the criteria in the next proposition will be needed.

**Proposition 4.3.4.** *Let $P^{(m)}$ be an $H$-prime ideal of $A^{(m)}$. $P^{(m)} \in \mathrm{Im}(\phi_m)$ if and only if one of two following conditions is satisfied.*

1. $X_m^{(m)} \notin P^{(m)}$.
2. $X_m^{(m)} \in P^{(m)}$ *and* $\Theta^{(m)}(\delta_m^{(m+1)}(X_i^{(m+1)})) \in P^{(m)}$ *for* $1 \leqslant i \leqslant m-1$ (*where* $\delta_m^{(m+1)}(X_i^{(m+1)}) = P_{m,i}^{(m+1)}(X_{i+1}^{(m+1)}, \ldots, X_{m-1}^{(m+1)})$ (*Lemma* 4.2.1) *and* $\Theta^{(m)} : \mathbb{K}\langle X_1^{(m+1)}, \ldots, X_{m-1}^{(m+1)}\rangle \to \mathbb{K}\langle X_1^{(m)}, \ldots, X_{m-1}^{(m)}\rangle$ *is the homomorphism which send* $X_l^{(m+1)}$ *to* $X_l^{(m)}$).

**Proof.** Assume that $P^{(m)} \in \mathrm{Im}(\phi_m)$, so that $P^{(m)} = \phi_m\ (P^{(m+1)})$ with $P^{(m+1)} \in \mathrm{Spec}(A^{(m+1)})$, and assume that condition 1. is not satisfied. This implies that $P^{(m)} = \ker(g)$ where $g : A^{(m)} \to$

$A^{(m+1)}/P^{(m+1)}$ is the homomorphism which sends $X_i^{(m)}$ to $x_i^{(m+1)} := X_i^{(m+1)} + P^{(m+1)}$. Let $1 \leqslant i \leqslant m-1$. Recall that $\delta_m^{(m+1)}(X_i^{(m+1)}) = P_{m,i}^{(m+1)}(X_{i+1}^{(m+1)}, \dots, X_{m-1}^{(m+1)})$ and that $\Theta^{(m)} : k\langle X_1^{(m+1)}, \dots, X_{m-1}^{(m+1)}\rangle \to k\langle X_1^{(m)}, \dots, X_{m-1}^{(m)}\rangle$ is the homomorphism which transforms each $X_l^{(m+1)}$ in $X_l^{(m)}$. Since $X_m^{(m)} \in P^{(m)}$, we have $X_m^{(m+1)} \in P^{(m+1)}$ [Cau03a, Proposition 4.3.1] and so, $\delta_m^{(m+1)}(X_i^{(m+1)}) \in P^{(m+1)}$. Now, we have $g(\Theta^{(m)}(\delta_m^{(m+1)}(X_i^{(m+1)}))) = g(\Theta^{(m)}(P_{m,i}^{(m+1)}(X_{i+1}^{(m+1)}, \dots, X_{m-1}^{(m+1)}))) = g(P_{m,i}^{(m+1)}(X_{i+1}^{(m)}, \dots, X_{m-1}^{(m)})) = P_{m,i}^{(m+1)}(x_{i+1}^{(m+1)}, \dots, x_{m-1}^{(m+1)}) = P_{m,i}^{(m+1)}(X_{i+1}^{(m+1)}, \dots, X_{m-1}^{(m+1)}) + P^{(m+1)} = 0$. This implies that $\Theta^{(m)}(\delta_m^{(m+1)}(X_i^{(m+1)})) \in \ker(g) = P^{(m)}$.

If condition 1. is satisfied, then $P^{(m)} \in \mathrm{Im}(\phi_m)$ by [Cau03a, Lemma 4.3.1].

Assume that condition 2. is satisfied. Let $1 \leqslant i \leqslant m-1$. Then we have, as previously, $P_{m,i}^{(m+1)}(X_{i+1}^{(m)}, \dots, X_{m-1}^{(m)}) = \Theta^{(m)}(\delta_m^{(m+1)}(X_i^{(m+1)})) \in P^{(m)}$. So, in $Q^{(m)} = A^{(m)}/P^{(m)}$, we have $P_{m,i}^{(m+1)}(x_{i+1}^{(m)}, \dots, x_{m-1}^{(m)}) = 0$.

Since $P_{m,i}^{(m)} = 0$ (see Lemma 4.2.1), we can write $x_m^{(m)}x_i^{(m)} - \lambda_{m,i}x_i^{(m)}x_m^{(m)} = P_{m,i}^{(m)}(x_{i+1}^{(m)}, \dots, x_{m-1}^{(m)}) = 0 = P_{m,i}^{(m+1)}(x_{i+1}^{(m)}, \dots, x_{m-1}^{(m)})$.

If $1 \leqslant i \leqslant j-1$ with $j \neq m$, it follows from Lemma 4.2.1 that:

$$x_j^{(m)}x_i^{(m)} - \lambda_{j,i}x_i^{(m)}x_j^{(m)} = P_{j,i}^{(m)}\big(x_{i+1}^{(m)}, \dots, x_{j-1}^{(m)}\big) = P_{j,i}^{(m+1)}\big(x_{i+1}^{(m)}, \dots, x_{j-1}^{(m)}\big).$$

So, by the universal property of algebras defined by generators and relations, there exists a (unique) homomorphism $\epsilon : A^{(m+1)} \to Q^{(m)}$ which sends $X_l^{(m+1)}$ to $x_l^{(m)}$ for all $l$. This homomorphism is surjective, and its kernel $\ker(\epsilon) = P^{(m+1)}$ is a prime ideal of $A^{(m+1)}$. We observe that, since $X_m^{(m)} \in P^{(m)}$, we have $X_m^{(m+1)} \in P^{(m+1)}$, and that $\epsilon$ induces an automorphism

$$\overline{\epsilon} : A^{(m+1)}/P^{(m+1)} \to Q^{(m)} = A^{(m)}/P^{(m)}$$

which sends $x_l^{(m+1)}$ to $x_l^{(m)}$ for all $l$. Recall that $f_m : A^{(m)} \to A^{(m)}/P^{(m)}$ denotes the canonical homomorphism. So, $g = (\overline{\epsilon})^{-1} \circ f_m : A^{(m)} \to A^{(m+1)}/P^{(m+1)}$ is the homomorphism which sends $X_l^{(m)}$ to $x_l^{(m+1)}$ for all $l$. As $\ker(g) = \ker(f_m) = P^{(m)}$, we conclude that $P^{(m)} = \phi_m(P^{(m+1)})$, as desired. $\square$

## 5. Cauchon diagrams in $\mathcal{U}_q^+(\mathfrak{g})$

In [Cau03b], Cauchon uses a combinatorial tool to describe "admissible diagrams" (which are called "Cauchon diagrams" here) for the algebra $O_q(M_n(k))$ of quantum matrices. Thanks to Lusztig admissible planes theory (see Section 3.2), results from Section 3.3 and the deleting derivation theory, we describe those diagrams for $\mathcal{U}_q^+(\mathfrak{g})$ (where $\mathfrak{g}$ is a simple Lie algebra of finite dimension over $\mathbb{C}$). The goal of this section is to prove the following statement:

**Theorem.** *A diagram $\Delta \subset \Phi^+$ satisfies all the implications from admissible planes (to be defined) if and only if $\Delta$ is a Cauchon diagram (in the sense of Definition 4.3.2).*

### 5.1. Implications in a diagram

**Lemma 5.1.1.** *Let $j \in [\![1, N]\!]$, $l \in [\![2, N]\!]$, $P^{(l+1)}$ be a prime ideal of $A^{(l+1)}$ and $P^{(l)} := \varphi_l(P^{(l+1)})$.*

1. *If $X_j^{(l+1)} \in P^{(l+1)}$, then $X_j^{(l)} \in P^{(l)}$.*
2. *If $X_j^{(l+1)} = X_j^{(l)}$ (this is in particular the case if $j \geqslant l$), then one has: $X_j^{(l+1)} \in P^{(l+1)}$ if and only if $X_j^{(l)} \in P^{(l)}$.*

**Proof.** The second point can be shown as in [Cau03a, Lemma 4.3.4]. Let us show the first point when $j < l$.

*1st case*: The pivot (in reference to Gaussian elimination) $\varpi := X_l^{(l+1)}$ belongs to $P^{(l+1)}$. Recall (see Section 4.2) that there is a surjective homomorphism of algebra

$$g : A^{(l)} \to \frac{A^{(l+1)}}{(\mathcal{P}^{(l+1)})}$$

which satisfies $g(X_i^{(l)}) = \overline{X_i^{(l+1)}}$ $(:= X_i^{(l+1)} + (\mathcal{P}^{(l+1)}))$ for all $i \in [\![1, N]\!]$. As $X_j^{(l+1)} \in P^{(l+1)}$, one has $g(X_j^{(l)}) \in \frac{P^{(l+1)}}{(X_l^{(l+1)})}$, so that $X_j^{(l)} \in g^{-1}(\frac{P^{(l+1)}}{(X_l^{(l+1)})}) =: P^{(l)}$.

*2nd case*: The pivot $\varpi = X_l^{(l+1)}$ does not belongs to $P^{(l+1)}$. Set $S_l := \{\varpi^n \mid n \in \mathbb{N}\}$. Recall (see Section 4.2) that we have $P^{(l)} = A^{(l)} \cap (P^{(l+1)} S_l^{-1})$.

Set $J := \bigcap_{h \in H^{(l+1)}} h(P^{(l+1)})$ and observe that $J$ is an $H^{(l+1)}$-invariant two-sided ideal by construction. As $A^{(l+1)}$ [Cau03a, Hypothesis 4.1.2] by Lemma 4.2.5, $X_j^{(l+1)}$ is an $H^{(l+1)}$-eigenvector. Thus, since $X_j^{(l+1)}$ belongs to $P^{(l+1)}$, it also belongs to $J$.

From Lemma 4.2.8, we deduce that $(\delta_l^{(l+1)})^n \circ (\sigma_l^{(l+1)})^{-n}(X_j^{(l+1)}) \in J \subset P^{(l+1)}$ for all $n \in \mathbb{N}$. As a result, we get:

$$X_j^{(l)} = \sum_{n=0}^{+\infty} \left[ \frac{(1-q_l)^{-n}}{[n]!_{q_l}} \big(\delta_l^{(l+1)}\big)^n \circ \big(\sigma_l^{(l+1)}\big)^{-n} \big(X_j^{(l+1)}\big) \right] \big(X_l^{(l+1)}\big)^{-n} \in P^{(l+1)} S_l^{-1}.$$

Thus, $X_j^{(l)} \in A^{(l)} \cap (P^{(l+1)} S_l^{-1}) = P^{(l)}$. $\square$

**Lemma 5.1.2.** *Let $l \in [\![2, N]\!]$ and $P^{(l+1)}$ be a prime ideal of $A^{(l+1)}$. Consider an integer $j$ with $2 \leqslant j < l$ and set $P^{(j)} = \varphi_j \circ \cdots \circ \varphi_l(P^{(l+1)})$.*

1. *Assume that $\beta_j$ is in the same box as $\beta_l$ or in the box before $\beta_l$'s one. Then*
   - $X_j^{(j+1)} = X_j^{(j+2)} = \cdots = X_j^{(l+1)}$,
   - $(X_j^{(j+1)} \in P^{(j+1)}) \Rightarrow (X_j^{(j+2)} \in P^{(j+2)}) \Rightarrow \cdots \Rightarrow (X_j^{(l+1)} \in P^{(l+1)})$.
2. *Assume that the boxes $B$ and $B'$ of $\beta_j$ and $\beta_l$ (respectively) are separated by a box $B''$ containing a unique element $\beta_e$ such that $X_e^{(e+1)} \in P^{(e+1)}$. Then $(X_j^{(j+1)} \in P^{(j+1)}) \Rightarrow (X_j^{(l+1)} \in P^{(l+1)})$.*

**Proof.**

1. Let $k \in [\![j+1, l]\!]$ so that $\beta_k$ is, in the same box as $\beta_j$, or in the same box as $\beta_l$. As these boxes are consecutive or equal, one has $X_k X_j = q^{-\langle \beta_k, \beta_j \rangle} X_j X_k$, so that by Lemma 4.2.1, we have $X_k^{(k+1)} X_j^{(k+1)} = q^{-\langle \beta_k, \beta_j \rangle} X_j^{(k+1)} X_k^{(k+1)}$. So one has $\delta_k^{(k+1)}(X_j^{(k+1)}) = 0$ and, by [Cau03a, Section 3.2], we get:

$$\begin{aligned} X_j^{(k)} &= \sum_{s=0}^{+\infty} \lambda_s \big(\delta_k^{(k+1)}\big)^s \circ \big(\sigma_k^{(k+1)}\big)^{-s} \big(X_j^{(k+1)}\big) \big(X_k^{(k+1)}\big)^{-s} \\ &= \sum_{s=0}^{+\infty} \lambda'_s \big(\delta_k^{(k+1)}\big)^s \big(X_j^{(k+1)}\big) \big(X_k^{(k+1)}\big)^{-s} = X_j^{(k+1)} \quad (\lambda_s, \lambda'_s \in \mathbb{K}). \end{aligned}$$

   This shows the first point. The second point follows from Lemma 5.1.1.

2. As $B$ and $B''$ are consecutive, 1. implies that $X_j^{(j+1)} = \cdots = X_j^{(e+1)}$ and that $(X_j^{(j+1)} \in P^{(j+1)}) \Rightarrow \cdots \Rightarrow (X_j^{(e+1)} \in P^{(e+1)})$. It just remains to show that $X_j^{(k)} \in P^{(k)} \Rightarrow X_j^{(k+1)} \in P^{(k+1)}$ for $e+1 \leqslant k \leqslant l$. We do that by induction on $k$. As in the previous point, we have

$$X_j^{(k)} = X_j^{(k+1)} + \sum_{s=1}^{+\infty} \lambda_s \big(\delta_k^{(k+1)}\big)^s \circ \big(\sigma_k^{(k+1)}\big)^{-s} \big(X_j^{(k+1)}\big)\big(X_k^{(k+1)}\big)^{-s} \quad (\lambda_s \in \mathbb{K}).$$

- If $\delta_k^{(k+1)}(X_j^{(k+1)}) = 0$, then one has $X_j^{(k)} = X_j^{(k+1)}$ and we conclude thanks to Lemma 5.1.1.
- Otherwise, one has $\delta_k^{(k+1)}(X_j^{(k+1)}) = \lambda(X_e^{(k+1)})^m$ $(m \in \mathbb{N}^\star, \lambda \in \mathbb{K}^\star)$ by Lemma 4.2.1 and, as $B'$ and $B''$ are consecutive,

$$\begin{aligned} \delta_k^{(k+1)}\big(X_e^{(k+1)}\big) = 0 \quad &\Rightarrow \quad \big(\delta_k^{(k+1)}\big)^s (X_j^{(k+1)}) = \lambda\big(\delta_k^{(k+1)}\big)^{s-1}\big((X_e^{(k+1)})^m\big) = 0 \quad \text{for } s > 1 \\ &\Rightarrow \quad X_j^{(k)} = X_j^{(k+1)} + \lambda'\big(X_e^{(k+1)}\big)^m \big(X_k^{(k+1)}\big)^{-1} \quad \text{with } \lambda' \in \mathbb{K}^\star. \end{aligned}$$

  - •• If $X_k^{k+1} \in P^{(k+1)}$, then consider the homomorphism $g : A^{(k)} \to \frac{A^{(k+1)}}{(P^{(k+1)})}$ which satisfies $g(X_i^{(k)}) = \overline{X_i^{(k+1)}}$ $(:= X_i^{(k+1)} + (P^{(k+1)}))$ for $i \in [\![1, N]\!]$ (see Section 4.2). By definition of $\phi_k$ (see [Cau03a, Notation 4.3.1.]), one has $P^{(k)} = g^{-1}(\frac{P^{(k+1)}}{(X_k^{(k+1)})})$. So $X_j^{(k)} \in P^{(k)} \Rightarrow g(X_j^{(k)}) = \overline{X_j^{(k+1)}} \in \frac{P^{(k+1)}}{(X_k^{(k+1)})} \Rightarrow X_j^{(k+1)} \in P^{(k+1)}$.
  - •• By 1., one has $X_e^{(e+1)} = \cdots = X_e^{(k)} = X_e^{(k+1)}$ and $(X_e^{(e+1)} \in P^{(e+1)}) \Rightarrow \cdots \Rightarrow (X_e^{(k)} \in P^{(k)}) \Rightarrow (X_e^{(k+1)} \in P^{(k+1)})$. Set, as in [Cau03a, Theorem 3.2.1], $S_k := \{(X_k^{(k+1)})^n \mid n \in \mathbb{N}\}$ so that $P^{(k+1)} = A^{(k+1)} \cap (P^{(k)} S_k^{-1})$ by definition of $\varphi_k$ [Cau03a, Notation 4.3.1.]. Then one has $X_j^{(k+1)} = X_j^{(k)} - \lambda'(X_e^{(k+1)})^m (X_k^{(k+1)})^{-1} = X_j^{(k)} - \lambda'(X_e^{(k)})^m (X_k^{(k+1)})^{-1} \in P^{(k)} S_k^{-1}$. As $X_j^{(k+1)}$ is also in $A^{(k+1)}$, one has $X_j^{(k+1)} \in P^{(k+1)}$, as claimed. $\square$

We use [Cau03a, Proposition 5.2.1] to determine the shape of Cauchon diagrams. Let us rewrite this proposition in our notation:

**Proposition 5.1.3.** *Let $\Delta$ be a Cauchon diagram and let $P \in \mathrm{Spec}(A)$. The ideal $P$ belongs to $\mathrm{Spec}_\Delta(A)$ if and only if it satisfies the following criteria*:

$$\big(\forall l \in [\![1, N]\!]\big) \quad \big(X_l^{(l+1)} \in P^{(l+1)} \Leftrightarrow \beta_l \in \Delta\big).$$

We can now prove the following proposition.

**Proposition 5.1.4.** *Let $\Delta$ be a Cauchon diagram and $\beta_l \in \Delta$ $(1 \leqslant l \leqslant N)$. Assume there is an integer $k \in [\![1, l-1]\!]$ such that $X_{\beta_l} X_{\beta_k} - q^{-(\beta_l, \beta_k)} X_{\beta_k} X_{\beta_l} = c X_{\beta_{i_1}} \ldots X_{\beta_{i_s}}$ with $c \in \mathbb{K}^\star$, $s \geqslant 1$ and $k < i_1 \leqslant \cdots \leqslant i_s < l$. Then one of the $\beta_{i_r}$ $(1 \leqslant r \leqslant s)$ belongs to $\Delta$.*

**Proof.** Let $P \in \mathrm{Spec}_\Delta(A)$. By Lemma 4.2.1, one has:

$$X_l^{(l+1)} X_k^{(l+1)} - q^{-(\beta_l, \beta_k)} X_k^{(l+1)} X_l^{(l+1)} = c X_{\beta_{i_1}}^{(l+1)} \ldots X_{\beta_{i_s}}^{(l+1)} := M.$$

By Proposition 5.1.3, one has $X_l^{(l+1)} \in P^{(l+1)}$ so that $M \in P^{(l+1)}$. As $P^{(l+1)}$ is a prime ideal of $A^{(l+1)}$, we know by [BG02, II.6.9] that $P^{(l+1)}$ is completely prime, so that there exists $r \in [\![1, s]\!]$ such that

$X_{\beta_{i_r}}^{(l+1)} \in P^{(l+1)}$. By Lemma 5.1.1, we deduce that $X_{i_r}^{(i_r+1)} \in P^{(i_r+1)}$ and, by Proposition 5.1.3, we obtain $\beta_{i_r} \in \Delta$. $\square$

**Convention.** We say that a diagram $\Delta$ satisfies the implication

1. $\beta_{j_0} \to \beta_{j_1}$ if $(\beta_{j_0} \in \Delta) \Rightarrow (\beta_{j_1} \in \Delta)$.
2. $$\begin{tikzcd}[row sep=small] & \beta_{j_1} \\ \beta_{j_0} \arrow[ur, dashed] \arrow[r, dashed] \arrow[dr, dashed] & \vdots \\ & \beta_{j_s} \end{tikzcd}$$ if $(\beta_{j_0} \in \Delta) \Rightarrow (\beta_{j_1} \in \Delta)$ or $\dots$ or $(\beta_{j_1} \in \Delta)$.

Proposition 5.1.4 can be rewritten as follows:

**Proposition 5.1.5.** *Let $\Delta$ be a Cauchon diagram and $\beta_l \in \Delta$ $(1 \leqslant l \leqslant N)$. Assume that there exists an integer $k \in [\![1, l-1]\!]$ such that $X_{\beta_l} X_{\beta_k} - q^{-(\beta_l, \beta_k)} X_{\beta_k} X_{\beta_l} = c X_{\beta_{i_1}}^{m_1} \dots X_{\beta_{i_s}}^{m_s}$ with $c \in \mathbb{K}^\star$, $s \geqslant 1$, $k < i_1 < \dots < i_s < l$ and $m_1, \dots, m_s \in \mathbb{N}^\star$.*

1. *If $s = 1$, then the solid arrow $\beta_l \to \beta_{i_1}$ is an implication.*
2. *If $s \geqslant 2$, then the system* $$\begin{tikzcd}[row sep=small] & \beta_{i_1} \\ \beta_l \arrow[ur, dashed] \arrow[r, dashed] \arrow[dr, dashed] & \vdots \\ & \beta_{i_s} \end{tikzcd}$$ *of dashed arrows is an implication.*

In the three following propositions, denotes by $\Delta$ a Cauchon diagram.

**Proposition 5.1.6.** *Let $1 \leqslant l \leqslant n$ and $\beta \in C_l$. If there exists $i \in [\![1, l-1]\!]$ such that $\beta + \alpha_i = m\beta'$ with $m \in \mathbb{N}^\star$ and $\beta' \in \Phi^+$, then $\beta \to \beta'$ is an implication.*

**Proof.** We know (see Proposition 3.2.5 when $\Phi$ is of type $G_2$, Corollary 3.2.7 when $\Phi$ is not of type $G_2$) that we have in this case a commutation relation of the type $E_\beta E_{\alpha_i} - q^{(\beta, \alpha_i)} E_{\alpha_i} E_\beta = k E_{\beta'}^m$ with $k \neq 0$ (where $E_\gamma$ are defined in Section 3.2).

Then, it follows from Proposition 3.4.8 that $X_\beta X_{\alpha_i} - q^{-(\beta, \alpha_i)} X_{\alpha_i} X_\beta = k' X_{\beta'}^m$ with $k' \neq 0$. So we deduce from Proposition 5.1.5 that $\beta \to \beta'$ is an implication. $\square$

**Proposition 5.1.7.** *Let $C_l$ $(1 \leqslant l \leqslant n)$ be an exceptional column. If $\beta \in C_l$ is in the box following the box of the exceptional root $\beta_{ex}$, then $\beta \to \beta_{ex}$ is an implication.*

**Proof.** Suppose first that $\Phi$ is of type $G_2$. With the notation of Proposition 3.2.5, one has $l = 2$, $\beta_{ex} = \beta_4$, $\beta = \beta_5$ and one has a commutation formula of the type $E_{\beta_5} E_{\beta_3} - q^{(\beta_3, \beta_5)} E_{\beta_3} E_{\beta_5} = k E_{\beta_4}$ with $k \in \mathbb{K}^\star$. It implies, by Proposition 5.1.5 that $\beta = \beta_5 \to \beta_{ex} = \beta_4$ is an implication.

Suppose now that $\Phi$ is not of type $G_2$. We know (see Proposition 2.2.12) that $h'(\beta_{ex}) = t + \frac{1}{2}$ $(t \in \mathbb{N}^\star)$, so that $h'(\beta) = h(\beta) = t$. We also know (see Proposition 2.2.3) that if $D = \mathrm{Vect}(\beta_{ex})$, one has $\beta' = s_D(\beta) = \beta_{ex} - \beta \in C_l$, so that $h'(\beta') = h(\beta') = h(\beta_{ex}) - h(\beta) = t + 1$. As a result, $P = \mathrm{Vect}(\beta, \beta')$ is an admissible plane of type (1.1) or (1.2). So, by Proposition 3.2.6, we have a commutation relation of the type $E_\beta E_{\beta'} - q^{(\beta, \beta')} E_{\beta'} E_\beta = k E_{\beta_{ex}}$ with $k \neq 0$. As in Proposition 5.1.6, this implies that $\beta \to \beta_{ex}$ is an implication. $\square$

**Proposition 5.1.8.** *Let $C_l$ $(1 \leqslant l \leqslant n)$ be an exceptional column and $\beta_{ex}$ be its exceptional root. Assume that there exists $i \in [\![1, l[\![$ such that $\beta_{ex} + \alpha_i = \beta'_{i_1} + \beta'_{i_2}$ with $\beta'_{i_1} \neq \beta'_{i_2}$ in the box which precedes $\beta_{ex}$. Then the system* $$\begin{tikzcd}[row sep=small] & \beta'_{i_1} \\ \beta_{ex} \arrow[ur, dashed] \arrow[dr, dashed] & \\ & \beta'_{i_2} \end{tikzcd}$$ *of dashed arrows is an implication.*

**Proof.** As, by hypothesis, $\beta'_{i_1} \neq \beta'_{i_2}$ are in the box preceding the box of $\beta_{ex}$, the root system is not of type $G_2$ (see Proposition 3.2.5).

As in the proof of Proposition 5.1.6, it is enough to prove that: $[E_{\beta_{ex}}, E_{\alpha_i}]_q := E_{\beta_{ex}} E_{\alpha_i} - q^{(\beta_{ex},\alpha_i)} E_{\alpha_i} E_{\beta_{ex}} = \lambda E_{\beta'_{i_1}} E_{\beta'_{i_2}}$ with $\lambda \in \mathbb{K}^\star$. Recall from Proposition 2.2.3 that $\beta_{ex} \perp \alpha_i$, so that:

$$\left(\alpha_i, \beta'_{i_1} + \beta'_{i_2}\right) = (\alpha_i, \beta_{ex} + \alpha_i) = \|\alpha_i\|^2 \quad \Rightarrow \quad \left(\alpha_i, \beta'_{i_1}\right) > 0 \text{ or } \left(\alpha_i, \beta'_{i_2}\right) > 0.$$

We can assume, without loss of generality, that $(\alpha_i, \beta'_{i_2}) > 0$, so that (Corollary 3.2.7) $[E_{\beta'_{i_2}}, E_{\alpha_i}]_q = 0$.

As in the proof of the previous proposition, one has:

- $h'(\beta_{ex}) = t + \frac{1}{2}$ $(t \in \mathbb{N}^\star)$ and $h'(\beta'_{i_1}) = h'(\beta'_{i_1}) = t + 1$,
- $\beta_{i_1} = s_D(\beta'_{i_1})$ and $\beta_{i_2} = s_D(\beta'_{i_2})$ belong to $C_l$ and satisfy $h'(\beta_{i_1}) = h'(\beta_{i_1}) = t$,
- $E_{\beta_{i_2}} E_{\beta'_{i_2}} - q^{(\beta_{i_2},\beta'_{i_2})} E_{\beta'_{i_2}} E_{\beta_{i_2}} = k E_{\beta_{ex}}$ with $k \neq 0$. $(\star)$

By definition of $\beta_{i_2}$, one has $\beta_{ex} = \beta_{i_2} + \beta'_{i_2}$, so that $\beta'_{i_1} + \beta'_{i_2} = \beta_{ex} + \alpha_i = \beta_{i_2} + \beta'_{i_2} + \alpha_i \Rightarrow \beta'_{i_1} = \beta_{i_2} + \alpha_i$. Thus, by Corollary 3.2.7, we have $[E_{\beta_{i_2}}, E_{\alpha_i}]_q := h E_{\beta'_{i_1}}$ $(h \neq 0)$. We know that $\mathcal{U}_q^+(\mathfrak{g})$ is $\mathbb{Z}\Phi$-graded. So there is a (unique) automorphism $\sigma$ of $\mathcal{U}_q^+(\mathfrak{g})$ such that for all $u \in \mathcal{U}_q^+(\mathfrak{g})$, homogeneous in degree $\beta$, $\sigma(u) = q^{(\beta,\alpha_i)} u$.

Denote by $\delta$ the interior right-sided $\sigma$-derivation associated to $E_{\alpha_i}$, so that $\delta(u) = u E_{\alpha_i} - E_{\alpha_i} \sigma(u)$ $(\forall u \in \mathcal{U}_q^+(\mathfrak{g}))$. If $\beta \in C_l$, one has $\delta(E_\beta) = E_\beta E_{\alpha_i} - q^{(\beta,\alpha_i)} E_{\alpha_i} E_\beta = [E_\beta, E_{\alpha_i}]_q$ and, this implies $\delta(E_{\beta'_{i_2}}) = 0$ and $\delta(E_{\beta_{i_2}}) = h E_{\beta'_{i_1}}$. We can show with $(\star)$ that:

$$\begin{aligned}
k[E_{\beta_{ex}}, E_{\alpha_i}]_q = k\delta(\beta_{ex}) &= \delta(E_{\beta_{i_2}} E_{\beta'_{i_2}}) - q^{(\beta_{i_2},\beta'_{i_2})} \delta(E_{\beta'_{i_2}} E_{\beta_{i_2}}) \\
&= E_{\beta_{i_2}} \delta(E_{\beta'_{i_2}}) + \delta(E_{\beta_{i_2}}) \sigma(E_{\beta'_{i_2}}) - q^{(\beta_{i_2},\beta'_{i_2})} \left(E_{\beta'_{i_2}} \delta(E_{\beta_{i_2}}) + \delta(E_{\beta'_{i_2}}) \sigma(E_{\beta_{i_2}})\right) \\
&= h\left[q^{(\beta'_{i_2},\alpha_i)} E_{\beta'_{i_1}} E_{\beta'_{i_2}} - q^{(\beta_{i_2},\beta'_{i_2})} E_{\beta'_{i_2}} E_{\beta'_{i_1}}\right].
\end{aligned}$$

As $\beta'_{i_2}$ and $\beta'_{i_1}$ are in the same box, we know (Corollary 3.4.7) that $E_{\beta'_{i_1}} E_{\beta'_{i_2}} = E_{\beta'_{i_2}} E_{\beta'_{i_1}}$, so that $k[E_{\beta_{ex}}, E_{\alpha_i}]_q = h(q^{(\beta'_{i_2},\alpha_i)} - q^{(\beta_{i_2},\beta'_{i_2})}) E_{\beta'_{i_1}} E_{\beta'_{i_2}}$. Since $\beta_{i_2} + \beta'_{i_2} = \beta_{ex}$, $P = \text{Vect}(\beta_{i_2}, \beta'_{i_2})$ is an admissible plane of type (1.1) or (1.2) (see Remark 2.2.14) with $\{\beta_{i_2}, \beta'_{i_2}\} = \{\beta, \beta'\}$, so that $(\beta_{i_2}, \beta'_{i_2}) \leqslant 0$. As we have assumed that $(\alpha_i, \beta'_{i_2}) > 0$, this implies that $[E_{\beta_{ex}}, E_{\alpha_i}]_q := E_{\beta_{ex}} E_{\alpha_i} - q^{(\beta_{ex},\alpha_i)} E_{\alpha_i} E_{\beta_{ex}} = \lambda E_{\beta'_{i_1}} E_{\beta'_{i_2}}$ with $\lambda \neq 0$. $\square$

### 5.2. *Implications from an admissible plane*

We define the notion of implications coming from an admissible plane $P$, and we verify that all Cauchon diagrams satisfy all implications from admissible planes. Let us begin by showing some precise results on the exceptional root and near boxes behaviour. First, let us recall some notation introduced in Sections 2 and 3.

**Notation.** $C_1, \ldots, C_n$ denote the columns of $\Phi^+$ (relative to the chosen Lusztig order). In the following, we consider a diagram $\Delta$, that is, $\Delta$ a subset of $\Phi^+$. For any integer $j \in [\![1, n]\!]$, we set $\Delta_j := \Delta \cap C_j = \{\beta_{u_1}, \ldots, \beta_{u_l}\} \subset C_j = \{\beta_k, \ldots, \beta_r\}$. If the column $C_j$ is exceptional, $\beta_{ex}$ denotes the exceptional root and $B_{ex} := \{\beta_{ex}\}$ is its box. Then $B_1$ denote the box of $C_j$ which precedes $B_{ex}$ and $B'_1$ the one which follows $B_{ex}$ in the Lusztig order; so that $s_D(B_1) = B'_1$.

In Propositions 5.1.6, 5.1.7 and 5.1.8, we proved the existence of implications thanks to admissible planes. We formalise this fact in the following definition of "implications coming from an admissible plane":

**Definition 5.2.1.** Let $\beta \in C_j$ with $h'(\beta) = l$ and, $P$ be an admissible plane.

1. If $\Phi_P^+ = \{\beta, \beta + \alpha_i, \alpha_i\}$ with $i < j$ type (2.1), then the implication coming from $P$ is $\beta \to \beta + \alpha_i$.
2. $\Phi_P^+ = \{\beta, \beta + \alpha_i, \beta + 2\alpha_i, \alpha_i\}$ with $i < j$ type (2.3), then the implications coming from $P$ are $\beta \to \beta + \alpha_i$ and $\beta + \alpha_i \to \beta + 2\alpha_i$.
3. $\Phi_P^+ = \{\beta, \beta + \beta', \beta\}$ with $i < j$, $\beta' \in C_j$ and $h(\beta') = h(\beta) + 1$ type (1.1), then the implication coming from $P$ is $\beta \to \beta + \beta'$.
4. $\Phi_P^+ = \{\alpha_i, \alpha_i + \beta, \alpha_i + 2\beta, \beta\}$ with $i < j$, $h'(\alpha_i + 2\beta) = \frac{2l+1}{2}$ and $h(\beta) = l$ type (1.2) or type (2.2), then the implications coming from $P$ are $\beta \to \alpha_i + \beta$, $\beta \to \alpha_i + 2\beta$ and $\alpha_i + 2\beta \to \alpha_i + \beta$.
5. $\Phi_P^+ = \{\beta, \alpha_i\}$ with $i < j$, $\alpha_i \perp \beta$ and there are $\beta_1$ and $\beta_2$ in $C_j$ such that $\beta + \alpha_i = \beta_1 + \beta_2$ type (2.4), then the implications coming from $P$ are
$$\begin{tikzcd}[row sep=small] & \beta_1 \\ \beta_{ex} \arrow[ur, dashed] \arrow[dr, dashed] & \\ & \beta_2 \end{tikzcd}.$$
6. $\Phi_P^+ = \Phi^+ = \{\beta_1, \ldots, \beta_6\}$ is the positive part of a roots system of type $G_2$ (see Proposition 3.2.5), then the implications coming from $P$ are $\beta_6 \to \beta_5$, $\beta_5 \to \beta_4$, $\beta_5 \to \beta_3$, $\beta_4 \to \beta_3$, $\beta_3 \to \beta_2$.

**Lemma 5.2.2.** *Let $\beta \in C_j$.*

1. *If $\beta$ belongs to a box which follows $\{\beta_{ex}\}$, then $\beta \to \beta_{ex}$ is an implication from an admissible plane.*
2. *If there is $i < j$ such that $\gamma = \beta + \alpha_i \in \Phi^+$ then $\beta \to \gamma$ is an implication from an admissible plane.*

**Proof.** The results holds in the case where $\Phi$ is of type $G_2$. From now on, we assume that $\Phi$ is not of type $G_2$.

1. Let $P = \langle \beta, \beta_{ex} \rangle$. It is an admissible plane of type 3 or 4 in the previous definition and in each case, $\beta \to \beta_{ex}$ is an implication coming from $P$.
2. Let $P = \langle \beta, \alpha_i \rangle$. It is an admissible plane of type 1,2 or 4 in the previous definition and in each case, $\beta \to \gamma$ is an implication coming from $P$. $\square$

**Proposition 5.2.3.** *Let $\Delta$ be a Cauchon diagram. Then $\Delta$ satisfies all the implication coming from admissible planes containing elements of $\Delta$.*

**Proof.** Let $\beta \in \Delta$ and $P$ be an admissible plane containing $\beta$. Recall (see Definition 5.2.1) that $\Phi_P^+ = \Phi^+ \cap P$.

1. If $\Phi_P^+ = \{\beta, \beta + \alpha_i, \alpha_i\}$ with $i < j$, then it follows from Proposition 5.1.6 that $\Delta$ satisfies the implication $\beta \to \beta + \alpha_i$.
2. If $\Phi_P^+ = \{\beta, \beta + \alpha_i, \beta + 2\alpha_i, \alpha_i\}$ with $i < j$, then applying Proposition 5.1.6 to $\beta$ and $\beta + \alpha_i$, we get that $\Delta$ satisfies the implications $\beta \to \beta + \alpha_i$ and $\beta + \alpha_i \to \beta + 2\alpha_i$.
3. If $\Phi_P^+ = \{\beta, \beta + \beta', \beta\}$ with $i < j$, $\beta' \in C_j$ and $h(\beta') = h(\beta) + 1$ then it follows from Proposition 5.1.7 that $\Delta$ satisfies the implication $\beta \to \beta + \beta'$.
4. If $\Phi_P^+ = \{\alpha_i, \alpha_i + \beta, \alpha_i + 2\beta, \beta\}$ with $i < j$ and $h'(\alpha_i + 2\beta) = \frac{2l+1}{2}$, then it follows from Propositions 5.1.6, 5.1.7 and 5.1.8 that $\Delta$ satisfies the implications $\beta \to \alpha_i + \beta$, $\beta \to \alpha_i + 2\beta$ and $\alpha_i + 2\beta \to \alpha_i + \beta$.

5. If $\Phi_P^+ = \{\beta, \alpha_i\}$ with $i < j$, $\alpha_i \perp \beta$ and there exist $\beta_1$ and $\beta_2$ in $C_j$ such that $\beta + \alpha_i = \beta_1 + \beta_2$, then it follows from Proposition 5.1.8 that $\Delta$ satisfies the implication $\begin{smallmatrix} & \nearrow \beta_1 \\ \beta_{ex} & \\ & \searrow \beta_2 \end{smallmatrix}$ .

6. If $\Phi_P^+ = \Phi^+$ is of type $G_2$, Proposition 5.1.6 implies that $\Delta$ satisfies the implications $\beta_6 \to \beta_5$, $\beta_5 \to \beta_3, \beta_4 \to \beta_3, \beta_3 \to \beta_2$. Moreover Proposition 5.1.7 implies that $\Delta$ satisfies the implication $\beta_5 \to \beta_4$. $\square$

*5.3. The converse*

The goal of this section is to prove the converse of Proposition 5.2.3, that is:

**Theorem 5.3.1.** *If $\Delta$ is a diagram which satisfies all the implications coming from admissible planes, then $\Delta$ is a Cauchon diagram.*

Let $\beta \in \Phi^+$ be a positive root of the column $C_j$. We denote by $B_0$ the box which contains $\beta$, by $B_1$ the box which precedes $B_0$ in the column $C_j$ (if it exists) and by $B_2$ the box which precedes $B_1$ in $C_j$ (if it exists).

Set $\Phi_\beta^+ = \{\alpha_i \mid i < j\} \cup \{\gamma < \beta \mid \gamma$ is in the box of $\beta\} \cup B_1 \cup (B_2$ if $B_1 = \{\beta_{ex}\})$. If $\gamma \in \Phi^+$, then there exists $k \in [\![1, N]\!]$ such that $\gamma = \beta_k$ and recall (see Section 4.1) that $X_\gamma = X_k$.

Set $D_\beta := \mathbb{K}\langle X_\gamma \mid \gamma < \beta\rangle$.

**Lemma 5.3.2.** $D_\beta = \mathbb{K}\langle X_\gamma \mid \gamma \in \Phi_\beta^+\rangle$.

**Proof.** Set $D'_\beta := \mathbb{K} < X_\gamma \mid \gamma \in \Phi_\beta^+ > \subset D_\beta$. Let us start by showing that, for $i < j$, we have $\{X_\gamma, \gamma \in C_i\} \subset D'_\beta$. If $\Phi$ is of type $G_2$, $\{X_\gamma, \gamma \in C_i\}$ is the empty set or it only contains $X_{\alpha_1} \in D'_\beta$. If $\Phi$ si not of type $G_2$, then we prove this result by induction on $h(\gamma)$.

*If $h(\gamma) = 1$,* then $\gamma = \alpha_i$ and $X_\gamma \in D'_\beta$ by definition of $\Phi_\beta^+$.

*If $h(\gamma) > 1$ and $\gamma$ ordinary,* then by Proposition 2.2.11, there exists $l < i$ such that $\gamma' = \gamma - \alpha_l \in \Phi^+$, so that, by Corollary 3.2.7 and Proposition 3.4.8, one has $X_\gamma \in \mathbb{K} < X_{\gamma'}, X_{\alpha_l} > \subset D'_\beta$ (by induction hypothesis).

*If $h(\gamma) > 1$ and $\gamma$ exceptional,* then we know (see Proposition 2.2.3) that in this case, there are two ordinary roots of $C_i$, denoted $\eta_1$ and $\eta_2$, such that $\eta_1 + \eta_2 = \gamma$ and $h(\eta_2) = h(\eta_1) + 1$. This implies by Corollary 3.2.7 and Proposition 3.4.8 that $X_\gamma \in \mathbb{K}\langle X_{\eta_1}, X_{\eta_2}\rangle \subset D'_\beta$ ($X_{\eta_1}$ and $X_{\eta_2}$ are in $D'_\beta$ because $\eta_1$ and $\eta_2$ are exceptional).

It just remains to show that $\{X_\gamma \mid \gamma \in C_j,\ \gamma < \beta\} \subset D'_\beta$.

If $h(\gamma) = h(\beta)$ with $\gamma < \beta$, then $\gamma \in \Phi_\beta^+$. So $X_\gamma \in D'_\beta$.

One uses again an induction to show that for each ordinary box $B$ of $C_j$ such that $B < B_0$ (i.e. all roots $\beta$ of $B$ are strictly less than all roots of $B_0$), one has $\{X_\gamma \mid \gamma \in B\} \subset D'_\beta$.

*Assume that $B_1$ ordinary.* The result is true for the box $B_1$ since $B_1 \subset \Phi_\beta^+$.

Let $B$ be an ordinary root of $C_j$ such that $h(B) > h(B_1)$ and $\gamma \in B$. By Proposition 2.2.11, there is $\alpha_l \in \Pi$ $(l < j)$ such that $\gamma - \alpha_l \in \Phi^+$. Then $\gamma' := \gamma - \alpha_l$ is in an ordinary box $B'$ of $C_j$ such that $h(B) = h(B') + 1 > h(B') \geqslant h(B_1) > h(B_0)$ and one has $X_{\gamma'} \in D'_\beta$ by induction hypothesis.

If $\Phi$ is not of type $G_2$, then we deduce from Corollary 3.2.7 and Proposition 3.4.8 that $[X_{\gamma'}, X_{\alpha_l}]_q = kX_\gamma$ with $k \in \mathbb{K}^\star$. As $X_{\alpha_l} \in D'_\beta$, this implies that $X_\gamma \in D'_\beta$.

If $\Phi$ is of type $G_2$, then we deduce from Propositions 3.2.5 and 3.4.8 that $[X_{\gamma'}, X_{\alpha_l}]_q = kX_\gamma$ with $k \in \mathbb{K}^\star$. As $X_{\alpha_l} \in D'_\beta$, this implies that $X_\gamma \in D'_\beta$.

*Assume that $B_1$ exceptional.* The results is true for $B_2$ since, in this case, $B_2 \subset \Phi^+_\beta$. This is the same proof as above with $B_1$ replaced by $B_2$.

It remains to prove that if $B = \{\beta_{ex}\}$ is an exceptional box of $C_j$ such that $B < B_0$, then one has $X_{\beta_{ex}} \in D'_\beta$.

If $B = B_1$, then one has $B \subset \Phi^+_\beta$, and the result is proved.

Assume that $B < B_1$. As above, one has $\beta_{ex} = \eta_1 + \eta_2$ with $\eta_1$ and $\eta_2$ two exceptional roots of $C_j$ such that $h(\eta_2) = h(\eta_1) + 1$. The boxes of $\eta_1$ and $\eta_2$ are ordinary, on each side of $B$, so less than or equal to $B_1$, so strictly less than $B_0$. As the result holds for ordinary boxes, $X_{\eta_1} \in D'_\beta$ and $X_{\eta_2} \in D'_\beta$.

If $\Phi$ is not of type $G_2$, then we deduce (as above) from Corollary 3.2.7 and Proposition 3.4.8 that $X_{\beta_{ex}} \in D'_\beta$.

If $\Phi$ is of type $G_2$, we deduce (as above) from Propositions 3.2.5 and 3.4.8 that $X_{\beta_{ex}} \in D'_\beta$.

So we can conclude that $D_\beta = D'_\beta$. $\square$

Let us recall that $A = \mathcal{U}_q^+(\mathfrak{g}) = \mathbb{K}\langle X_{\beta_i} \mid i \in [\![1, N]\!]\rangle := \mathbb{K}\langle X_i \mid i \in [\![1, N]\!]\rangle$. Let $\beta_r$ and $\beta_{r+1}$ ($1 \leqslant r \leqslant N-1$) be two consecutive roots of $\Phi^+$ ($\beta_r < \beta_{r+1}$). Recall that $A^{(r+1)} = \mathbb{K}\langle X_i^{(r+1)}\rangle$ and $A^{(r)} = \mathbb{K}\langle X_i^{(r)}\rangle$ ($1 < r < N$) are the algebras deduced from $A$ by the deleting derivation algorithm of Section 4.

**Lemma 5.3.3.** *Let $\beta_r \in \Phi^+$ be a positive root of the column $C_j$ and $D_{\beta_r}^{(r+1)} := \mathbb{K} < X_\gamma^{(r+1)} \mid \gamma \langle \beta_r \rangle$. Then $D_{\beta_r}^{(r+1)} = \mathbb{K}\langle X_\gamma^{(r+1)} \mid \gamma \in \Phi^+_{\beta_r}\rangle$.*

**Proof.** By Lemma 4.2.1, the commutation relations between the $X_\gamma^{(r+1)}$ with $\gamma \leqslant \beta_r$ are the same as the commutation relations between the $X_\gamma$ with $\gamma \leqslant \beta_r$. So the proof is the same as the proof of Lemma 5.3.2 but with $X_\gamma$ replaced by $X_\gamma^{(r+1)}$. $\square$

*Denote, as in Section* 4, $\varphi : \operatorname{Spec} A \hookrightarrow \operatorname{Spec}(\overline{A})$ ($\overline{A} = A^{(2)}$) *the canonical injection, that is, the composition of canonical injections* $\varphi_r : \operatorname{Spec}(A^{(r+1)}) \hookrightarrow \operatorname{Spec}(A^{(r)})$ *for* $r \in [\![2, N]\!]$. *Recall that a subset $\Delta$ of $\Phi^+$ is a Cauchon diagram if and only if* $(\exists P \in \operatorname{Spec}(A))$ $(\varphi(P) = \langle T_\gamma \mid \gamma \in \Delta\rangle)$.

**Proof of Theorem 5.3.1.** Let $\Delta \subset \Phi^+$ be a diagram satisfying the implications coming from the admissible planes. Set $Q := \langle T_\gamma | \gamma \in \Delta\rangle$. By [Cau03a, Section 5.5], this is an $H^{(2)}$-prime ideal, so completely prime, of $A^{(2)} = \overline{A}$ and, if $\beta \in \Phi^+ \setminus \Delta$, then $T_\beta$ is regular modulo $Q$. So, $Q \cap \Phi^+ = \{T_\gamma \mid \gamma \in \Delta\}$. Let us show by induction, that for each $r \in [\![2, N+1]\!]$, there exists $P^{(r)} \in \operatorname{Spec}(A^{(r)})$ such that $Q = \varphi_2 \circ \cdots \circ \varphi_{r-1}(P^{(r)})$.

If $r = 2$, then in this case, one has $\varphi_2 \circ \cdots \circ \varphi_{r-1} = Id_{\operatorname{Spec}(\overline{A})}$ and $P^{(2)} = Q$.

*Consider an integer* $r \in [\![2, N]\!]$, *assume that there exists* $P^{(r)} \in \operatorname{Spec}(A^{(r)})$ *such that* $\varphi_2 \circ \cdots \circ \varphi_{r-1}(P^{(r)}) = Q$ *and let us show there is* $P^{(r+1)} \in \operatorname{Spec}(A^{(r+1)})$ *such that* $\varphi_r(P^{(r+1)}) = P^{(r)}$ (*so that* $\varphi_2 \circ \cdots \circ \varphi_r(P^{(r+1)}) = Q$).

- If $X_r^{(r)} \notin P^{(r)}$, then this follows from Proposition 4.3.4.
- Assume now that $X_r^{(r)} \in P^{(r)}$. From the second point of Proposition 4.3.4, it is enough to show that $\Theta^{(r)}(\delta_r^{(r+1)}(X_i^{(r+1)})) \in P^{(r)}$ for $1 \leqslant i \leqslant r-1$.

**Observation.** It is enough to prove that $\Theta^{(r)}(\delta_r^{(r+1)}(X_i^{(r+1)})) \in P^{(r)}$ for $i \in [\![1, r-1]\!]$ such that $\beta_i \in \Phi^+_{\beta_r}$.

**Proof of the observation.** Let $i \in [\![1, r-1]\!]$. It follows from Corollary 5.3.3 that $X_i^{(r+1)} = \sum_{j_1,\dots,j_s \in \Gamma} m_{i,r+1} X_{j_1}^{(r+1)} \dots X_{j_s}^{(r+1)}$ where $\Gamma := \{j \in [\![1, r-1]\!] \mid \beta_j \in \Phi^+_{\beta_r}\}$. Thus

$$
\begin{aligned}
\delta_r^{(r+1)}\big(X_i^{(r+1)}\big) &= \sum m_{i,r+1}\delta_r^{(r+1)}\big(X_{j_1}^{(r+1)}\dots X_{j_s}^{(r+1)}\big) \\
&= \sum m_{i,r+1}\big[\delta_r^{(r+1)}\big(X_{j_1}^{(r+1)}\big)X_{j_2}^{(r+1)}\dots X_{j_s}^{(r+1)} \\
&\quad + \sigma_r^{(r+1)}\big(X_{j_1}^{(r+1)}\big)\delta_r^{(r+1)}\big(X_{j_2}^{(r+1)}\big)\dots X_{j_s}^{(r+1)} \\
&\quad + \dots + \sigma_r^{(r+1)}\big(X_{j_1}^{(r+1)}\dots X_{j_{s-1}}^{(r+1)}\big)\delta_r^{(r+1)}\big(X_{j_s}^{(r+1)}\big)\big] \\
&= \sum m_{i,r+1}\big[\delta_r^{(r+1)}\big(X_{j_1}^{(r+1)}\big)X_{j_2}^{(r+1)}\dots X_{j_s}^{(r+1)} + \lambda_{r,j_1}X_{j_1}^{(r+1)}\delta_r^{(r+1)}\big(X_{j_2}^{(r+1)}\big)\dots X_{j_s}^{(r+1)} \\
&\quad + \dots + \lambda_{r,j_1}\dots\lambda_{r,j_{s-1}}X_{j_1}^{(r+1)}\dots X_{j_{s-1}}^{(r+1)}\delta_r^{(r+1)}\big(X_{j_s}^{(r+1)}\big)\big].
\end{aligned}
$$

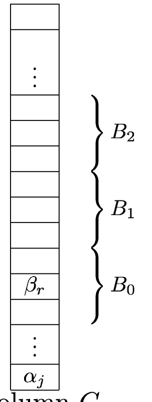
Then, $\Theta^{(r)}(\delta_r^{(r+1)}(X_i^{(r+1)})) = \sum m_{j_1,\dots,j_s}[\Theta^{(r)}(\delta_r^{(r+1)}(X_{j_1}^{(r+1)}))X_{j_2}^{(r)}\dots X_{j_s}^{(r)} + \lambda_{r,j_1}X_{j_1}^{(r)}\Theta^{(r)}(\delta_r^{(r+1)}(X_{j_2}^{(r+1)}))\dots X_{j_s}^{(r)} + \dots + \lambda_{r,j_1}\dots\lambda_{r,j_{s-1}}X_{j_1}^{(r)}\dots X_{j_{s-1}}^{(r)}\Theta^{(r)}(\delta_r^{(r+1)}(X_{j_s}^{(r+1)}))]$. As each $\Theta^{(r)}(\delta_r^{(r+1)}(X_{j_l}^{(r+1)})) \in P^{(r)}$ by hypothesis, one has $\Theta^{(r)}(\delta_r^{(r+1)}(X_i^{(r+1)})) \in P^{(r)}$. $\square$

**Back to the proof of Theorem 5.3.1.** For each $s \in [\![2, r-1]\!]$, set $P^{(s)} = \varphi_s \circ \dots \varphi_{r-1}(P^{(r)})$.

**Observation.** $\beta_r \in \Delta$.

Indeed, as $X_r^{(r)} \in P^{(r)}$, Lemma 5.1.1 implies successively that $X_r^{(r-1)} \in P^{(r-1)}, \dots, X_r^{(2)} \in P^{(2)} = Q$. Hence $T_{\beta_r} = X_r^{(2)} \in Q$ and so $\beta_r \in \Delta$.

Recall that, if $\beta_r \in C_j$, then $\Phi_{\beta_r}^+ = \{\alpha_i \mid i < j\} \cup \{\gamma < \beta_r \mid \gamma \in B_0\} \cup B_1 \cup (B_2 \text{ if } B_1 = \{\beta_{ex}\})$ ($B_0$ is the box containing $\beta_r$, $B_1$ is the box preceding $B_0$ if $C_j$ if exists and $B_2$ is the box preceding $B_1$ in $C_j$ if exists).



Column $C_j$

Let $i \in [\![1, r-1]\!]$ such that $\beta_i \in \Phi_{\beta_r}^+$.

- If $\beta_i \in B_0 \cup B_1$, then Theorem 3.4.3 implies that $\delta_r^{(r+1)}(X_i^{(r+1)}) = 0$. Hence $\Theta^{(r)}(\delta_r^{(r+1)}(X_i^{(r+1)})) = 0 \in P^{(r)}$.
- Let us assume that $B_1 = \{\beta_{ex}\}$ with $\beta_{ex} = \beta_e$ ($e < r$), and that $\beta_i \in B_2$.
  By Theorem 3.4.3, $\delta_r^{(r+1)}(X_i^{(r+1)}) = P_{r,i}^{(r+1)}$ is homogeneous of weight $\beta_r + \beta_i$ and the variables $X_l^{(r+1)}$ which appear in $P_{r,i}^{(r+1)}$ are such that $\beta_l \in B_1 = \{\beta_e\}$. So $P_{r,i}^{(r+1)}$ is equal to zero or is of the form $\lambda X_e^m$ with $\lambda \in \mathbb{K}^\star$ and $m\beta_{ex} = \beta_r + \beta_i$, so that (by comparing the coefficient on $\alpha_j$) one has $m = 1$.

If $P^{(r+1)}_{r,i} = 0$, then one has $\Theta^{(r)}(\delta^{(r+1)}_r(X^{(r+1)}_i)) = 0 \in P^{(r)}$.
Otherwise, assume that $P^{(r+1)}_{r,i} = \lambda X^m_e$. As $\Delta$ satisfies the implications from admissible planes, Lemma 5.2.2 implies that $\Delta$ satisfies the implication $\beta_r \to \beta_{ex}$ and, as $\beta_r \in \Delta$, one has $\beta_{ex} \in \Delta$. Then $X^{(2)}_e \in Q = P^{(2)}$ and by Lemma 5.1.1, $X^{(e+1)}_e \in P^{(e+1)}$. As $\beta_e$ and $\beta_r$ are in consecutive boxes by construction, Lemma 5.1.2 shows that $X^{(e+1)}_e \in P^{(e+1)} \Rightarrow X^{(r)}_e \in P^{(r)}$. So, we deduce that $\Theta^{(r)}(\delta^{(r+1)}_r(X^{(r+1)}_i)) = \Theta^{(r)}(\lambda X^{(r+1)}_e) = \lambda X^{(r)}_e \in P^{(r)}$.

- Consider now the case where $\beta_i = \alpha_k$ with $k < j$. If $\delta^{(r+1)}_r(X^{(r+1)}_i) = 0$, then one has $\Theta^{(r)}(\delta^{(r+1)}_r(X^{(r+1)}_i)) = 0 \in P^{(r)}$. Assume that $\delta^{(r+1)}_r(X^{(r+1)}_i) \neq 0$. From Theorem 3.4.3, we get that $\delta^{(r+1)}_r(X^{(r+1)}_i) = \sum_{i<j_1\leqslant\cdots\leqslant j_s<r} c_{j_1,\dots,j_s} X^{(r+1)}_{j_1} \dots X^{(r+1)}_{j_s}$ $(c_{j_1,\dots,j_s} \in \mathbb{K})$. Thus $c_{j_1,\dots,j_s} \in \mathbb{K}^* \Rightarrow$ $(\beta_{j_1} + \cdots + \beta_{j_s} = \beta_r + \alpha_k$ and $\beta_{j_1},\dots,\beta_{j_s} \notin B_0)$. This implies that $\Theta^{(r)}(\delta^{(r+1)}_r(X^{(r+1)}_i)) = \sum_{i<j_1\leqslant\cdots\leqslant j_s<r} c_{j_1,\dots,j_s} X^{(r)}_{j_1} \dots X^{(r)}_{j_s}$ and that is enough to show that, if $c_{j_1,\dots,j_s} \in \mathbb{K}^*$, then one has $X^{(r)}_{j_1} \dots X^{(r)}_{j_s} \in P^{(r)}$.
  So, take $(j_1,\dots j_s)$ such that $i < j_1 \leqslant \cdots \leqslant j_s < r$ and let us assume that $c_{j_1,\dots,j_s} \neq 0$. Considering the coefficient of $\alpha_j$ in the following equality

$$\beta_{j_1} + \cdots + \beta_{j_s} = \beta_r + \alpha_k, \tag{9}$$

  we deduce that $\beta_{j_s} \in C_j$. As $\beta_{j_s} \notin B_0$ and $j_s < r$, the box $B_1$ exists. The proof splits into three cases.
  - •• $B_0$ and $B_1$ are ordinaries. As $j_s < r$ and $\beta_{j_s} \notin B_0$, one has $h(\beta_r) < h(\beta_{j_s})$. By (9), $h(\beta_{j_s}) \leqslant h(\beta_r + \alpha_k) = h(\beta_r) + 1$. As a result, $s = 1$ and $\beta_{j_s} \in B_1$. That is why, on has (Lemma 5.2.2) the implication $\beta_r \to \beta_{j_s}$. Since $\beta_r \in \Delta$, one has $\beta_{j_s} \in \Delta$ and, as above, $X^{(j_s+1)}_{j_s} \in P^{(j_s+1)}$, so that $X^{(r)}_{j_s} \in P^{(r)}$. Hence the considered monomial whose coefficient $c_{j_1,\dots,j_s} \neq 0$ is in $P^{(r)}$.
  - •• $B_0$ is ordinary and $B_1$ is exceptional so that $B_2$ exists. As in the previous case, one checks that $s = 1$ and $\beta_{j_s} \in B_2$. So from Lemma 5.2.2, there exists an implication $\beta_r \to \beta_{j_s}$. Also, from Lemma 5.2.2, one has the implication $\beta_r \to \beta_e$. Since $\beta_r \in \Delta$, one has $\beta_e, \beta_{j_s} \in \Delta$, so that $X^{(e+1)}_e \in P^{(e+1)}$ and $X^{(j_s+1)}_{j_s} \in P^{(j_s+1)}$. By the second point of Lemma 5.1.2, one deduces that $X^{(r)}_{j_s} \in P^{(r)}$. Thus the considered monomial is in $P^{(r)}$.
  - •• $B_0$ is exceptional. Since $\beta_{j_s} \notin B_0$, $\beta_{j_s}$ is ordinary in $C_j$. By the equality (9), one has $s \geqslant 2$ and $\beta_{j_{s-1}}$ is also ordinary in $C_j$. Set $h(\beta_r) := 2l + 1$ $(l \geqslant 1)$. We knows that $h(\beta_{j_{s-1}}) \geqslant l + 1$, $h(\beta_{j_s}) \geqslant l + 1$ and $h(\beta_r + \alpha_k) = 2l + 2$. This implies that $s = 2$ and $\beta_{j_{s-1}}, \beta_{j_s} \in B_1$. The equality (9) can be then written as $\beta_r + \alpha_k = \beta_{j_{s-1}} + \beta_{j_s}$.
    - ••• Assume $\beta_{j_{s-1}} \neq \beta_{j_s}$, so that $\beta_{j_{s-1}}$ and $\beta_{j_s}$ are in the same box $B_1$, so they are orthogonal. As a result, $\Phi$ is not of the type $G_2$ (in the $G_2$ case, the boxes contain only one element). Set $P := \langle \beta_{j_s}, \beta_{j_{s-1}} \rangle$ the plane spanned by $\beta_{j_s}$, $\beta_{j_{s-1}}$, and assume $\Phi^+_P \neq \{\beta_{j_{s-1}}, \beta_{j_s}\}$. So, since $\Phi_P$ is not of type $G_2$, $\Phi_P$ is of type $A_2$ or $B_2$. As $\beta_{j_{s-1}}$ and $\beta_{j_s}$ are orthogonal, $\Phi_P$ is of type $B_2$ and there exists $\beta \in \Phi^+$ such that $\beta_r + \alpha_k = \beta_{j_{s-1}} + \beta_{j_s} = m\beta$ with $m = 1$ or 2.
      If $m = 1$, then $\beta$ and $\beta_r$ are two distinct exceptional roots of $C_j$, which is impossible.
      Hence $m = 2$ and so $\beta_r + \alpha_k = \beta_{j_{s-1}} + \beta_{j_s} = 2\beta$. This implies that $h(\beta) = l + 1$, so that $\beta$ is an element of $B_1$ too, different from $\beta_{j_{s-1}}$ and $\beta_{j_s}$. As a result, $\beta, \beta_{j_{s-1}}, \beta_{j_s}$ are pairwise orthogonal, which is a contradiction with the equality $\beta_{j_{s-1}} + \beta_{j_s} = 2\beta$.

      So one has $\Phi^+_P = \{\beta_{j_{s-1}}, \beta_{j_s}\}$ and so we have the implication $\begin{tikzcd}[row sep=small] & \beta_{j_{s-1}} \\ \beta_r \arrow[ur, dashed] \arrow[dr, dashed] & \\ & \beta_{j_s} \end{tikzcd}$. Hence one of the two roots $\beta_{j_s}$, $\beta_{j_{s-1}}$ is in $\Delta$. If, for example, $\beta_{j_s} \in \Delta$, one has, as in the first case, $X^{(j_s+1)}_{j_s} \in P^{(j_s+1)}$ and $X^{(r)}_{j_s} \in P^{(r)}$. The considered monomial is in $P^{(r)}$ as claimed.

- • • If $\beta_{j_{s-1}} = \beta_{j_s}$, then the equality (9) becomes $\beta_r + \alpha_k = 2\beta_{j_s}$. Set $\beta = s_D(\beta_{j_s}) = \beta_r - \beta_{j_s} \in \Phi^+$ and substract $\beta_{j_s}$ to each part of the previous equality, to obtain $\beta + \alpha_k = \beta_{j_s}$. Denote by $P$ the plane spanned by $\beta_r$ and $\beta_{j_s}$.
  Assume that $\Phi$ is of type $G_2$. Then one has $\beta_r = \beta_4$, $\alpha_k = \beta_1$ and $\beta_{j_s} = \beta_3$. By Definition 5.2.1, we have the implication $\beta_r \to \beta_{j_s}$.
  Assume that $\Phi$ is not of type $G_2$. The equality $\beta_r + \alpha_k = 2\beta_{j_s}$ implies that $\Phi_P$ is of type $B_2$, so that $\Phi_P^+ = \{\alpha_k, \alpha_k + \beta = \beta_{j_s}, \alpha_k + 2\beta = \beta_r, \beta\}$ with $h(\beta) = h(\beta_r) - h(\beta_{j_s}) = 2l + 1 - (l + 1) = l$. So $P$ is an admissible plane of type 4 in the sense of Definition 5.2.1. So we have again the implication $\beta_r \to \beta_{j_s}$.
  Thus, in all cases, one has $\beta_{j_s} \in \Delta$. So we have, as in the first case, $X_{j_s}^{(j_s+1)} \in P^{(j_s+1)}$ and $X_{j_s}^{(r)} \in P^{(r)}$. The considered monomial is again in $P^{(r)}$, as desired. $\square$

## 6. Cauchon diagrams for a particular decomposition of $w_0$

In this section, we give an explicit description of Cauchon diagrams for a chosen decomposition of $w_0$ in each type of simple Lie algebra of finite dimension. Denote by $\mathcal{D}$ the set of Cauchon diagrams. For all $\beta \in \Phi^+$, we give the list of implications of the type $\beta \to \beta'$ with $\beta' \in \Phi^+$.

**Definition 6.0.4.** Let $\beta \in \Phi^+$. An implication from the root $\beta$ is an implication from an admissible plane of the type $\beta \to \beta'$ or
$$\begin{tikzcd}[row sep=small] & \beta'_1 \\ \beta \arrow[ur, dashed] \arrow[r, dashed] \arrow[dr, dashed] & \vdots \\ & \beta'_s \end{tikzcd}$$
(Definition 5.2.1).

**Observation.** The implications from all admissible planes coincide with the implications from all the positive roots.

**Lemma 6.0.5.** *Suppose that $\Phi$ is a root system which is not of type $G_2$.*

1. *Let $C_l$ be an ordinary column. If $\beta \in C_l$, then the implications from $\beta$ are $\beta \to \beta'$ with $\beta' \in C_l$, $\beta' = \beta + \alpha_i$ $(i < l)$.*
2. *Let $C_l$ be an exceptional column and $\beta \in C_l$.*
   (a) *If $\beta \neq \beta_{ex}$ and if $\beta$ is not in $B$, the box after $\{\beta_{ex}\}$, then the implications from $\beta$ are $\beta \to \beta'$ with $\beta' \in C_l$, $\beta' = \beta + \alpha_i$ $(i < l)$.*
   (b) *If $\beta \in B$, the box after $\{\beta_{ex}\}$, then the implications from $\beta$ are $\beta \to \beta'$ with $\beta' \in C_l$, $\beta' = \beta + \alpha_i$ $(i < l)$ and $\beta \to \beta_{ex}$.*
3. *Let $C_l$ be an exceptional column with exceptional root $\beta_{ex}$ and $B_1$ the box before $\{\beta_{ex}\}$. Then the implications from $\beta$ are*:
   - *$\beta_{ex} \to \beta'$ with $\beta' \in B_1$ such that $P = \langle \beta_{ex}, \beta' \rangle$ is an admissible plan of type 2.2 (i.e. $\Phi_P^+ = \{\beta, \beta_{ex} = \epsilon_i + 2\beta, \beta' = \epsilon_i + \beta, \epsilon_i\}$ with $i < l$ and $\beta \in B$ the box after $\{\beta_{ex}\}$).*
   - $$\begin{tikzcd}[row sep=small] & \beta'_1 \\ \beta_{ex} \arrow[ur, dashed] \arrow[dr, dashed] & \\ & \beta'_2 \end{tikzcd}$$ *with $\beta'_1, \beta'_2 \in B_1$, $\beta'_1 + \beta'_2 = \beta_{ex} + \epsilon_i$ $(i < l)$ and $P = \langle \beta_{ex}, \epsilon_i \rangle$ is an admissible plane of type 2.4 (i.e. $\Phi_P^+ = \{\beta_{ex}, \epsilon_i\}$).*

**Proof.**

1. Let $\beta' \in C_l$ with $\beta' = \beta + \alpha_i$ and $i < l$. From Lemma 5.2.2, $\beta \to \beta'$ is an implication from an admissible plane. So this is an implication from $\beta$.
   Conversely, consider an implication $\beta \to \beta'$ from $\beta$, so that $\beta' \in C_l$ (Lemma 5.2.2). As $C_l$ is ordinary, $\beta$ and $\beta'$ are ordinary roots and, by Lemma 5.2.2, one has $\beta' = \beta + \epsilon_i$ with $i < l$.

2. (a) Let $\beta' \in C_l$ with $\beta' = \beta + \alpha_i$ and $i < l$. From Lemma 5.2.2, $\beta \to \beta'$ is an implication from an admissible plane. So this is an implication from $\beta$.
   Conversely, consider an implication $\beta \to \beta'$ from $\beta$, so that $\beta' \neq \beta$, $P = \langle \beta, \beta' \rangle$ is an admissible plane and $\beta \to \beta'$ is an implication from $P$. From Lemma 5.2.2, we know that $\beta' \in C_l$.
   Suppose that $\beta' = \beta_{ex}$, so that the type of $P$ is in the following list:
   - type 1.1 with $\Phi_P^+ = \{\beta_1, \beta_{ex} = \beta_1 + \beta_2, \beta_2\}$, $\beta_1 > \beta_{ex} > \beta_2$.
   - type 1.2 with $\Phi_P^+ = \{\beta_1, \beta_{ex} = 2\beta_1 + \alpha_i, \beta_2 = \beta_1 + \alpha_i, \alpha_i\}$ $(i < l)$, and $\beta_1 > \beta_{ex} > \beta_2 > \epsilon_i$.

   As $\beta \to \beta_{ex} = \beta'$ is an implication, we deduce from Definition 5.2.1 that $\beta = \beta_1$. Then Definition 2.2.13 permits to claim that $\beta$ is in the box after $\beta_{ex}$, which contradict the hypothesis. So $\beta' \neq \beta_{ex}$. Moreover $\beta \neq \beta_{ex}$, it comes from Lemma 5.2.2 that $\beta' = \beta + \alpha_i$ with $i < l$.

   (b) As $\beta \in B$, the implication $\beta \to \beta_{ex}$ comes from Lemma 5.2.2. If $\beta' = \beta + \alpha_i$ is a root, the implication $\beta \to \beta'$ also comes from Lemma 5.2.2.
   Conversely, let $\beta \to \beta'$ an implication from $\beta$. By Lemma 5.2.2, we know that $\beta' \in C_l$.
   If $\beta' = \beta_{ex}$, there is nothing to prove. Otherwise, as $\beta \neq \beta_{ex}$, one has $\beta' = \beta + \alpha_i$ with $i < l$ by Lemma 5.2.2.
3. If $\beta' \in B_1$ satisfies the hypothesis, Definition 5.2.1 permits to claim that $\beta_{ex} \to \beta'$ is an implication from $P$. If $\beta'_1$, $\beta'_2$ belong to $B_1$ and satisfy the hypothesis, Definition 5.2.1 permits to claim that

   $$\begin{tikzcd}[row sep=tiny] & \beta'_1 \\ \beta_{ex} \arrow[ur, dashed] \arrow[dr, dashed] & \\ & \beta'_2 \end{tikzcd}$$ is an implication from $P$.

   Moreover, Definition 5.2.1 permits to claim that all implications from $\beta_{ex}$ come from an admissible plane $P$ of type 1.2 or 2.4.
   - If $P$ is of type 1.2, one has $\Phi_P^+ = \{\beta, \beta_{ex} = 2\beta + \alpha_1, \beta' = \beta + \alpha_i, \alpha_i\}$ $(i < l)$ and $\beta > \beta_{ex} > \beta' > \alpha_i$. In this case, the only implication from $\beta_{ex}$ and from $P$, is $\beta_{ex} \to \beta'$ with $\langle \beta_{ex}, \beta' \rangle = P$ admissible plane of type 1.2.
   - If $P$ is of type 2.4, one has $P = \langle \beta_{ex}, \alpha_i \rangle$ and $\Phi_P^+ = \{\beta_{ex}, \alpha_i\}$ $(i < l)$. Definition 5.2.1 permits to claim that all the implication from $P$ are of the shape $$\begin{tikzcd}[row sep=tiny] & \beta'_1 \\ \beta_{ex} \arrow[ur, dashed] \arrow[dr, dashed] & \\ & \beta'_2 \end{tikzcd}$$ , where $\beta'_1$ and $\beta'_2$ belong to $B_1$ and satisfy $\beta'_1 + \beta'_2 = \beta_{ex} + \alpha_i$. $\square$

### *6.1. Infinite series*

#### *6.1.1. Type $A_n$, $n \geqslant 1$*

**Convention.** The numbering of simple roots in the Dynkin diagram is as follow: $\alpha_1 - \alpha_2 - \cdots - \alpha_{n-1} - \alpha_n$. We know (see for example [Lit98, Section 5]) that $s_{\alpha_1} \circ (s_{\alpha_2} \circ s_{\alpha_1}) \cdots \circ (s_{\alpha_n} \circ s_{\alpha_{n-1}} \circ \cdots \circ s_{\alpha_1})$ is a reduced decomposition of $w_0$ which induces the following order on positive roots. (We have arranged the roots in columns.)

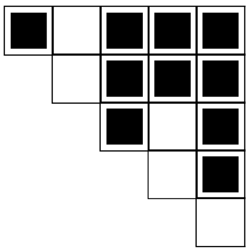
| $C_1$ | $C_2$ | | $C_n$ |
|---|---|---|---|
| $\beta_1 = \alpha_1$ | $\beta_2 = \alpha_1 + \alpha_2$ | $\cdots$ | $\beta_{N-n+1} = \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n$ |
| | $\beta_3 = \alpha_2$ | $\cdots$ | $\vdots$ |
| | | $\ddots$ | $\vdots$ |
| | | | $\beta_N = \alpha_n$ |

This is a Lusztig order and none of the columns $C_1, \dots, C_n$ is exceptional. Moreover, if two roots $\beta > \beta'$ are in the same column $C_l$ then: $\beta' = \beta + \alpha_i$ $(i < l)$ $\Leftrightarrow$ $\beta'$ and $\beta$ are consecutive.

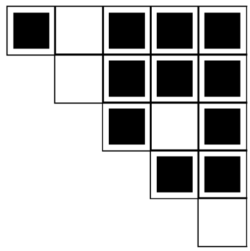
**Proposition 6.1.1.** *Let $\Delta$ be a diagram, $\Delta$ is a Cauchon diagram if and only if it satisfies all the implications $\beta_{j+1} \to \beta_j$ where $\beta_j$ and $\beta_{j+1}$ are two consecutive roots of the same column $C_l$.*

**Convention.** If $C_l = \{\beta_s, \beta_{s+1}, \ldots, \beta_r = \alpha_l\}$ is the column $l$ with $1 \leqslant l \leqslant n$, the truncated columns contained in $C_l$ are the following subsets $\{\beta_s, \beta_{s+1}, \ldots, \beta_t\}$, $t \in [\![s, r]\!]$.

Proposition 6.1.1 permits to claim that the Cauchon diagrams are the diagrams $\Delta$ which are unions of truncated columns. In the following picture a positive roots $\beta$, belonging to the diagram $\Delta$, is represented by a black box in the location of $\beta$ in the previous tabular of the order induced by the choosen reduced decomposition of $w_0$. This convention will be used in the rest of this article.



$\in \mathcal{D}$



$\notin \mathcal{D}$

**Remark 6.1.2.** The set of Cauchon diagrams $\mathcal{D}$ has the same cardinality as the Weyl group $W$.

**Proof.** As $\mathcal{D}$ is the set of all diagrams $\Delta$ which are unions of truncated columns, one has $|\mathcal{D}| = (n+1)! = |W|$.

*6.1.2. Type $B_n$, $n \geqslant 2$*

**Convention.** The numbering of simple roots in the Dynkin diagram is as follow $\alpha_1 \Leftarrow \alpha_2 - \cdots - \alpha_{n-1} - \alpha_n$. We know (see for example [Lit98, Section 6]) that $s_{\alpha_1} \circ (s_{\alpha_2} \circ s_{\alpha_1} \circ s_{\alpha_2}) \cdots \circ (s_{\alpha_n} \circ s_{\alpha_{n-1}} \circ \cdots \circ s_{\alpha_2} \circ s_{\alpha_1} \circ s_{\alpha_2} \circ \cdots \circ s_{\alpha_n})$ is a reduced decomposition of $w_0$ which induces the following order on positive roots.

| | | | |
|---|---|---|---|
| $\beta_1 = \epsilon_1$ | $\beta_2 = 2\epsilon_1 + \epsilon_2$ | | $\beta_{(n-1)^2+1} = 2\epsilon_1 + \cdots + 2\epsilon_{n-1} + \epsilon_n$ |
| | $\beta_3 = \epsilon_1 + \epsilon_2$ | | $\vdots$ |
| | $\beta_4 = \epsilon_2$ | | $\vdots$ |
| | | | $\beta_{N-n} = 2\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1} + \epsilon_n$ |
| | | | $\beta_{N-n+1} = \epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n$ |
| | | | $\beta_{N-n+2} = \epsilon_2 + \cdots + \epsilon_{n-1} + \epsilon_n$ |
| | | | $\vdots$ |
| | | | $\beta_N = \epsilon_n$ |

This is a Lusztig order and none of the columns $C_1, \ldots, C_n$ is exceptional. Moreover, if two roots $\beta > \beta'$ are in the same column $C_l$ then: $\beta' = \beta + \alpha_i$ $(i < l)$ $\Leftrightarrow$ $\beta'$ and $\beta$ are consecutive.

**Proposition 6.1.3.** *Let $\Delta$ be a diagram, $\Delta$ is a Cauchon diagram if and only if it satisfies all the implications $\beta_{j+1} \to \beta_j$ where $\beta_j$ and $\beta_{j+1}$ are two consecutive roots of the same column $C_l$.*

**Convention.** If $C_l = \{\beta_s, \beta_{s+1}, \ldots, \beta_r = \alpha_l\}$ is the column $l$ with $1 \leqslant l \leqslant n$, the truncated columns contained in $C_l$ are the following subsets $\{\beta_s, \beta_{s+1}, \ldots, \beta_t\}$, $t \in [\![s, r]\!]$.

Proposition 6.1.3 permits to claim that the Cauchon diagrams are the diagrams $\Delta$ which are unions of truncated columns.

**Remark 6.1.4.** The set of Cauchon diagrams $\mathcal{D}$ has the same cardinality as the Weyl group $W$.

**Proof.** As $\mathcal{D}$ is the set of all diagrams $\Delta$ which are unions of truncated columns, one has $|\mathcal{D}| = 2^{n+1}(n+1)! = |W|$. $\square$

*6.1.3. Type $C_n$, $n \geqslant 3$*

**Convention.** The numbering of simple roots in the Dynkin diagram is $\alpha_1 \Rightarrow \alpha_2 - \cdots - \alpha_{n-1} - \alpha_n$. We know (see for example [Lit98, Section 6]) that $s_{\alpha_1} \circ (s_{\alpha_2} \circ s_{\alpha_1} \circ s_{\alpha_2}) \cdots \circ (s_{\alpha_n} \circ s_{\alpha_{n-1}} \circ \cdots \circ s_{\alpha_2} \circ s_{\alpha_1} \circ s_{\alpha_2} \circ \cdots \circ s_{\alpha_n})$ is a reduced decomposition of $w_0$ which induces the following order on positive roots.

| | | | |
|---|---|---|---|
| $\beta_1 = \epsilon_1$ | $\beta_2 = \epsilon_1 + \epsilon_2$ | | $\beta_{(n-1)^2+1} = \epsilon_1 + 2\epsilon_2 + \cdots + 2\epsilon_{n-1} + \epsilon_n$ |
| | $\beta_3 = \epsilon_1 + 2\epsilon_2$ | | $\vdots$ |
| | $\beta_4 = \epsilon_2$ | | $\vdots$ |
| | | | $\beta_{N-n} = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1} + \epsilon_n$ |
| | | | $\beta_{N-n+1} = \epsilon_1 + 2\epsilon_2 + \cdots + 2\epsilon_{n-1} + 2\epsilon_n$ |
| | | | $\beta_{N-n+2} = \epsilon_2 + \cdots + \epsilon_{n-1} + \epsilon_n$ |
| | | | $\vdots$ |
| | | | $\beta_N = \epsilon_n$ |

This is a Lusztig order and all the columns $C_2, \ldots, C_n$ are exceptional, the first one $C_1$ is ordinary. We obtain the same result as for $B_n$, the proof is a bit more technical due to the exceptional columns and is left to the reader.

**Proposition 6.1.5.** *Let $\Delta$ be a diagram, $\Delta$ is a Cauchon diagram if and only if it satisfies all the implications $\beta_{j+1} \to \beta_j$ where $\beta_j$ and $\beta_{j+1}$ are two consecutive roots of the same column $C_l$.*

Proposition 6.1.5 permits to claim that the Cauchon diagrams are the diagrams $\Delta$ which are unions of truncated columns.

**Remark 6.1.6.** The set of Cauchon diagrams $\mathcal{D}$ has the same cardinality as the Weyl group $W$.

**Proof.** As $\mathcal{D}$ is the set of all diagrams $\Delta$ which are unions of truncated columns, one has $|\mathcal{D}| = 2^{n+1}(n+1)! = |W|$. $\square$

*6.1.4. Type $D_n$, $n \geqslant 4$*

**Convention.** The numbering of simple roots in the Dynkin diagram is

$$\begin{array}{ccccccccc} \alpha_1 & & & & & & & & \\ & \diagdown & & & & & & & \\ & & \alpha_3 & - & \alpha_4 & - & \cdots & - & \alpha_{n-1} & - & \alpha_n\,. \\ & \diagup & & & & & & & \\ \alpha_2 & & & & & & & & \end{array}$$

We know (see for example [Lit98, Section 6]) that $s_{\alpha_1} \circ s_{\alpha_2} \circ (s_{\alpha_3} \circ s_{\alpha_1} \circ s_{\alpha_2} \circ s_{\alpha_3}) \cdots \circ (s_{\alpha_n} \circ s_{\alpha_{n-1}} \circ \cdots \circ s_{\alpha_3} \circ s_{\alpha_1} \circ s_{\alpha_2} \circ s_{\alpha_3} \circ \cdots \circ s_{\alpha_n})$ is a reduced decomposition of $w_0$ which induces the following order on positive roots.

| | | | | |
|---|---|---|---|---|
| $\beta_1 = \epsilon_1$ | $\beta_2 = \epsilon_2$ | $\beta_3 = \epsilon_1 + \epsilon_2 + \epsilon_3$ | $\cdots$ | $\beta_{N-2n+1} = \epsilon_1 + \epsilon_2 + 2\epsilon_3 \cdots + 2\epsilon_{n-1} + \epsilon_n$ |
| | | $\beta_4 = \epsilon_2 + \epsilon_3$ | | $\vdots$ |
| | | $\beta_5 = \epsilon_1 + \epsilon_3$ | | $\vdots$ |
| | | $\beta_6 = \epsilon_3$ | | $\beta_{N-n-1} = \epsilon_1 + \epsilon_2 + \epsilon_3 \cdots + \epsilon_{n-1} + \epsilon_n$ |
| | | | | $\beta_{N-n} = \epsilon_1$ or$^\star$ $\epsilon_2$ $+ \epsilon_3 \cdots + \epsilon_{n-1} + \epsilon_n$ |
| | | | | $\beta_{N-n+1} = \epsilon_2$ or$^\star$ $\epsilon_1$ $+ \epsilon_3 \cdots + \epsilon_{n-1} + \epsilon_n$ |
| | | | | $\beta_{N-n+2} = \epsilon_3 + \cdots + \epsilon_{n-1} + \epsilon_n$ |
| | | | | $\vdots$ |
| | | | | $\beta_N = \epsilon_n$ |

$\star$: depends on columns' parity

This is a Lusztig order and all the columns are ordinary.

**Observation.** Let $l \geqslant 3$.

- The column $C_l$ has an even number of roots, so that there is $s \in \mathbb{N}$ $(s = l - 1)$ such that $C_l = \{\beta_{u_1} < \cdots < \beta_{u_s} < \beta_{u_{s+1}} < \cdots < \beta_{u_{2s}}\}$.
- Let $\beta$ an element of $C_l$ different from $\beta_{u_1}$.
  - •• If $\beta = \beta_{u_{s+2}}$, there is exactly 2 roots in $C_l$ of the shape $\beta' = \beta + \alpha_i$ $(i < l)$, namely $\beta_{u_s}$ and $\beta_{u_{s+1}}$.
  - •• If $\beta \neq \beta_{u_{s+2}}$, there is only one root in $C_l$ of the shape $\beta' = \beta + \alpha_i$ $(i < l)$, namely $\beta'$ is the root before $\beta$ if $\beta \neq \beta_{u_{s+1}}$ or $\beta' = \beta_{u_{s-1}}$ if $\beta = \beta_{u_{s+1}}$.

As there is no exceptional column, we deduce from Theorem 5.3.1 and Lemma 6.0.5(1.),

**Proposition 6.1.7.** *Let $\Delta$ be a diagram, $\Delta$ is a Cauchon diagram if and only if it satisfies all the implications below, for all integers $l \in [\![3, ]\!]$, denote $C_l = \{\beta_{u_1}, \ldots, \beta_{u_s}, \beta_{u_{s+1}}, \ldots, \beta_{u_{2s}}\}$ (with $s = l - 1$):*

$$\begin{tikzcd}
 & & & & \beta_{u_s} \arrow[dr] & & & & \\
\beta_{u_{2s}} \arrow[r] & \beta_{u_{2s-1}} \arrow[r] & \ldots \arrow[r] & \beta_{u_{s+2}} \arrow[ur] \arrow[dr] & & \beta_{u_{s-1}} \arrow[r] & \ldots \arrow[r] & \beta_{u_2} \arrow[r] & \beta_{u_1} \\
 & & & & \beta_{u_{s+1}} \arrow[ur] & & & &
\end{tikzcd}$$

Proposition 6.1.7 permits to claim that Cauchon diagrams are the sets $\Delta = \bigsqcup_{l \in [\![1,n]\!]} \Delta_l$, where $\Delta_1$ is a truncated column from $C_1$, $\Delta_2$ is a truncated column from $C_2$ and, for $l \in [\![3, n]\!]$, denote $C_l = \{\beta_{u_1} < \cdots < \beta_{u_s} < \beta_{u_{s+1}} < \cdots < \beta_{u_{2s}}\}$ $(s = l - 1)$, $\Delta_l$ is a truncated column $\{\beta_{u_1} < \cdots < \beta_{u_{j-1}} < \beta_{u_j}\}$ from $C_l$, or the set $\{\beta_{u_1} < \cdots < \beta_{u_{s-1}} < \beta_{u_{s+1}}\} \subset C_l$.

**Proposition 6.1.8.** *The set $\mathcal{D}$ of Cauchon diagrams has the same cardinality as the Weyl group $W$.*

**Proof.** $\Delta_1$ can be two sets ($\emptyset$ or $C_1$) as $\Delta_2$ ($\emptyset$ or $C_2$). If $l \in [\![3, n]\!]$, one has $|C_3| = 2l - 2$. One can then extract $2l - 1$ truncated columns from $C_l$ so that there is $2l$ possibilities for $\Delta_l$. As a result $|\mathcal{D}| = 2 \times 2 \times 6 \times \cdots \times 2n = 4 \times 6 \times 8 \times \cdots \times 2n = 2^{n-1}(n!) = |W|$. $\square$

*6.2. Exceptional cases*

*6.2.1. Type $G_2$*

**Convention.** The numbering of simple roots in the Dynkin diagram is: $\alpha_1 \Lleftarrow \alpha_2$. We know that $s_{\alpha_1} \circ s_{\alpha_2} \circ s_{\alpha_1} \circ s_{\alpha_2} \circ s_{\alpha_1} \circ s_{\alpha_2}$ is a reduced decomposition of $w_0$ which induces the following order on positive roots: $\beta_1 = \alpha_1, \beta_2 = 3\alpha_1 + \alpha_2, \beta_3 = 2\alpha_1 + \alpha_2, \beta_4 = 3\alpha_1 + 2\alpha_2, \beta_5 = \alpha_1 + \alpha_2, \beta_6 = \alpha_2$.

**Lemma 6.2.1.** *One has the following implications*: $\beta_6 \longrightarrow \beta_5 \longrightarrow \beta_4 \longrightarrow \beta_3 \longrightarrow \beta_2$.

**Proof.** To prove this implications, we apply Propositions 5.1.6, 5.1.7 and 5.1.8 with the following equalities ($\beta_4$ is an exceptional root): $\beta_6+\alpha_1=\beta_5$, $h'(\beta_5)+1=\beta_4$, $\beta_4+\alpha_1=2\beta_3$, $\beta_3+\alpha_1=\beta_2$. $\square$

**Convention.** $\mathcal{D}$ is the set of Cauchon diagrams, they satisfy implications from Lemma 6.2.1.

**Remark 6.2.2.** The set of Cauchon diagrams $\mathcal{D}$ has the same cardinality as the Weyl group $W$.

*6.2.2. Type $F_4$*

**Convention.** The numbering of simple roots in the Dynkin diagram is: $\alpha_1 - \alpha_2 \Rightarrow \alpha_3 - \alpha_4$. We choose the following reduced decomposition of $w_0$: $s_4s_3s_4s_2s_3s_4s_2s_3s_2s_1s_2s_3s_4s_2s_3s_1s_2s_1s_3s_4s_2s_3s_2s_1$. This decomposition induces the following order on positive roots:

Column 1: $\beta_1(0,0,0,1)$

Column 2: $\beta_2(0,0,1,1)$, $\beta_3(0,0,1,0)$

Column 3: $\beta_4(0,1,2,2)$, $\beta_5(0,1,2,1)$, $\beta_6(0,1,1,1)$, $\beta_7(0,1,2,0)$, $\beta_8(0,1,1,0)$, $\beta_9(0,1,0,0)$

Column 4: $\beta_{10}(1,3,4,2)$, $\beta_{11}(1,2,4,2)$, $\beta_{12}(1,2,3,2)$, $\beta_{13}(1,2,3,1)$, $\beta_{14}(1,2,2,2)$, $\beta_{15}(1,2,2,1)$, $\beta_{16}(1,1,2,2)$, $\beta_{17}(2,3,4,2)$, $\beta_{18}(1,2,2,0)$, $\beta_{19}(1,1,2,1)$, $\beta_{20}(1,1,1,1)$, $\beta_{21}(1,1,2,0)$, $\beta_{22}(1,1,1,0)$, $\beta_{23}(1,1,0,0)$, $\beta_{24}(1,0,0,0)$

One checks that each column is ordinary or exceptional and then computes $h'(\beta_i)$ for all roots to verify that the order is a Lusztig one. We already know the form of diagrams for the two first columns. Thanks to commutation relations, Propositions 5.1.6, 5.1.7 and 5.1.8, we obtain the following result:

**Proposition 6.2.3.** *Let $\Delta$ be a diagram, $\Delta$ is a Cauchon diagram if and only if it satisfies the following implications*:

$$3 \to 2, \qquad \begin{tikzcd} & & 6 \arrow[dr] & & \\ 9 \arrow[r] & 8 \arrow[ur] \arrow[dr] & & 5 \arrow[r] & 4, \\ & & 7 \arrow[ur] & & \end{tikzcd} \qquad \begin{tikzcd} & & & 20 \arrow[r] & 19 \arrow[rr] \arrow[dr] & & 16 \arrow[r] & 14 \arrow[dr] & & & \\ 24 \arrow[r] & 23 \arrow[r] & 22 \arrow[ur] \arrow[dr] & & & 17 \arrow[dr] & & & 12 \arrow[r] & 11 \arrow[r] & 10 \\ & & & 21 \arrow[r] \arrow[uur] & 18 \arrow[ur] & & 15 \arrow[r] \arrow[uur] & 13 \arrow[ur] & & & \end{tikzcd}$$

This permits to claim that the Cauchon diagrams are the sets $\Delta = \bigsqcup_{l\in[\![1,4]\!]} \Delta_l$ where $\Delta_1$ is a truncated column from $C_1$, $\Delta_2$ is a truncated column from $C_2$, $\Delta_3$ are $\Delta_4$ subsets of $C_3$ and $C_4$ respectively which satisfy the implication from Proposition 6.2.3. By counting the possibilities, one obtains:

**Proposition 6.2.4.** *The set $\mathcal{D}$ of Cauchon diagrams has same cardinality as the Weyl group $W$.*

*6.2.3. Type $E_6$*

**Convention.** The numbering of simple roots in the Dynkin diagram is:
$$\begin{array}{ccccccccc} & & & & \alpha_2 & & & & \\ & & & & | & & & & \\ \alpha_1 & - & \alpha_3 & - & \alpha_4 & - & \alpha_5 & - & \alpha_6 \end{array}.$$
To describe the chosen reduced decomposition of $w_0$, we remark that the roots $\alpha_1$ to $\alpha_5$ span a roots system of $D_5$. Denote by $\tau$, the longest Weyl word used for $D_5$ then the decomposition $\tau s_6s_5s_4s_2s_3s_1s_4s_3s_5s_4s_6s_2s_5s_4s_3s_1$ is a reduced decomposition of $w_0$ which induces the following order on positive roots, the first five columns are the same as in $D_5$ and the sixth is:

$\beta_{21} = (1, 2, 2, 3, 2, 1)$, $\beta_{22} = (1, 1, 2, 3, 2, 1)$, $\beta_{23} = (1, 1, 2, 2, 2, 1)$, $\beta_{24} = (1, 1, 2, 2, 1, 1)$,

$\beta_{25} = (1, 1, 1, 2, 2, 1)$, $\beta_{26} = (0, 1, 1, 2, 2, 1)$, $\beta_{27} = (1, 1, 1, 2, 1, 1)$, $\beta_{28} = (0, 1, 1, 2, 1, 1)$,

$\beta_{29} = (1, 1, 1, 1, 1, 1)$, $\beta_{30} = (0, 1, 1, 1, 1, 1)$, $\beta_{31} = (1, 0, 1, 1, 1, 1)$, $\beta_{32} = (0, 1, 0, 1, 1, 1)$,

$\beta_{33} = (0, 0, 1, 1, 1, 1)$, $\beta_{34} = (0, 0, 0, 1, 1, 1)$, $\beta_{35} = (0, 0, 0, 0, 1, 1)$, $\beta_{36} = (0, 0, 0, 0, 0, 1)$.

We obtain, by Lemma 6.0.5(1.) and Theorem 5.3.1,

**Proposition 6.2.5.** *Let $\Delta$ be a diagram, $\Delta$ is a Cauchon diagram if and only if it satisfies all the implications from Proposition* 6.1.7 *for the five first columns and the following implications for the last one*:

$$\begin{tikzcd}
 & & & 33 \arrow[r] \arrow[drr] & 31 \arrow[r] & 29 \arrow[r] & 27 \arrow[r] \arrow[drr] & 24 \arrow[dr] & & & \\
36 \arrow[r] & 35 \arrow[r] & 34 \arrow[ur] \arrow[dr] & & & & & & 23 \arrow[r] & 22 \arrow[r] & 21 \\
 & & & 32 \arrow[r] & 30 \arrow[r] \arrow[uur] & 28 \arrow[r] \arrow[uur] & 26 \arrow[r] & 25 \arrow[ur] & & &
\end{tikzcd}$$

**Proposition 6.2.6.** *The set $\mathcal{D}$ of Cauchon diagrams has same cardinality as the Weyl group $W$.*

*6.2.4. Type $E_7$*

**Convention.** The numbering of simple roots in the Dynkin diagram is:

$$\begin{array}{ccccccccccccc}
 & & & & \alpha_2 & & & & & & \\
 & & & & | & & & & & & \\
\alpha_1 & - & \alpha_3 & - & \alpha_4 & - & \alpha_5 & - & \alpha_6 & - & \alpha_7
\end{array}$$

As the roots $\alpha_1$ to $\alpha_6$ span a roots system of type $E_6$, denote by $\sigma$ the longest Weyl word used for the type $E_6$. The decomposition $\sigma s_7 s_6 s_5 s_4 s_2 s_3 s_1 s_4 s_3 s_5 s_4 s_6 s_2 s_5 s_7 s_4 s_6 s_3 s_5 s_1 s_4 s_2 s_3 s_4 s_5 s_6 s_7$ is a reduced decomposition of $w_0$ which induces the following order on positive roots (only the last column is given). We already know the form of diagrams for the first six columns. Proposition 5.1.6 permits to find the implications in the last columns of type $E_7$.

| $\beta_i$ | $h'(\beta_i)$ |
|---|---|
| $\beta_{37}(2, 2, 3, 4, 3, 2, 1)$ | 17 |
| $\beta_{38}(1, 2, 3, 4, 3, 2, 1)$ | 16 |
| $\beta_{39}(1, 2, 2, 4, 3, 2, 1)$ | 15 |
| $\beta_{40}(1, 2, 2, 3, 3, 2, 1)$ | 14 |
| $\beta_{41}(1, 1, 2, 3, 3, 2, 1)$ | 13 |
| $\beta_{42}(1, 2, 2, 3, 2, 2, 1)$ | 13 |
| $\beta_{43}(1, 2, 2, 3, 2, 1, 1)$ | 12 |
| $\beta_{44}(1, 1, 2, 3, 2, 2, 1)$ | 12 |
| $\beta_{45}(1, 1, 2, 3, 2, 1, 1)$ | 11 |
| $\beta_{46}(1, 1, 2, 2, 2, 2, 1)$ | 11 |
| $\beta_{47}(1, 1, 2, 2, 2, 1, 1)$ | 10 |
| $\beta_{48}(1, 1, 1, 2, 2, 2, 1)$ | 10 |
| $\beta_{49}(1, 1, 2, 2, 1, 1, 1)$ | 9 |
| $\beta_{50}(1, 1, 1, 2, 2, 1, 1)$ | 9 |

| $\beta_i$ | $h'(\beta_i)$ |
|---|---|
| $\beta_{51}(0, 1, 1, 2, 2, 2, 1)$ | 9 |
| $\beta_{52}(1, 1, 1, 2, 1, 1, 1)$ | 8 |
| $\beta_{53}(0, 1, 1, 2, 2, 1, 1)$ | 8 |
| $\beta_{54}(1, 1, 1, 1, 1, 1, 1)$ | 7 |
| $\beta_{55}(0, 1, 1, 2, 1, 1, 1)$ | 7 |
| $\beta_{56}(1, 0, 1, 1, 1, 1, 1)$ | 6 |
| $\beta_{57}(0, 1, 1, 1, 1, 1, 1)$ | 6 |
| $\beta_{58}(0, 1, 0, 1, 1, 1, 1)$ | 5 |
| $\beta_{59}(0, 0, 1, 1, 1, 1, 1)$ | 5 |
| $\beta_{60}(0, 0, 0, 1, 1, 1, 1)$ | 4 |
| $\beta_{61}(0, 0, 0, 0, 1, 1, 1)$ | 3 |
| $\beta_{62}(0, 0, 0, 0, 0, 1, 1)$ | 2 |
| $\beta_{63}(0, 0, 0, 0, 0, 0, 1)$ | 1 |

$$\begin{tikzcd}
 & & 63 \arrow[d] & & \\
 & & 62 \arrow[d] & & \\
 & & 61 \arrow[d] & & \\
 & & 60 \arrow[dl] \arrow[dr] & & \\
 & 59 \arrow[d] \arrow[drr] & & 58 \arrow[d] & \\
 & 56 \arrow[d] & & 57 \arrow[d] \arrow[dll] & \\
 & 54 \arrow[d] & & 55 \arrow[d] \arrow[dll] & \\
 & 52 \arrow[dl] \arrow[dr] & & 53 \arrow[dl] \arrow[dr] & \\
49 \arrow[dr] & & 50 \arrow[dl] \arrow[dr] & & 51 \arrow[dl] \\
 & 47 \arrow[d] \arrow[drr] & & 48 \arrow[d] & \\
 & 45 \arrow[d] \arrow[drr] & & 46 \arrow[d] & \\
 & 43 \arrow[d] & & 44 \arrow[d] \arrow[dll] & \\
 & 42 \arrow[dr] & & 41 \arrow[dl] & \\
 & & 40 \arrow[d] & & \\
 & & 39 \arrow[d] & & \\
 & & 38 \arrow[d] & & \\
 & & 37 & &
\end{tikzcd}$$

**Proposition 6.2.7.** *The set $\mathcal{D}$ of Cauchon diagrams has same cardinality as the Weyl group $W$.*

*6.2.5. Type $E_8$*

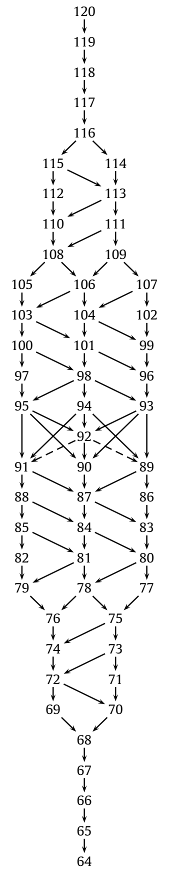
**Convention.** The numbering of simple roots in the Dynkin diagram is:

$$\begin{array}{ccccccccccccc} & & & & \alpha_2 & & & & & & & & \\ & & & & | & & & & & & & & \\ \alpha_1 & - & \alpha_3 & - & \alpha_4 & - & \alpha_5 & - & \alpha_6 & - & \alpha_7 & - & \alpha_8 \end{array}$$

$\{\alpha_1, \dots, \alpha_7\}$ span a roots system of type $E_7$. Denote by $\sigma_7$, the longest Weyl word used for type $E_7$. The decomposition $\sigma_7 s_8 s_7 s_6 s_5 s_4 s_2 s_3 s_1 s_4 s_3 s_5 s_4 s_6 s_2 s_5 s_7 s_4 s_6 s_8 s_3 s_5 s_7 s_1 s_4 s_6 s_3 s_2 s_5 s_4 s_5 s_2 s_3 s_6 s_1 s_4 s_7 s_3 s_5 s_8$ $s_4 s_6 s_2 s_5 s_7 s_4 s_6 s_3 s_5 s_1 s_4 s_2 s_3 s_4 s_5 s_6 s_7 s_8$ is a reduced decomposition of $w_0$ which induces the following order on positive roots of the last column.

| $\beta_i$ | $h'(\beta_i)$ |
|---|---|
| $\beta_{64}(2,3,4,6,5,4,3,1)$ | 28 |
| $\beta_{65}(2,3,4,6,5,4,2,1)$ | 27 |
| $\beta_{66}(2,3,4,6,5,3,2,1)$ | 26 |
| $\beta_{67}(2,3,4,6,4,3,2,1)$ | 25 |
| $\beta_{68}(2,3,4,5,4,3,2,1)$ | 24 |
| $\beta_{69}(2,2,4,5,4,3,2,1)$ | 23 |
| $\beta_{70}(2,3,3,5,4,3,2,1)$ | 23 |
| $\beta_{71}(1,3,3,5,4,3,2,1)$ | 22 |
| $\beta_{72}(2,2,3,5,4,3,2,1)$ | 22 |
| $\beta_{73}(1,2,3,5,4,3,2,1)$ | 21 |
| $\beta_{74}(2,2,3,4,4,3,2,1)$ | 21 |
| $\beta_{75}(1,2,3,4,4,3,2,1)$ | 20 |
| $\beta_{76}(2,2,3,4,3,3,2,1)$ | 20 |
| $\beta_{77}(1,2,2,4,4,3,2,1)$ | 19 |
| $\beta_{78}(1,2,3,4,3,3,2,1)$ | 19 |
| $\beta_{79}(2,2,3,4,3,2,2,1)$ | 19 |
| $\beta_{80}(1,2,2,4,3,3,2,1)$ | 18 |
| $\beta_{81}(1,2,3,4,3,2,2,1)$ | 18 |
| $\beta_{82}(2,2,3,4,3,2,1,1)$ | 18 |
| $\beta_{83}(1,2,2,3,3,3,2,1)$ | 17 |
| $\beta_{84}(1,2,2,4,3,2,2,1)$ | 17 |
| $\beta_{85}(1,2,3,4,3,2,1,1)$ | 17 |
| $\beta_{86}(1,1,2,3,3,3,2,1)$ | 16 |
| $\beta_{87}(1,2,2,3,3,2,2,1)$ | 16 |
| $\beta_{88}(1,2,2,4,3,2,1,1)$ | 16 |
| $\beta_{89}(1,1,2,3,3,2,2,1)$ | 15 |
| $\beta_{90}(1,2,2,3,2,2,2,1)$ | 15 |
| $\beta_{91}(1,2,2,3,3,2,1,1)$ | 15 |
| $\beta_{92}(2,3,4,6,5,4,3,2)$ | 29/2 |

| $\beta_i$ | $h'(\beta_i)$ |
|---|---|
| $\beta_{93}(1,1,2,3,2,2,2,1)$ | 14 |
| $\beta_{94}(1,1,2,3,3,2,1,1)$ | 14 |
| $\beta_{95}(1,2,2,3,2,2,1,1)$ | 14 |
| $\beta_{96}(1,1,2,2,2,2,2,1)$ | 13 |
| $\beta_{97}(1,2,2,3,2,1,1,1)$ | 13 |
| $\beta_{98}(1,1,2,3,2,2,1,1)$ | 13 |
| $\beta_{99}(1,1,1,2,2,2,2,1)$ | 12 |
| $\beta_{100}(1,1,2,3,2,1,1,1)$ | 12 |
| $\beta_{101}(1,1,2,2,2,2,1,1)$ | 12 |
| $\beta_{102}(0,1,1,2,2,2,2,1)$ | 11 |
| $\beta_{103}(1,1,2,2,2,1,1,1)$ | 11 |
| $\beta_{104}(1,1,1,2,2,2,1,1)$ | 11 |
| $\beta_{105}(1,1,2,2,1,1,1,1)$ | 10 |
| $\beta_{106}(1,1,1,2,2,1,1,1)$ | 10 |
| $\beta_{107}(0,1,1,2,2,2,1,1)$ | 10 |
| $\beta_{108}(1,1,1,2,1,1,1,1)$ | 9 |
| $\beta_{109}(0,1,1,2,2,1,1,1)$ | 9 |
| $\beta_{110}(1,1,1,1,1,1,1,1)$ | 8 |
| $\beta_{111}(0,1,1,2,1,1,1,1)$ | 8 |
| $\beta_{112}(1,0,1,1,1,1,1,1)$ | 7 |
| $\beta_{113}(0,1,1,1,1,1,1,1)$ | 7 |
| $\beta_{114}(0,1,0,1,1,1,1,1)$ | 6 |
| $\beta_{115}(0,0,1,1,1,1,1,1)$ | 6 |
| $\beta_{116}(0,0,0,1,1,1,1,1)$ | 5 |
| $\beta_{117}(0,0,0,0,1,1,1,1)$ | 4 |
| $\beta_{118}(0,0,0,0,0,1,1,1)$ | 3 |
| $\beta_{119}(0,0,0,0,0,0,1,1)$ | 2 |
| $\beta_{120}(0,0,0,0,0,0,0,1)$ | 1 |




We already know the shape of diagrams from the first seven columns. Thanks to Propositions 5.1.6, 5.1.7 and 5.1.8, one obtains the implications for the last column.

In particular, we obtain implications such as ($\mathbf{i} \Rightarrow \mathbf{j}$ or $\mathbf{k}$):

$$(92 \Rightarrow 91 \text{ or } 90) \quad \text{and} \quad (92 \Rightarrow 90 \text{ or } 89) \quad \text{and} \quad (92 \Rightarrow 91 \text{ or } 89).$$

**Proposition 6.2.8.** *The set $\mathcal{D}$ of Cauchon diagrams has same cardinality as the Weyl group $W$.*

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## References

[AD08] N. Andruskiewitsch, F. Dumas, On the automorphisms of $u_q^+(\mathfrak{g})$, in: Enriquez Benjamin (Ed.), Quantum Groups, in: IRMA Lect. Math. Theor. Phys., vol. 12, Eur. Math. Soc., Zürich, 2008, pp. 108–133.
[BG02] K.A. Brown, K.R. Goodearl, Lectures on Algebraic Quantum Groups, Adv. Courses Math. CRM Barcelona, Birkhäuser-Verlag, Basel, 2002.
[Bou68] N. Bourbaki, Éléments de mathématique, Fasc. XXXIV, Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines, Actualités Scientifiques et Industrielles, vol. 1337, Hermann, Paris, 1968.
[Cau03a] G. Cauchon, Effacement des dérivations et spectres premiers des algèbres quantiques, J. Algebra 260 (2) (2003) 476–518.
[Cau03b] G. Cauchon, Spectre premier de $O_q(M_n(k))$ : Image canonique et séparation normale, J. Algebra 260 (2) (2003) 519–569.
[CM08] G. Cauchon, A. Mériaux, Admissible diagrams and combinatoric properties of Weyl groups, preprint, 2008.
[DCKP95] C. De Concini, V.G. Kac, C. Procesi, Some quantum analogues of solvable Lie groups, in: Geometry and Analysis, Bombay, 1992, Tata Inst. Fund. Res., Bombay, 1995, pp. 41–65.
[GL00] K.R. Goodearl, E.S. Letzter, The Dixmier–Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras, Trans. Amer. Math. Soc. 352 (3) (2000) 1381–1403.
[Gor00] M. Gorelik, The prime and the primitive spectra of a quantum Bruhat cell translate, J. Algebra 227 (1) (2000) 211–253.
[Hum78] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Grad. Texts in Math., vol. 9, Springer-Verlag, New York, 1978, second printing, revised.
[Jan96] J.C. Jantzen, Lectures on Quantum Groups, Grad. Stud. Math., vol. 6, Amer. Math. Soc., Providence, RI, 1996.
[Lau07a] S. Launois, On the automorphism groups of $q$-enveloping algebras of nilpotent lie algebras, ED. CIM 28 (2007) 125.
[Lau07b] S. Launois, Primitive ideals and automorphism group of $U_q^+(B_2)$, J. Algebra Appl. 6 (1) (2007) 21–47.
[Lit98] P. Littelmann, Cones, crystals, and patterns, Transform. Groups 3 (2) (1998) 145–179.
[LLR06] S. Launois, T.H. Lenagan, L. Rigal, Quantum unique factorisation domains, J. Lond. Math. Soc. (2) 74 (2) (2006) 321–340.
[LS91] S. Levendorskiĭ, Y. Soibelman, Algebras of functions on compact quantum groups, Schubert cells and quantum tori, Comm. Math. Phys. 139 (1) (1991) 141–170.
[Lus90] G. Lusztig, Quantum groups at roots of 1, Geom. Dedicata 35 (1–3) (1990) 89–113.
[MR04] R.J. Marsh, K. Rietsch, Parametrizations of flag varieties, Represent. Theory 8 (2004) 212–242 (electronic).
[Mér08] A. Mériaux, Les diagrammes de cauchon pour $\mathcal{U}_q^+(\mathfrak{g})$, http://arxiv.org/abs/0807.1012, 2008.
[Pap94] P. Papi, A characterization of a special ordering in a root system, Proc. Amer. Math. Soc. 120 (3) (1994) 661–665.