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Correspondence Theorems for Modules and Their Endomorphism Rings

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1. INTRODUCTION

Let ${}_RM$ be a left R-module and let $B = \text{End}\ {}_RM$ be its endomorphism ring. When investigating the relationship between properties of B and properties of M, one very useful and well-known technique makes use of two natural dual Galois connections, $G1$ and $G2$, which exist between the lattice, L, of submodules of M and the lattices, L_r or L_l, of right or left ideals of B. The Galois connection, $G1$, is given by the maps $r_B: L \to L_r$ and $l_M: L_r \to L$, where $r_B(U) = \{b \in B: Ub = 0\}$, for $U \subseteq M$, and $l_M(J) = \{m \in M: mJ = 0\}$, for $J \subseteq B$. Here, the restrictions, $\bar{r}_B$ and $\bar{l}_M$, of r_B and l_M to the Galois objects of $G1$, $\bar{L} = \{U \in L: U = l_M(J)$, for $J \subseteq B\}$, and $\bar{L}_r = \{J \in L_r: J = r_B(U)$, for $U \subseteq M\}$, are mutually inverse bijections between $\bar{L}$ and $\bar{L}_r$. When studying a given property of B via $G1$, the first thing one needs to do is to determine the class of ideals of B which is associated with this property and to establish that these ideals are Galois objects of $G1$. Once that is done, all that remains is to identify those submodules of M which correspond, via $\bar{l}_M$, to the relevant ideals. In this way, we obtain a correspondence theorem involving the particular ideals which concern us, and from this correspondence theorem it becomes possible to deduce conditions on M which are necessary and sufficient in order for B to possess the property under investigation.

To illustrate, consider the property that B is right noetherian; here, the relevant ideals of B are all the finitely generated (f.g.) right ideals, since B is right noetherian if and only if it satisfies the ascending chain condition (a.c.c.) on f.g. right ideals. Now, when M is quasi-injective, every f.g. right ideal of B is a Galois object of $G1$. Moreover, $\bar{r}_B$ and $\bar{l}_M$ induce order-reversing bijections between the f.g. right ideals of B and the finitely closed submodules of M; hence, we can deduce that B is right noetherian if and only if M has the d.c.c. on finitely closed submodules [1, Corollary 4.3(1)].

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The beauty of this technique lies in the fact that it allows one to group a class of related properties of B and to treat them in a systematic and unified manner. In [1], for example, Albu and Nastasescu used this technique to study chain conditions on B and their relationship to chain conditions on M, and were consequently able to derive, in a unified and simplified manner, some classical results. In this paper, we use this same technique to give a unified and systematic treatment of properties of B associated with annihilators (such as the property that B is a Baer ring or a right or left upper or lower Levitzki ring), and of properties of B associated with left complements in B (such as the property that B is a left CS ring or a ring with finite left Goldie dimension).

As we observed above, the key result one needs, when studying properties of B via this technique, is a correspondence theorem between those ideals of B which are associated with the property under investigation and certain submodules of M. Consequently, we direct our efforts here towards the establishment of correspondence theorems for annihilators and of correspondence theorems for left complements, and we find that, in this way, we can coordinate a variety of well-known results, as well as deduce some new results, all within one unified, simplified framework.

In Section 2, we establish the correspondence theorems for annihilators (Theorems 2.2 and 2.5), and from these we can immediately deduce, for example, necessary and sufficient conditions on M in order that B should be Baer (Corollaries 2.3, 2.4, and 2.6) or Levitzki (Corollary 2.7). In Section 3, we establish the correspondence theorems for left complements (Theorems 3.5, 3.7, and 3.10), which, analogously, give us as corollaries necessary and sufficient conditions on M in order for B to be left Goldie (Corollaries 3.4 and 3.14) or left CS (Theorem 3.6 and Corollaries 3.8, 3.9, and 3.11). In some cases, the correspondence theorems are applied together, and we try, as much as possible, to compare the various hypotheses on M which are sometimes required for the establishment of these correspondence theorems. We end the paper with a small diagram summarizing the relationships between the various types of modules used in our correspondence theorems.

2. Correspondence Theorems for Annihilators

Throughout this paper, unless otherwise indicated, R denotes an associative ring with 1, ${}_RM$ a left R-module, and B the ring of R-endomorphisms of ${}_RM$. The action of homomorphisms will be written on the side opposite to that of the scalars.

The right (left) annihilator in B of a subset, H, of B will be denoted by

$\mathscr{R}(H)(\mathscr{L}(H))$, while r and l will be used for annihilators in M of subsets of B, or in B of subsets of M:

$$l_M(H) = \{m \in M: mh = 0, \text{ for each } h \in U\}, \qquad \text{for } H \subseteq B$$

$$r_B(U) = \{b \in B: ub = 0, \text{ for each } u \in U\}, \qquad \text{for } U \subseteq M.$$

Also, let $I_B(U) = \{b \in B: Mb \subseteq U\}$, for any submodule, U, of M, and

$$MH = S_M(H) = \left\{\sum_{i=1}^{n} m_i h_i: m_i \in M,\ h_i \in H\right\}, \qquad \text{for any subset } H \text{ of } B.$$

The following Lemma is straightforward:

LEMMA 2.1. (i) *$S_m I_B(U) \subseteq U$ and $U \subseteq l_M r_B(U)$, for any submodule, U, of M.*

(ii) *$I_B l_M(H) = \mathscr{L}(H)$ and $r_B S_M(H) = \mathscr{R}(H)$, for $H \subseteq B$.*

Let L be a complete lattice. A *closure operator* on L is a mapping $\varphi: L \to L$, written $\varphi(a) = a^c$, such that:

(c1) $a \leqslant b$ implies $a^c \leqslant b^c$;

(c2) $a \leqslant a^c$;

(c3) $(a^c)^c = a^c$.

An element a is *closed* under φ if $a = a^c$.

Let L' be another complete lattice. *A Galois connection* between L and L' is a pair of mappings $\sigma: L \to L'$ and $\tau: L' \to L$ satisfying:

(1) $x_1 \leqslant x_2$ implies $\sigma(x_1) \geqslant \sigma(x_2)$ for $x_1, x_2 \in L$.

(2) $y_1 \leqslant y_2$ implies $\tau(y_1) \geqslant \tau(y_2)$ for $y_1, y_2 \in L'$.

(3) $x \leqslant \tau\sigma(x)$ and $y \leqslant \sigma\tau(y)$ for $x \in L$, $y \in L'$.

Given a Galois connection, it can be shown that $\sigma\tau\sigma(x) = \sigma(x)$ and $\tau\sigma\tau(y) = \tau(y)$ for $x \in L$, $y \in L'$, so that the maps $\sigma\tau$ and $\tau\sigma$ are closure operators on L and L', respectively. The closed elements, or *Galois objects*, in L (resp. L') are those which are of the form $\tau(y)$ for some $y \in L'$ (resp. $\sigma(x)$ for some $x \in L$). Set $\bar{L} = \tau(L')$ and $\bar{L}' = \sigma(L)$, and let $\bar{\sigma}: \bar{L} \to \bar{L}'$ and $\bar{\tau}: \bar{L}' \to \bar{L}$ denote the restrictions of σ and τ to the sets of Galois objects of L and L', respectively. Then it is straightforward to show that $\bar{\sigma}$ and $\bar{\tau}$ are inverse bijections to one another.

For ${}_RM$ a left R-module and $B = \operatorname{End}_R M$, let L denote the lattice of submodules of M, L_r the lattice of right ideals of B, and L_l the lattice of left ideals of B; for any lattice, A, let A^{op} denote its opposite lattice, i.e., A ordered by the opposite order. Then it is easily seen that:

— the mappings r_B and l_M form a Galois connection $G1$, between L and L_r;

— the mappings l_B and S_M form a Galois connection $G2$, between L and $(L_l)^{\text{op}}$; and

— the mappings $\mathscr{L}$ and $\mathscr{R}$ form a Galois connection between L_r and L_l.

The Galois objects $l_M(H)$ of M are called the *a-closed* submodules of M, and the Galois objects $S_M(H)$ of M are called the *M-cotorsionless* submodules of M. The Galois objects $\mathscr{L}(H)$ and $\mathscr{R}(H)$ are, of course, the left and right annihilator ideals of B.

The following observation is easily verified and will be used without comment in the sequel: If U is a direct summand in M, then U is a-closed and M-cotorsionless.

If J is a right ideal of B which is generated by an idempotent $e \in B$, i.e., $J = eB$, then J is a right annihilator ideal in B, in fact, $J = \mathscr{R}[B(1-e)]$, and similarly for left ideals. In case ${}_RM$ is a vector space, then we have, conversely:

(a) if J is a right annihilator ideal in B, then J is generated by an idempotent in B; and

(b) if H is a left annihilator ideal in B, then H is generated by an idempotent in B.

Moreover, in any ring B with 1, (a) and (b) are equivalent properties. Any ring having properties (a) and (b) is called a *Baer ring*.

Baer endomorphism rings have been investigated by several authors, using the three Galois connections mentioned above (cf. e.g., [8, 7, 5]). These Galois connections are especially appropriate for the investigation of when B is a Baer ring, since, by Lemma 2.1(ii), any right or left annihilator in $B = \text{End}\ {}_RM$ is a Galois object. Consequently, according to the technique outlined under Introduction, the main task here is to identify those submodules of M which correspond to the left or right annihilators in B. For simplicity and conciseness in the statement of our theorems, recall that if we have two partially ordered sets, then a bijection between them which is order-preserving (resp. order-reversing) is called a *projectivity* (resp. a *duality*).

THEOREM 2.2. *The maps $U \to r_B(U)$ and $J \to l_M(J)$ determine a duality between $S_1 = \{U \subseteq M: U = l_M r_B[S_M l_B(U)]\}$ and the right annihilators $\mathscr{A}(B.) = \{J \subseteq B: J = \mathscr{R}(H),\ H \subseteq B\}$.*

Proof. For any $U \in S_1$, we have $r_B(U) = r_B[S_M l_B(U)] = \mathscr{R}[l_B(U)]$, by Lemma 2.1(ii); i.e., $r_B(U)$ is a right annihilator in B. On the other hand, let

$J=\mathscr{R}(H)$ be a right annihilator in B, and let $U=l_M(J)$. Then, by Lemma 2.1(ii),

$$l_M r_B S_M I_B(U)=l_M\mathscr{R}I_B(l_M(J))=l_M\mathscr{R}\mathscr{L}(J)=l_M\mathscr{R}\mathscr{L}\mathscr{R}(H)$$
$$=l_M\mathscr{R}(H)=l_M(J)=U;$$

i.e., $U\in S_1$. Since any $U\in S_1$ is a-closed and any right annihilator, $J=\mathscr{R}(H)$, is a Galois object $J=r_B S_M(H)$, we have: $U=l_M r_B S_M I_B(U)\to r_B(U)=\mathscr{R}[I_B(U)]\to l_M r_B(U)=U$, and $J=\mathscr{R}(H)=r_B S_M(H)\to l_M(J)=l_M r_B S_M(H)\to r_B l_M(J)=J$. Hence the two order-reversing mappings are inverses of each other and so determine a duality. ∎

COROLLARY 2.3. *B is a Baer ring if and only if every $U\in S_1$ is a direct summand in M.*

Proof. Assume that every $U\in S_1$ is a direct summand in M, and let $J=\mathscr{R}(H)$ be a right annihilator in B. Then $U=l_M(J)$ is in S_1, so there is an idempotent $e\in B$ such that $U=Me$. It follows that $J=r_B l_M(J)=r_B(U)=r_B(Me)=\mathscr{R}(e)=(1-e)B$, i.e., J is generated by an idempotent in B, proving that B is Baer.

Conversely, assume that B is a Baer ring and let $U\in S_1$. Then $J=r_B(U)$ is a right annihilator in B, so $J=eB$, for $e=e^2\in B$. It follows that $U=l_M r_B(U)=l_M(J)=l_M(eB)=l_M(e)=M(1-e)$; i.e., U is a direct summand in M. ∎

COROLLARY 2.4. *If ${}_RM$ is semisimple, then B is a Baer ring.*

If every submodule, U, of M is M-cotorsionless, i.e., $U=S_M I_B(U)$, then M is said to be a *self-generator* [9]. Any free module and any semisimple module is a self-generator; another example of a self-generator is any infinitely generated projective module containing a unimodular element (cf. [6]). The property of being a self-generator is very useful, and often plays a crucial role, in establishing correspondences between the a-closed submodules of M and the right annihilators of B (cf., e.g., [6–8]). However, it is not necessary for M to be a self-generator in order to have such a correspondence theorem; rather, an approximation of self-generation is sufficient—and necessary—in order that r_B and l_M should induce a lattice anti-isomorphism between $\mathscr{C}^a=\{U\subseteq M\colon U=l_M r_B(U)\}$ and the right annihilators $\mathscr{A}(B.)$.

DEFINITION 1. *Using the notation* $l_M r_B(U)=U^a$ for the "a-closure" operator $l_M r_B$, we will call M an *a-self-generator* if, for every a-closed submodule, U, of M, we have $U=[S_M I_B(U)]^a$. It turns out that this condition on M is also equivalent to obtaining a lattice isomorphism, via I_B

and S_M, between $\mathscr{C}^a$ and the left annihilators $\mathscr{A}(.B)=\{H\subseteq B\colon H=\mathscr{L}(J), J\subseteq B\}$, for, we have:

THEOREM 2.5. *The following are equivalent*: (1) ${}_RM$ *is an a-self-generator.*

(2) *The maps* $U\to r_B(U)$ *and* $J\to l_M(J)$ *determine a lattice anti-isomorphism between* $\mathscr{C}^a$ *and* $\mathscr{A}(B.)$.

(3) *The maps* $U\to I_B(U)$ *and* $H\to[S_M(H)]^a$ *determine a lattice isomorphism between* $\mathscr{C}^a$ *and* $\mathscr{A}(.B)$.

Proof. (1)$\Rightarrow$(2) Always: $S_1\subseteq\mathscr{C}^a$; M is an a-self-generator if and only if $\mathscr{C}^a\subseteq S_1$. Clearly, $\mathscr{C}^a\subseteq S_1$—or, equivalently, $\mathscr{C}^a=S_1$—implies, by Theorem 2.2 that r_B and l_M determine an anti-isomorphism between $\mathscr{C}^a$ and $\mathscr{A}(B.)$.

(2)$\Rightarrow$(1) If r_B and l_M give such an anti-isomorphism, then, for any $U\in\mathscr{C}^a$, we have $r_B(U)=\mathscr{R}(H)$, for $H\subseteq B$, hence

$$\begin{aligned}l_Mr_B[S_MI_B(U)]&=l_Mr_B[S_MI_Bl_Mr_B(U)]=l_M\mathscr{R}\mathscr{L}\mathscr{R}(H)\\&=l_M\mathscr{R}(H)=l_Mr_B(U)=U;\end{aligned}$$

i.e., $U\in S_1$.

(1)$\Rightarrow$(3) Note first that, for any $U\in\mathscr{C}^a$, $I_B(U)$ is in $\mathscr{A}(.B)$, since $I_B(U)=I_Bl_Mr_B(U)=\mathscr{L}r_B(U)$, and, clearly, $[S_M(H)]^a\in\mathscr{C}^a$ for any $H\subseteq B$. Now assume (1); then, we have, for any $U\in\mathscr{C}^a$, $U\to I_B(U)\to[S_MI_B(U)]^a=U$, and, for any $H\in\mathscr{A}(.B)$, $H\to[S_M(H)]^a\to I_Bl_Mr_BS_M(H)=\mathscr{L}\mathscr{R}(H)=H$. Hence the two order-preserving maps are inverses of each other, and so determine a lattice isomorphism.

(3)$\Rightarrow$(1) If the two maps are inverses of each other, then we have $U=[S_MI_B(U)]^a$ for $U\in\mathscr{C}^a$. ∎

COROLLARY 2.6. *Let* ${}_RM$ *be an a-self-generator. Then B is a Baer ring if and only if every a-closed submodule of M is a direct summand in M.*

Remarks. (1) Any self-generator is clearly an a-self-generator. In Section 3, we shall give an example of an a-self-generator which is not a self-generator.

(2) Corollary 2.4 is Theorem 6 of [8]. If M is a free module, then the a-closed submodules coincide with the dual-closed submodules and Corollary 2.6 gives Theorem 9 of [8] or Theorem 2 of [7]. The a-closed submodules and the dual-closed submodules coincide also in case M is an infinitely generated projective containing a unimodular element, in which case Theorem 2.5 gives Theorem 3.8 of [6].

(3) Theorems 2.2 and 2.5 can be applied to investigate other properties of B having to do with annihilators. Recall that B is said to be right upper (lower) *Levitzki* if it satisfies the a.c.c. (d.c.c.) on annihilator right ideals. Left Levitzki rings are defined analogously. It follows immediately, for example, that:

COROLLARY 2.7. *Let ${}_RM$ be an a-self-generator. Then B is right upper (lower) Levitzki if and only if M satisfies the* d.c.c. (a.c.c.) *on a-closed submodules, and B is left upper (lower) Levitzki if and only if M satisfies the* a.c.c. (d.c.c.) *on a-closed submodules.*

Another class of rings which is defined in terms of annihilators, and is closely related to Baer rings, is the class of Rickart rings. Recall that B is a left (resp. right) *Rickart* ring if the left (resp. right) annihilator of any element of B is generated by an idempotent in B. Just as in the case of Baer rings, we can use our Galois connections to find out when $B = \text{End}\ {}_RM$ is right or left Rickart. Using similar proofs, we easily find, for example, that:

PROPOSITION 2.8. *The maps $U \to r_B(U)$ and $J \to l_M(J)$ determine a duality between $K_1 = \{U \subseteq M\colon U = (Mb)^a, \text{ for } b \in B\}$ and $\mathscr{A}_1(B.) = \{J \subseteq B\colon J = \mathscr{R}(b), b \in B\}$, and B is a right Rickart ring if and only if every $U \in K_1$ is a direct summand in M.*

PROPOSITION 2.9. *Let ${}_RM$ be an a-self-generator. Then the maps $U \to I_B(U)$ and $H \to [S_M(H)]^a$ determine a projectivity between $K_2 = \{U \subseteq M\colon U = l_M(b) = \ker b, \text{ for } b \in B\}$ and $\mathscr{A}_1(.B) = \{H \subseteq B\colon H = \mathscr{L}(b), b \in B\}$, and B is a left Rickart ring if and only if every $U \in K_2$ is a direct summand in M.*

If we take M to be a free module, then Propositions 2.8 and 2.9 give Theorem 3 of [7].

3. CORRESPONDENCE THEOREMS FOR LEFT COMPLEMENTS

As mentioned under Introduction, in order to use our Galois connections to study a property of B which is defined in terms of a certain class of ideals of B, we need to establish that these ideals are Galois objects of $G1$ or $G2$. In Section 2, this relationship already existed since, for any ${}_RM$, any right or left annihilator is a Galois object in $G1$ or $G2$. When dealing with ideals other than annihilators, however, we need to choose M in such a way that the required relationship, between the ideals in question and the Galois objects of $G1$ or $G2$, can be established. For example, when the relevant ideals are f.g. right ideals, M is taken to be quasi-injective since, for such an ${}_RM$, $J = r_B l_M(J)$, for any f.g. right ideal, J, of B [1, Proposition 4.1]. In this section, since we are interested in properties of B defined in terms of left complements, we shall be dealing mostly with nondegenerate modules (which we shall define in a moment), for, when M is nondegenerate, every left complement, H, in B is a Galois element of $G2$; i.e., $H = I_B S_M(H)$.

Before defining a nondegenerate module, we recall some notation: $M^* = \mathrm{Hom}_R(M, R)$ denotes, as usual, the *dual module* of M, $T = (M, M^*) = \{\sum_{i=1}^n m_i f_i : m_i \in M, f_i \in M^*\}$ denotes the *trace* of M in R, and $(R, {}_RM_B, {}_BM_R^*, B)$ is the *standard Morita context* for M, with the $R-R$-bimodule homomorphism $(\, , \,): M \otimes_B M^* \to R$ given by $(m, f) = mf$, for $m \in M$, $f \in M^*$, and the $B-B$-bimodule homomorphism $[\, , \,]$: $M^* \otimes_R M \to B$ defined by $m_1[f, m] = (m_1, f)m$, for $m, m_1 \in M$, $f \in M^*$.

Also, ${}_RU \subset' {}_RM$ denotes that U is an *essential* R-submodule of M, i.e., that U has nonzero intersection with each nonzero R-submodule of M.

DEFINITION 2. We will say that ${}_RM$ is *nondegenerate* if:

$$Tm = 0 \Rightarrow m = 0, \qquad \text{for any } m \in M.$$

The following Proposition is easily verified.

PROPOSITION 3.1. *For any ${}_RM$, the following are equivalent*:

(1) *${}_RM$ is nondegenerate.*

(2) *For each $m \in M$, $[M^*, m] = 0$ implies $m = 0$.*

(3) *For each submodule, ${}_RU$, of ${}_RM$, $TU \subset' {}_RU$.*

Any free module, in fact, any generator, is nondegenerate. A self-generator, on the other hand, need not be nondegenerate: the $\mathbb{Z}$-module $\mathbb{Z}/p^n\mathbb{Z}$ is a self-generator which is not nondegenerate. M can also be nondegenerate without being a self-generator: let R be the ring of all Cauchy sequences in $\mathbb{Q}$ with component-wise multiplication and M the ideal of zero sequences; then M is nondegenerate but not a self-generator.

Recall that ${}_RM$ is said to be *nonsingular* if, for any $m \in M$, $Im = 0$, with ${}_RI$ an essential left ideal of R, implies $m = 0$. From Definition 2, it is clear that any nonsingular module with essential trace is nondegenerate. Another example of nondegenerate modules is: any torsionless module over a semiprime ring.

Nondegenerate modules possess several properties which make them especially appropriate for establishing correspondence theorems for left complements in B. In particular, the property that, for M nondegenerate, every left complement, H, in B satisfies $H = I_B S_M(H)$ will follow directly from (3) of the next proposition. Before stating Proposition 3.2, we remark that the following well-known property of essential submodules will be used without comment in the sequel: If ${}_RU \subset {}_RV \subset {}_RM$, then ${}_RU \subset' {}_RM$ if and only if ${}_RU \subset' {}_RV$ and ${}_RV \subset' {}_RM$.

PROPOSITION 3.2. *Let ${}_RM$ be nondegenerate. Then:*

(1) *For any nonzero submodule, U, of M, we have: $I_B(U) \neq 0$.*

(2) *If H and J are left ideals of B such that $H \subseteq J$, then we have:*

$$ {}_BH \subset' {}_BJ \qquad \text{if and only if } S_M(H) \subset' S_M(J). $$

(3) *For any left ideal, H, in B, we have: ${}_BH \subset' I_BS_M(H)$.*

(4) *If U and V are submodules of M such that $U \subseteq V$, then we have:*

$$ {}_RU \subset' {}_RV \qquad \text{if and only if } I_B(U) \subset' I_B(V). $$

(5) *M is an a-self-generator.*

Proof. (1) Let U be a nonzero submodule of M, and let $0 \neq u \in U$. Then, since M is nondegenerate, $[M^*, u] \neq 0$ (Proposition 3.1). From $M[M^*, u] = (M, M^*)u \subseteq Ru$, we see that $[M^*, u] \subseteq I_B(U)$; hence $I_B(U) \neq 0$.

(2) Assume that ${}_BH \subset' {}_BJ$ and let $0 \neq m = \sum_{i=1}^n m_i j_i \in S_M(J)$, with $m_i \in M$ and $j_i \in J$ for $i = 1, ..., n$. Then $0 \neq [M^*, m] = \sum_{i=1}^n [M^*, m_i] j_i \subseteq J$, hence $[M^*, m] \cap H \neq 0$. We have: $0 \neq M(H \cap [M^*, m]) \subseteq MH \cap M[M^*, m] = MH \cap (M, M^*)m \subseteq MH \cap Rm$; therefore, $MH = S_M(H) \subset' S_M(J)$.

Conversely, assume that $S_M(H) \subset' S_M(J)$, for $H \subseteq J$, and let $0 \neq c \in J$. Then, $Mc \neq 0$ implies $Mc \cap S_M(H) \neq 0$, and this implies:

$$ 0 \neq [M^*, Mc \cap S_M(H)] \subseteq [M^*, Mc] \cap [M^*, S_M(H)] \subseteq Bc \cap H; $$

hence ${}_BH \subset' {}_BJ$.

(3) From $S_M(H) = S_MI_BS_M(H)$, for any left ideal, ${}_BH$, in B, we have, in particular, that $S_M(H) \subset' S_MI_BS_M(H)$, hence by (2), $H \subset' I_BS_M(H)$.

(4) Observe first that it follows from (1) that, for any nonzero submodule, U, of M, we have $S_MI_B(U) \subset' U$; for, if $0 \neq u \in U$, then there is $0 \neq b \in I_B(Ru)$. Hence, since $I_B(Ru) \subseteq I_B(U)$, we have: $0 \neq Mb \subseteq Ru \cap S_MI_B(U)$, so that $S_MI_B(U) \subset' {}_RU$.

Now let U and V be nonzero submodules of M such that $U \subseteq V$ (the cases when one or both of U and V is 0 are trivial), and assume first that ${}_RU \subset' {}_RV$. Then, we have: $I_B(U) \subseteq I_B(V)$, $S_MI_B(U) \subset S_MI_B(V) \subset' V$, and $S_MI_B(U) \subset' U \subset' V$; therefore, $S_MI_B(U) \subset' V$ and hence $S_MI_B(U) \subset' S_MI_B(V)$, and this last implies, by (2), that $I_B(U) \subset' I_B(V)$.

Assume now that $I_B(U) \subset' I_B(V)$. Then, using (2), we have $S_M I_B(U) \subset' S_M I_B(V) \subset' V$, hence $S_M I_B(U) \subset' V$. But $S_M I_B(U) \subset' U \subset V$, hence $S_M I_B(U) \subset' V$ implies that $U \subset' V$.

(5) Let U be an a-closed submodule of M. It will suffice to show that $r_B(U) = r_B[S_M I_B(U)]$, for then $U = l_M r_B(U) = l_M r_B[S_M I_B(U)]$. Clearly, since $S_M I_B(U) \subseteq U$, we always have $r_B(U) \subseteq r_B[S_M I_B(U)]$. For the reverse inclusion, let $b \in r_B[S_M I_B(U)]$, so that $[S_M I_B(U)]b = 0$ and consequently $[I_B(U)]b = 0$. If u is any nonzero element in U, we have, by nondegeneracy, $[M^*, u] \neq 0$. But, as we noted in (1), $[M^*, u] \subseteq I_B(U)$, so that $[M^*, ub] = [M^*, u]b \subseteq [I_B(U)]b = 0$, which last implies, by nondegeneracy again, that $ub = 0$. Hence $Ub = 0$ and $b \in r_B(U)$, completing the proof. ▌

Property (1) of Proposition 3.2 is of interest in itself:

Definition 3. We will say that $_RM$ is *retractable* if, for any nonzero submodule, U, of M, we have: $I_B(U) \neq 0$. Moreover, if "c" is any closure operator on L, then we will say that M is *c-retractable* if, for any nonzero, c-closed $U \subseteq M$, we have: $I_B(U) \neq 0$.

It follows easily from the proof of Proposition 3.2(4) that M is retractable if and only if $S_M I_B(U) \subset' {}_RU$ for each nonzero submodule, U, of M. Combining Proposition 3.2(5) with Corollaries 2.6 and 2.7, we get:

Corollary 3.3. *Let $_RM$ be nondegenerate. Then*: (1) *B is a Baer ring if and only if every a-closed submodule of M is a direct summand in M*; (2) *B is right upper* (*lower*) *Levitzki if and only if M satisfies the d.c.c.* (*a.c.c.*) *on a-closed submodules*; *and* (3) *B is left upper* (*lower*) *Levitzki if and only if M satisfies the a.c.c.* (*d.c.c.*) *on a-closed submodules.*

Before proving our first correspondence theorem for left complements, we recall that: A submodule, $_RU$, of M is said to be a *complement* in M if U has no proper essential extension in M or, equivalently, if there is a submodule, V, of M such that U is maximal with respect to the property $U \cap V = 0$.

M is said to be a *CS module* if every complement in M is a direct summand in M. A ring R is said to be a left (right) CS ring if $_RR(R_R)$ is a CS module. Injective and quasi-injective modules are CS modules, as are semisimple and uniform modules; for other examples, see, e.g., [2]. M is said to be *finite-dimensional*, in the sense of Goldie—notation: $d(_RM) < \infty$—if M satisfies the a.c.c. on complement submodules. A ring R has finite left (right) Goldie dimension if $_RR(R_R)$ is finite-dimensional. R is a *left Goldie ring* if it satisfies the a.c.c. on left annihilators and on left complements.

It is known that, if ${}_RM$ is nondegenerate, then $d({}_RM) = d({}_BB)$, so that M is finite-dimensional if and only if B has finite left Goldie dimension (cf. e.g., [4, Proposition 3]). Combining this with Corollary 3.3(3), we obtain:

COROLLARY 3.4. *Let ${}_RM$ be nondegenerate. Then B is a left Goldie ring if and only if M satisfies the a.c.c. on a-closed and on complement submodules.*

THEOREM 3.5. *Let ${}_RM$ be nondegenerate. The maps $U \to I_B(U)$ and $H \to S_M(H)$ determine a projectivity between $S_2 = \{U \subseteq M: U = S_M I_B(U)$ and $I_B(U)$ is a complement left ideal in $B\}$ and the complement left ideals $C(.B) = \{H \subseteq B: H$ is a complement left ideal of $B\}$.*

Proof. Clearly, by definition, if $U \in S_2$ then $I_B(U) \in C(.B)$. On the other hand, if $H \in C(.B)$, then $S_M(H)$ is M-cotorsionless and $I_B S_M(H) = H$ by Proposition 3.2(3), hence $S_M(H) \in S_2$. We have: for $U \in S_2$, $U \to I_B(U) \to S_M I_B(U) = U$, and, for $H \in C(.B)$, $H \to S_M(H) \to I_B S_M(H) = H$. Hence, the two order-preserving maps are inverses of each other and so determine a projectivity. ∎

THEOREM 3.6. *Let ${}_RM$ be nondegenerate. Then: (1) B is a left CS ring if and only if every $U \in S_2$ is a direct summand in M; (2) If M satisfies the a.c.c. on M-cotorsionless submodules, then B is a left Goldie ring.*

Proof. (1) Assume that every $U \in S_2$ is a direct summand in M, and let H be a complement left ideal of B. Then $S_M(H) \in S_2$, hence, there is an idempotent, $e = e^2$, in B such that $S_M(H) = Me$, and $H = I_B S_M(H) = I_B(Me)$. But $I_B(Me) = Be$, for, clearly, $e \in I_B(Me)$, hence $Be \subseteq I_B(Me)$, and, conversely, for $b \in I_B(Me)$, we have, for any $m \in M$, $mb = m_1 e$ for some $m_1 \in M$, hence $mbe = m_1 e^2 = mb$; i.e., $b = be \in Be$. It follows that $H = Be$; i.e., H is a direct summand in B.

Conversely, assume that B is left CS and let $U \in S_2$. Then $I_B(U)$ is a left complement in B, so $I_B(U) = Be$, where $e = e^2 \in B$. It follows that $U = S_M I_B(U) = S_M(Be) = Me$, so U is a direct summand in M.

(2) Assume that M satisfies the a.c.c. on M-cotorsionless submodules. Then, in particular, M satisfies the a.c.c. on the elements of S_2, and this implies, by Theorem 3.5, that B satisfies the a.c.c. on complement left ideals. Next, let $\mathscr{L}(K_1) \subseteq \mathscr{L}(K_2) \subseteq \cdots$ be an ascending chain of left annihilators in B. Then $S_M[\mathscr{L}(K_1)] \subseteq S_M[\mathscr{L}(K_2)] \subseteq \cdots$ is an ascending chain of M-cotorsionless submodules of M; hence, there is an $n > 0$ such that $S_M[\mathscr{L}(K_n)] = S_M[\mathscr{L}(K_{n+j})]$, for $j \geq 1$. Then, using Lemma 2.1(ii), we have $\mathscr{R}\mathscr{L}(K_n) = r_B S_M[\mathscr{L}(K_n)] = r_B S_M[\mathscr{L}(K_{n+j})] = \mathscr{R}\mathscr{L}(K_{n+j})$, for $j \geq 1$, and therefore $\mathscr{L}(K_n) = \mathscr{L}(K_{n+j})$ for $j \geq 1$, proving that B satisfies the a.c.c. on left annihilators. ∎

Remark. From Theorem 3.6(2) and Corollary 3.4, we see that, for a nondegenerate M, if M satisfies the a.c.c. on M-cotorsionless submodules, then M satisfies the a.c.c. on a-closed and on complement submodules.

Things become more interesting when the nondegenerate module is, in addition, a self-generator or a CS module, for, in both of these cases, the submodules of S_2 are precisely the complements of M.

THEOREM 3.7. *Let ${}_RM$ be nondegenerate. If M is a self-generator or a CS module, then the maps I_B and S_M determine a projectivity between the complement submodules of M and the complement left ideals of B.*

Proof. By Theorem 3.5, it will be sufficient to show that $U \in S_2$ if and only if U is a complement in M.

Let $U \in S_2$, so that $U = S_M I_B(U)$ and $I_B(U)$ is a left complement in B. Suppose that ${}_RU \subset' {}_RV$; we may assume, using Zorn's Lemma, that V is a complement in M. By Proposition 3.2(4), $I_B(U) \subset' I_B(V)$, hence, since $I_B(U)$ is a left complement, $I_B(U) = I_B(V)$ and $U = S_M I_B(U) = S_M I_B(V)$. If M is a self-generator, then every submodule is M-cotorsionless, so $V = S_M I_B(V) = U$, and U is a complement; and if M is CS, then every complement is a direct summand hence M-cotorsionless, so again $V = S_M I_B(V) = U$ and U is a complement.

Conversely, let U be a complement in M. Then if M is a self-generator or a CS module, U is M-cotorsionless. To see that $I_B(U)$ is a left complement in B, suppose that $I_B(U) \subset' J$, where J may be taken to be a left complement in B (using Zorn's Lemma). Then, by Proposition 3.2(2), $S_M I_B(U) \subset' S_M(J)$, so $U = S_M I_B(U) \subset' S_M(J)$, hence $U = S_M(J)$ since U is a complement. But this implies $J \subseteq I_B(U)$, hence $J = I_B(U)$, proving that $I_B(U)$ is a left complement in B. This shows that $U \in S_2$ and completes the proof. ∎

COROLLARY 3.8. *Let ${}_RM$ be a nondegenerate self-generator. Then B is a left CS ring if and only if M is a CS module.*

COROLLARY 3.9. *Let ${}_RM$ be a nondegenerate CS module. Then B is a left CS ring.*

From these last results we see that the existence of a projectivity between the complements of M and the left complements of B is a very useful thing to have, since the existence of such a projectivity means that B is left CS iff M is CS. We shall now show that, for a nondegenerate M, it is possible to have such a projectivity without requiring M to be a self-generator or a CS module, but simply by taking M to be nonsingular. In this case, however, the maps which give the projectivity must be adjusted slightly in order to make up for the fact that we do not have as many M-cotorsionless submodules at our disposal as we had in the case of CS or self-generating modules. We recall here that, when M is nonsingular, then, to any submodule, U, of M, there corresponds a unique complement, U^e, in M such that $U \subset' U^e$ (cf. [3, p. 61, Proposition 7]). The map $U \to U^e$ defines a closure operator, called the *essential closure* (or e-closure) operator, on the lattice L of submodules of M.

THEOREM 3.10. *Let ${}_RM$ be nondegenerate and nonsingular. Then the maps $U \to I_B(U)$ and $H \to [S_M(H)]^e$ determine a projectivity between the complement submodules of M and the complement left ideals of B.*

Proof. Let U be a complement submodule of M; we will show that $I_B(U)$ is a complement left ideal of B. Suppose that $I_B(U) \subset' H$, where we may assume, by Zorn's Lemma, that H is a complement left ideal of B. Then, by Proposition 3.2(2), $S_M I_B(U) \subset' S_M(H)$, and this implies that $[S_M I_B(U)]^e = [S_M(H)]^e$. But, by Proposition 3.2(1), $S_M I_B(U) \subset' U$, so that $U = [S_M I_B(U)]^e = [S_M(H)]^e$, and this implies, in particular, that $S_M(H) \subseteq U$ hence that $H \subseteq I_B(U)$. Therefore, $H = I_B(U)$, so $I_B(U)$ is a left complement in B. Conversely, it is clear that $[S_M(H)]^e$ is a complement in M for any left ideal, H, in B.

If H is any left ideal of B, we have $S_M(H) \subset' [S_M(H)]^e$, and, since M is nondegenerate, we have, by Proposition 3.2(4), $I_B S_M(H) \subset' I_B\{[S_M(H)]^e\}$, and, by Proposition 3.2(3), $H \subseteq' I_B S_M(H)$. If H is a complement left ideal in B, then we have $H = I_B S_M(H) = I_B\{[S_M(H)]^e\}$. We have $U \to I_B(U) \to [S_M I_B(U)]^e = U$, for each complement, U, of M, and $H \to [S_M(H)]^e \to I_B\{[S_M(H)]^e\} = H$, for each complement left ideal, H, of B. Hence the two order-preserving maps are inverses of each other and so determine a projectivity. ▌

COROLLARY 3.11. *Let ${}_RM$ be nondegenerate and nonsingular. Then B is a left CS ring if and only if M is a CS module.*

Proof. Assume that M is a CS module and let H be a left complement in B. Then $[S_M(H)]^e = Me$, for $e = e^2 \in B$, and $H = I_B\{[S_M(H)]^e\} = I_B(Me) = Be$. Hence H is a direct summand in B, proving that B is a left CS ring.

Conversely, assume that B is a left CS ring, and let U be a complement in M. Then $I_B(U) = Be$, for $e = e^2 \in B$, and $U = [S_M I_B(U)]^e = [S_M(Be)]^e = [Me]^e = Me$, the last equality since every direct summand is e-closed. Hence U is a direct summand in M and M is a CS module. ▌

As is clear from the last three Corollaries, complements in M play an important role in determining when B is left CS. Our next theorem shows that complements can also play an important role in determining when B is

Baer. We will first prove a Lemma which will enable us to use the results of Section 2 for the proof of Theorem 3.13. Using Definition 3, we will say M is *e-retractable* if $I_B(U) \neq 0$, for each nonzero complement, U, in M.

LEMMA 3.12. *Let ${}_RM$ be a nonsingular, e-retractable module. Then M is an a-self-generator. Consequently, B is a Baer ring if and only if every a-closed submodule of M is a direct summand in M.*

Proof. Let $U = l_M r_B(U)$ be an a-closed submodule of M. We will show first that, since M is nonsingular, U is a complement. In order to do this, we will make use of the following well-known property of essential extensions: If ${}_RN \subset' {}_RK$ and $0 \neq k \in K$, then the left ideal, ${}_RI$, of R defined by ${}_RI = [N: k] = \{r \in R: rk \in N\}$ is essential in ${}_RR$ (cf., e.g., [3, p. 46, Lemma 3]). Let $V = U^e$, so that, in particular, $U \subset' V$; then, clearly, $r_B(V) \subseteq r_B(U)$. Conversely, let $b \in r_B(U)$; then we will show that $b \in r_B(V)$. Let $v \in V$; then the left ideal ${}_RI = [U: v] = \{r \in R: rv \in U\} \subset' {}_RR$, and we have $Ivb = 0$. Since ${}_RM$ is nonsingular, this implies $vb = 0$, i.e., $b \in r_B(V)$, and $r_B(U) = r_B(V)$. We have $V \subseteq l_M r_B(V) = l_M r_B(U) = U$, hence $U = V$, and U is a complement in M.

In the preceding paragraph, we showed that, if U and V are any two submodules of M such that ${}_RU \subset' {}_RV$, then $r_B(U) = r_B(V)$. We will show next that, since M is e-retractable, we have $S_M I_B(C) \subset' C$, for any complement submodule, C, of M. Let $0 \neq c \in C$; then we have $0 \neq Y = (Rc)^e \subseteq C$ and $0 \neq I_B(Y) \subseteq I_B(C)$. Let $0 \neq b \in I_B(Y)$; then $0 \neq Mb \subseteq Y$ implies $0 \neq Mb \cap Rc \subseteq S_M I_B(C) \cap Rc$, proving that $S_M I_B(C) \subset' C$.

Now, if U is any a-closed submodule of M, then U is a complement and we have $S_M I_B(U) \subset' U$. Hence $r_B[S_M I_B(U)] = r_B(U)$, and $U = l_M r_B(U) = l_M r_B[S_M I_B(U)]$, proving that M is an a-self-generator. The second assertion of the Lemma now follows directly from Corollary 2.6. ∎

THEOREM 3.13. *Let ${}_RM$ be a nonsingular, CS module. Then B is a Baer ring.*

Proof. If $0 \neq U$ is a complement in M, then, since M is CS, U is a direct summand in M, so $U = Me$ for some $e = e^2 \in B$. Hence, $I_B(U) \neq 0$; i.e., a CS module is e-retractable. Also, since a complement is a direct summand, every complement in M is a-closed. On the other hand, we know from the proof of Lemma 3.12 that, when M is nonsingular, every a-closed submodule is a complement. Hence, in a nonsingular CS module, M, U is a complement if and only if it is a-closed. Since M is, in particular, nonsingular and e-retractrable, we know from Lemma 3.12 that B is a Baer ring if and only if every a-closed submodule is a direct summand in M, i.e., if and only if every complement submodule is a direct summand in M, i.e., if and only if M is CS. Hence B is Baer and the proof is complete.

Combining the fact that, in a nonsingular module, every a-closed submodule is a complement, with Corollary 3.4, we get:

COROLLARY 3.14. *Let $_RM$ be nondegenerate and nonsingular. Then B is a left Goldie ring if and only if M satisfies the a.c.c. on complement submodules, i.e., iff $d(_RM) < \infty$.*

Combining Corollary 3.9 and Theorem 3.13, we have:

COROLLARY 3.15. *Let $_RM$ be nondegenerate, nonsingular, and CS. Then B is Baer and left CS.*

A comparison of the two results,

M nondegenerate $CS \Rightarrow B$ left CS (Corollary 3.9); and

M nonsingular $CS \Rightarrow B$ Baer (Theorem 3.13),

naturally brings up the questions: what is the relationship between Baer rings and left CS rings and what is the relationship between nondegenerate modules and nonsingular modules?

In order to answer the first question, we recall first that a ring R is said to be left co-nonsingular if every left ideal of R which has zero right annihilator is essential. Note that a Baer ring is always both left and right nonsingular, whereas a left CS ring need not be left nonsingular. It is known that: A ring is left nonsingular, left CS if and only if it is a left co-nonsingular Baer ring ([2, Theorem 2.1]).

With regard to the second question, we have:

PROPOSITION 3.16. *Let $_RM$ be nonsingular. Then $_RM$ is nondegenerate if and only if $_RM$ is retractable and $TM \subset' {}_RM$.*

Proof. Assume that the nonsingular M is retractable and satisfies $TM \subset' M$. We will show that M is nondegenerate by proving that $TU \subset' U$ for any nonzero submodule, U, of M, for then this implies $0 \neq TU = (M, M^*)U = M[M^*, U]$ and hence $[M^*, U] \neq 0$.

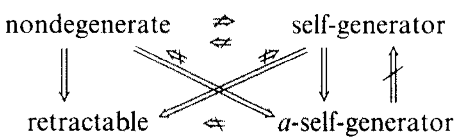
Note first that $M[M^*, U] = (M, M^*)U \subseteq U$, so that $[M^*, U] \subseteq I_B(U)$. We have: $(M, M^*)U = M[M^*, U] \subseteq S_M I_B(U) \subseteq U$, and, by retractability, $S_M I_B(U) \subset' U$, so it remains to show that $(M, M^*)U \subset' S_M I_B(U)$. Let $0 \neq u \in S_M I_B(U)$, and write $u = \sum_{i=1}^n m_i b_i$, with $m_i \in M$, $b_i \in I_B(U)$, and $m_i b_i \neq 0$, for $i = 1, ..., n$. Set $K_i = [(M, M^*)M : m_i] = \{r \in \mathrm{R} : rm_i \in (M, M^*)M\}$; then $_RK_i \subset' {}_RR$ and $0 \neq K_i m_i b_i \subseteq (M, M^*) Mb_i \cap Rm_i b_i$, for $i = 1, ..., n$. Let $J = l_R(u) = \{r \in R : ru = 0\}$; then, since M is nonsingular and $u \neq 0$, J is not an essential left ideal of R. Let $0 \neq {}_RN$ be a left ideal of R such that $N \cap J = 0$. Then, since $\bigcap_{i=1}^n K_i \subset' {}_RR$, there is $0 \neq k \in N \cap (\bigcap_{i=1}^n K_i)$. We have: $0 \neq ku = k(\sum_{i=1}^n m_i b_i) = \sum_{i=1}^n km_i b_i \in$

$\sum_{i=1}^{n} (M, M^*) Mb_i$, since $k \in \bigcap K_i$, therefore since $Mb_i \subseteq U$, $0 \neq ku \in (M, M^*) U \cap Ru$, which shows that $(M, M^*)U \subset' S_M I_B(U)$, and M is nondegenerate.

Conversely, if M is nondegenerate, then we know by Proposition 3.2(1) that M is retractable. Moreover, it is easy to see that $(M, M^*)M \subset' {}_RM$, for, if $0 \neq m \in M$, then, by nondegeneracy, $0 \neq (M, M^*)m \subseteq (M, M^*)M \cap Rm$. ∎

If R is a ring which is not left nonsingular, then ${}_RR$ is nondegenerate but not nonsingular, as is any free left R-module, ${}_RF$. On the other hand, in Example 3.2 of [2], we have a nonsingular, projective module which is not retractable (not even e-retractable) and hence not nondegenerate.

In Example 3.4 of [5], ${}_RM$ is a nonsingular, projective, e-retractable module which is not retractable, hence not nondegenerate. However, by Lemma 3.12, M is an a-self-generator. This gives us an example of an a-self-generator which is not nondegenerate. Also, since a self-generator is clearly retractable, this shows that an a-self-generator need not be a self-generator. Finally, note that the example mentioned earlier in this section of a nondegenerate module which is not a self-generator also shows that a retractable module need not be a self-generator. The relationships between self-generators, a-self-generators, nondegenerate and retractable modules may be summarized in the following diagram, where the symbol "$\not\Rightarrow$" indicates the counter-examples mentioned throughout this section:



It would be interesting to have an example of a rectractable module which is not an a-self-generator, in order to complete the diagram.

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