A strong convergence theorem for a modified Krasnoselskii iteration method and its application to seepage theory in Hilbert spaces

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Krasnoselskii iteration; Strong convergence; Minimum norm solution; Pseudomonotone mappings; Lipschitzian mappings; Seepage theory

Abstract

Inspired by the modified iteration method devised by He and Zhu [1], the purpose of this paper is to present a modified Krasnoselskii iteration via boundary method. A strong convergence theorem of this iteration for finding minimum norm solution of nonlinear equation of the form \( S(x) = 0 \), where \( S(x) \) is a nonlinear mapping of \( C \) into itself and \( h \) is a function of \( C \) into \( [0, 1] \), is then proved in Hilbert spaces. In the same vein, an application to the stationary problem of seepage theory is also presented. The results of this paper are extensions and improvements of some earlier theorems of Saddeek et al. [2].

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1. Introduction

Fixed point theory is an interesting topic with multiple applications in various branches of mathematics. For example in fluid mechanics, the development of iteration methods for finding fixed points of nonlinear self-mappings has historically been an important enterprise.

The Krasnoselskii iteration method (KIM) is one of the most salient examples of these methods.

Let \( X \) be a real Hilbert space and let \( C \) be a nonempty, closed and convex subset of \( X \). Let \( T : C \rightarrow C \) be a self-mapping.

Then the KIM (see, for example, [3]) generates, for any \( x_0 \in C \), a sequence \( \{x_n\} \) in \( C \) by

\[
x_{n+1} = (1 - \tau)x_n + \tau Tx_n, \quad n \geq 0,
\]

where \( \tau \in [0, 1] \).

It should be noted that for \( \tau = 1 \), the KIM reduces to the Picard iteration (successive iteration) method (see, for example, [4]) that is \( x_{n+1} = TX_n, n \geq 0 \).

The KIM has been studied extensively by many authors (see, for example, [5–11]).

In a recent paper [2, Theorem 2] Saddeek et al. have shown that, under certain appropriate conditions imposed on the mapping \( T \), the entire sequence of KIM (1) converges weakly...
to a fixed point of $T$ in a real Hilbert space setting. The results is then applied to a problem of fluid mechanics.

An interesting problem is how to appropriately modify the KIM (1) so as to have strong convergence? For this purpose, in this paper, inspired by He and Zhu [1], we introduce a modified Krasnoselskii iteration method MKIM (2) below with strong convergence (by boundary point method) for finding the minimum norm solution of nonlinear equation of the form $S(x_k)(x) = 0$, where $S(x_k)$ is a nonlinear mapping of $C$ into itself and $h$ is a function of $C$ into $[0,1]$ defined below. Furthermore, we apply this result to the stationary problem of seepage theory. The results obtained in this paper represent an extension as well as refinement of some earlier theorems of [2].

2. Preliminaries

Let $C$ be a nonempty, closed and convex subset of $X$ and let $T$ be a mapping of $C$ into itself.

In the sequel we use $F(T)$ to denote the nonempty set of fixed points of $T$, $\rightarrow$ (to denote strong (weak) convergence, and $\mathcal{W}(x_n)$ to denote the set of weak cluster points of $(x_n)$ (i.e., $\mathcal{W}(x_n) = \{y \in X : \{x_n - y\} \subseteq x_n \rightarrow y\}$). Denote by $\text{proj}_C(x)$ the metric projection mapping from $X$ onto $C$ (i.e. $\text{proj}_C(x) = \{y \in X : \|y - x\| = \inf_{z \in C}\|x - z\|\}$).

The projection mapping is characterized as follows (see, for example, [12]):

**Proposition 2.1.** Given $x \in X$ and $z \in C$. Then $z = \text{proj}_C(x)$ if and only if

$$(x - z, y - z) \leq 0, \quad \forall y \in X.$$ 

The notions $U(x; \delta)$ and $\partial C$ are used to denote, respectively, the spherical neighborhood of $x$ of radius $\delta > 0 : U(x; \delta) = \{y \in X : \|y - x\| < \delta\}$ and the boundary of $C$ (whenever $C$ is closed) : $\partial C = \{x \in X : U(x; \delta) \cap (X - C) \neq \emptyset\}.$

Since $F(T)$ is a nonempty, closed and convex subset of $X$ (see, for example, [13]), there exists a unique $\hat{x} \in F(T)$ satisfies the following:

$$\|\hat{x}\| = \min\{\|x\| : x \in F(T)\}.$$ 

That is, $\hat{x}$ is the minimum norm fixed point of $T$. In other words, $\hat{x}$ is the metric projection of the origin onto $F(T)$, that is, $\hat{x} = \text{proj}_C(0)$.

**Definition 2.1** (see, for example, [14–16]). For any $x, y \in C$ the mapping $T : C \rightarrow C$ is said to be as follows:

(i) pseudomonotone, if it is bounded and if for every sequence $\{x_n\} \subseteq C$ such that $x_n \rightarrow x$ and $\limsup_{n \rightarrow \infty} \langle Tx_n, x_n - x \rangle \leq 0$,

implies that

$$\lim_{n \rightarrow \infty} \inf\langle Tx_n, x_n - y \rangle \geq \langle Tx, x - y \rangle;$$

(ii) coercive, if

$$\langle Tx, x \rangle \geq \rho(\|x\|\|x\|, \lim_{n \rightarrow \infty} \rho(\xi) = +\infty;$$

(iii) Lipschitzian, if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|;$$

(iv) potential, if

$$\int_0^1 (\langle T(t(x + y), x + y) - \langle T(tx), x \rangle)dt = \int_0^1 (T(x + ty), y)dt;$$

(v) demiclosed at 0, if for every sequence $\{x_n\} \subseteq C$ the assumptions

$$x_n \rightarrow x \text{ and } Tx_n \rightarrow 0, \text{ as } n \rightarrow \infty$$

implies that

$$x \in C \text{ and } Tx = 0.$$ 

3. A MKIM by boundary point method

In order to state our algorithm we introduce, as in [1], $h : C \rightarrow [0, 1]$ by

$$h(x) = \inf\{x \in [0,1] : xz \in C\}, \quad \forall x \in C.$$ 

The function $h(x)$ is well defined because $C$ is closed and convex.

In [1] it has been noted that if $0 \notin C$, then $h(x) = 0$ for all $x \in C$ and if $0 \notin C$, then $h(x)x \in \partial C$ and $h(x) > 0$ for every $x \in C$ (for a contradiction, suppose that $h(x)x \notin \partial C$ in the case where $0 \notin C$, it is easy to verify that $h(x)x$ is an inner point in $C$, there exists sufficiently small $\delta > 0$ such that $U(h(x)x); \delta \subseteq C$, we have $[h(x) - \delta x \in C$. This contradicts the definition of $h(x)$. Notice that $C$ is closed and convex, hence, $h(x)x \in \partial C$). For more details of the properties of $h(x)$, the reader is refereed to [1].

Let $T : C \rightarrow C$ be a self mapping. Then the MKIM by boundary point is given by $x_0 = x \in C$, and

$$x_{n+1} = (1 - \tau(h(x_n)))x_n + \tau T, \quad n \geq 0,$$ 

where $\tau \in (0,1), \quad T = (1 - \tau)I + \tau T \text{ and } \sum_{n=0}^{\infty} h(x_n) = \infty.$

**Remark 3.1.**

(i) If $0 \notin C$ and $h(x_0) = 1$, then the MKIM $\{x_n\}$ given by (2) is exactly the KIM corresponding to the associated mapping $T$.

(ii) If $0 \in C$ (i.e., $h(x_n) = 0$ for all $n \geq 0$), then the MKIM $\{x_n\}$ given by (2) is exactly the Picard iteration corresponding to the associated mapping $I + \tau T$.

(iii) Since $C$ is closed and convex, it follows from the definition of $h(x)$ that, for any given $x \in C$, $\forall x \in C$ holds for every $y \in [h(x)x], \gamma \text{ which imply that } \{x_n\} \subseteq C$ is guaranteed.

(iv) In the case where $0 \notin C$, calculating the value $h(x_n)$ implies determining $h(x_n)x_n$ (a boundary point of $C$), so our modification method is called boundary point method.

**Lemma 3.1** (see, for example [17,18]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying

$$a_{n+1} \leq (1 - \omega_n)a_n + \omega_n b_n + c_n, \quad \forall n \geq 0,$$

where, $\omega_n \in (0,1)$. If $\sum_{n=0}^{\infty} \omega_n = \infty$, either $\limsup_{n \rightarrow \infty} b_n \leq 0$ or

$$\sum_{n=0}^{\infty} |\omega_n b_n| < \infty \text{ and } \sum_{n=0}^{\infty} \omega_n c_n < \infty, \text{ then } \lim_{n \rightarrow \infty} a_n = 0.$$

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Lemma 3.2 (see, for example [19]). For all $x, y \in X$ the following result holds:

$$
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle.
$$

4. Strong convergence theorem

In this section, we prove a strong convergence theorem which is our main result.

Theorem 4.1. Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $X$ and let $T$ be a mapping of $C$ into itself. For the function $h(x)$ defined above let $S_{h(x)} : C \to C$ be a nonlinear mapping defined by $S_{h(x)}x = h(x)x - Tx$, $\forall x \in C$. Suppose that $S_{h(x)}$ is demiclosed at 0, coercive, potential and bounded, and suppose that there exists a nonnegative real-valued function $r(x, y)$ such that $\sup_{x, y \in C} r(x, y) = \zeta < \infty$ and

$$
r(x, y) \|x - y\| \geq \|S_{h(x)}x - S_{h(y)}y\|, \quad \forall x, y \in C.
$$

Then the MKIM $\{x_n\} \subset C$ generated by (2) with $\sum_{n=0}^{\infty} h(x_n) = \infty$ and $0 < \zeta = \min \{1, \frac{1}{2}\}$, converges strongly to $\bar{x} = \text{proj}_{\bar{C}}(0)$, where $\bar{C} = \{x \in C : S_{h(x)}x = 0\}$.

Proof. We first prove that $\{x_n\}$ is bounded. For this it suffices to prove that

$$
\{x_n\} \subset S_0, \|x_n\| \leq R_0, \quad n \geq 0,
$$

where $S_0 = \{x \in C : F_1(x) \subseteq F_1(x_0)\}, \quad R_0 = \sup_{x \in S_0} \|x\|,$ and $F_1 : X \to (-\infty, \infty]$ is a functional defined by

$$
F_1(x) = \int_0^1 \langle S_{h(t)}(tx), x \rangle dt, \quad \forall x \in X.
$$

Observe that the coercivity of the mapping $S_{h(x)}$ implies that the functional $F_1$ is coercive, and hence $R_0 < \infty$. Moreover, observe that the boundedness of the mapping $S_{h(x)}$ implies that $\sup_{x \in S_0} \|S_{h(x)}x\| < \infty$. Therefore, the MKIM is well defined.

By induction. Suppose that $x_n \in S_0$ for $n \geq 1$. Then $F_1(x_n) \leq F_1(x_0)$.

Using (3) and writing $r$ for $r(x_n, x_{n+1})$, we have

$$
\|S_{h(x_0)}(x_{n+1} + t(x_n - x_{n+1})) - S_{h(x_0)}(x_n)\|
\leq r \|(t - 1)(x_n - x_{n+1})\| \leq r \|x_n - x_{n+1}\|,
$$

for $t \in [0, 1]$. This leads to

$$
\|S_{h(x_0)}(x_{n+1} + t(x_n - x_{n+1})) - S_{h(x_0)}(x_n)x_n - x_{n+1}\|
\leq r \|x_n - x_{n+1}\|^2.
$$

As $S_{h(x)}$ is potential, by (4) we get

$$
F_1(x_n) - F_1(x_{n+1}) = \int_0^1 \langle S_{h(t)}(tx_n), x_n - (S_{h(t)}(tx_n), x_{n+1}) \rangle dt
\leq \int_0^1 \langle (S_{h(t)}(tx_n + t(x_n - x_{n+1})), x_n - x_{n+1}) \rangle dt
\leq \int_0^1 \langle (S_{h(t)}(tx_n + t(x_n - x_{n+1})), x_n - x_{n+1}) \rangle dt
\leq \int_0^1 \langle (S_{h(t)}(tx_n, x_n - x_{n+1}) + (S_{h(t)}(tx_n), x_n - x_{n+1}) \rangle dt
\geq -\int_0^1 \langle (S_{h(t)}(tx_n + t(x_n - x_{n+1})), x_n - x_{n+1}) \rangle dt

This together with (2) and (6) give that

$$
F_1(x_n) - F_1(x_{n+1}) \geq -r\|x_n - x_{n+1}\|^2 + \tau^{-1}\langle x_n - x_{n+1}, x_n - x_{n+1} \rangle
\geq \lambda\|x_n - x_{n+1}\|^2, \quad \lambda = \tau^{-1} - \zeta.
$$

Therefore, $F_1(x_{n+1}) \leq F_1(x_n) \leq F_1(x_0)$, that is $x_{n+1} \in S_0$. Having in mind $x_n \in S_0$ (by the definition of $S_0$), the assertion $x_n \in S_0$, $n \geq 0$ follows by induction. Thus, $\{x_n\}$ is bounded. Hence, $\{S_{h(x_0)}x_n\}$ and $\{F_1(x_n)\}$ are also bounded (by (2) and (4)).

Since by (7), the sequence $\{F_1(x_n)\}$ is monotone, it therefore follows that $\lim_{n \to \infty} F_1(x_n)$ exists. This, together with (7), implies that

$$
\lim_{n \to \infty} \|x_n - x_{n+1}\| = 0.
$$

Therefore, from (2), we have

$$
\lim_{n \to \infty} \|S_{h(x_0)}x_n\| = 0.
$$

Now, we prove that all weak cluster points of $\{x_n\}$ are elements of $\bar{C}$. The boundedness of $\{x_n\}$ implies there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\lim_{n_k \to \infty} \|x_{n_k} - x^*\|$ exists to some $x^* \subset C$. This, together with (9) and the demiclosedness of $S_{h(x)}$, implies that $x^* \subset \bar{C}$.

Now using the same argument as in [20, p.70], it can be show that all weakly convergent subsequences of $\{x_n\}$ have the same weak limit. Then $\lim_{n \to \infty} \|x_n - x^*\|$ exists and hence

$$
\mathfrak{M}(x_n) \subset \bar{C}.
$$

Now we will show

$$
\lim \sup_{n \to \infty} \langle -\hat{x}, x_{n+1} - \hat{x} \rangle \leq 0.
$$

Indeed, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \to x^*$, $\lim \sup_{n \to \infty} \langle -\hat{x}, x_{n_k+1} - \hat{x} \rangle = \lim_{k \to \infty} \langle -\hat{x}, x_{n_k+1} - \hat{x} \rangle.

Since $\hat{x}$ is the metric projection of the origin onto $\bar{C}$, we obtain from $\hat{x} \subset \bar{C}$ and Proposition 2.1 that

$$
\lim_{n_k \to \infty} \langle -\hat{x}, x_{n_k+1} - \hat{x} \rangle = \lim_{k \to \infty} \langle -\hat{x}, x_{n_k+1} - \hat{x} \rangle
\geq \langle -\hat{x}, x^* - \hat{x} \rangle \leq 0.
$$

Finally, we prove that $\lim_{n \to \infty} \|x_n - \hat{x}\| = 0$. From (2) and Lemma 3.2, we have
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Let \( A \) be a pseudomonotone equation. This equation occurs in seapage theory.

We assume that \( g \) is continuous nonnegative, equal to zero when \( \xi = 0 \) (\( \beta \geq 0 \) is the limit gradient), strictly increasing when \( \eta \), and there exist \( c_2 > 0 \) such that

\[
g(\xi) \leq c_1(\xi - \beta), \quad \xi > 0.
\]

Under the conditions imposed above, the mapping defined by (14) is pseudomonotone, potential, coercive and Lipschitzian (see, for example, [21]).

Theorem 5.1. Let \( A : C \to C \) be a pseudomonotone, potential, coercive and Lipschitzian mapping. Then the sequence \( \{x_n\} \) generated by \( x_{n+1} = x_n - \tau A(x_n) \), \( n \geq 0 \),

\[
x_{n+1} = x_n - \tau A(x_n) - f, \quad n \geq 0,
\]

where \( 0 < \tau = \min \{1, \frac{1}{L}\} \), \( L \) is the Lipschitz constant for \( A \), converges strongly to the minimum norm solution of the Eq. (13). Provided that \( \sum \infty |\hat{h}(x_n)| = \infty \).

Proof. Our assumptions ensure that the set \( N = \{x \in C : Ax = f\} \) is nonempty, closed and convex (see, for example, [14]).

Let \( x^* \in \overline{N} \) and let \( S_{\hat{h}(\xi)}(x) = Ax - f \). Then we have \( S_{\hat{h}(\xi)}(x^*) = Ax^* - f \), that is \( \overline{N} \neq \emptyset \).

Using the Lipschitzian of \( A \), it follows that condition (3) is satisfied with \( r(x, y) = L \).

Now, we show that \( S_{\hat{h}(\xi)} \) is demiconvex at 0.

Let \( \{x_n\} \) be a sequence in \( C \) with \( \|Ax_n - f\| \to 0 \) as \( n \to \infty \) and \( x^* \in \overline{N} \).

Then we must show \( x^* \in \overline{N} \). Suppose that

\[
\limsup_{n \to \infty} (Ax_n, x_n - x^*) \leq 0.
\]

By the pseudomonotonicity of \( A \), we have

\[
\liminf_{n \to \infty} (Ax_n, x_n - y) \geq (Ax^*, x^* - y), \quad \forall y \in C.
\]

Let us now prove (16)

\[
\limsup_{n \to \infty} (Ax_n - f, x_n - x^*) \leq \limsup_{n \to \infty} \|Ax_n - f\| \|x_n - x^*\| \leq 0.
\]

This shows that \( x^* \) is a solution of the variational inequality \( \langle Ax^* - f, y - x^* \rangle \geq 0, \forall y \in C \), and consequently (see, for example, [14]), \( x^* \in \overline{N} \). Therefore, we have \( S_{\hat{h}(\xi)} \) is demiconvex at 0 and \( x^* \in \overline{N} \). The results follows from Theorem 4.1. □

References


