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On the fractional Adams method[☆]

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Dedicated to Professor Weiqiu Zhu on the occasion of his 70th birthday

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ABSTRACT

The generalized Adams–Bashforth–Moulton method, often simply called "the fractional Adams method", is a useful numerical algorithm for solving a fractional ordinary differential equation: $D_*^\alpha y(t) = f(t, y(t)), y^{(k)}(0) = y_0^{(k)}, k = 0, 1, \ldots, n-1$, where $\alpha > 0, n = \lceil\alpha\rceil$ is the first integer not less than α, and $D_*^\alpha y(t)$ is the αth-order fractional derivative of $y(t)$ in the Caputo sense. Although error analyses for this fractional Adams method have been given for (a) $0 < \alpha, D_*^\alpha y(t) \in C^2[0, T]$, (b) $\alpha > 1, y \in C^{1+\lceil\alpha\rceil}[0, T]$, (c) $0 < \alpha < 1, y \in C^2[0, T]$, (d) $\alpha > 1, f \in C^3(G)$, there are still some unsolved problems—(i) the error estimates for $\alpha \in (0, 1), f \in C^3(G)$, (ii) the error estimates for $\alpha \in (0, 1)$, $f \in C^2(G)$, (iii) the solution $y(t)$ having some special forms. In this paper, we mainly study the error analyses of the fractional Adams method for the fractional ordinary differential equations for the three cases (i)–(iii). Numerical simulations are also included which are in line with the theoretical analysis.

1. Introduction

In recent decades, the fractional calculus and fractional differential equations have attracted much attention and increasing interest (see [1–4] and many references cited therein) due to their potential applications in science and engineering (see the introduction parts of Refs. [5,6]). Here we study a fractional differential equation in the following form:

$$D_*^\alpha y(t) = f(t, y(t)), \qquad y^{(k)}(0) = y_0^{(k)}, \quad k = 0, 1, \ldots, n-1, \tag{1.1}$$

where $\alpha > 0$ and $n := \lceil\alpha\rceil$ is the first integer not less than α. $D_*^\alpha y(t)$ is the αth-order (always fractional) derivative of $y(t)$ in the Caputo sense, which is defined by

$$D_*^\alpha z(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} z^{(n)}(\tau) \mathrm{d}\tau, \quad n-1 < \alpha < n \in Z^+,$$

where $z^{(n)}$ denotes the derivative of integer nth order of z.

If we require the function f to be continuous and satisfy a Lipschitz condition with respect to the second argument y with Lipschitz constant L on a suitable set G, then the initial value problem (1.1) determines a unique solution on some interval $[0, T]$, by use of Theorems 2.1 and 2.2 of [7]. Throughout the paper, we always assume that f fulfils the above condition, so Eq. (1.1) has one and only one solution defined on $[0, T]$. This solution solves the following Volterra integral equation:

$$y(t) = \sum_{k=0}^{n-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u, y(u)) \mathrm{d}u, \quad t \le T. \tag{1.2}$$

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Now numerical integration of differential equation (1.1) is transformed into numerical quadrature of an associated integral equation, (1.2).

The fractional Adams method for solving Eq. (1.1) (or (1.2)) was first studied by Diethelm, Ford and Freed [5]. They worked on a uniform grid $\{t_j = jh : j = 0, 1, \ldots, N\}$ with some integer N and step length $h = T/N$, and let $y_j \approx y(t_j)$. In detail, their derived computation scheme is as follows:

$$\begin{cases} y_{k+1}^P = \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j), \\ y_{k+1} = \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}^P) \right), \end{cases} \tag{1.3}$$

where

$$a_{j,k+1} = \frac{h^\alpha}{\alpha(\alpha+1)} \cdot \begin{cases} (k^{\alpha+1} - (k-\alpha)(k+1)^\alpha) & \text{if } j = 0, \\ ((k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}) & \text{if } 1 \le j \le k, \\ 1 & \text{if } j = k+1, \end{cases}$$

and

$$b_{j,k+1} = \frac{h^\alpha}{\alpha} \left((k+1-j)^\alpha - (k-j)^\alpha \right), \quad j = 0, 1, 2, \ldots, k.$$

This computational scheme is very useful and efficient for numerical integration of fractional differential equations. In particular, it is successfully applied in computing chaotic attractors of fractional systems; for example, see [8].

The remainder of this article is organized as below. In Section 2, we simply recall the main results from [5]. In Section 3, we further study the error analysis of the fractional Adams scheme (1.3). A numerical example is included which is in line with the associated theoretical results, in the last section.

2. Known error analyses for the fractional Adams method

In their error analysis, Diethelm et al. applied the following theorem which is attributed to Lubich [9].

Theorem 2.1. (a) *Assume that $f \in C^2(G)$. Define $\nu_1 = \lceil 1/\alpha \rceil - 1$. Then there exists a function $\psi \in C^1[0, T]$ and some $c_1, \ldots, c_{\nu_1} \in \mathbf{R}$ such that the solution y of (1.1) can be expressed in the form*

$$y(t) = \psi(t) + \sum_{\nu=1}^{\nu_1} c_\nu t^{\alpha\nu}.$$

(b) *Assume that $f \in C^3(G)$. Define $\nu_1 = \lceil 2/\alpha \rceil - 1$ and $\nu_2 = \lceil 1/\alpha \rceil - 1$. Then there exists a function $\psi \in C^2[0, T]$ and some $c_1, \ldots, c_{\nu_1} \in \mathbf{R}$ and $d_1, \ldots, d_{\nu_2} \in \mathbf{R}$ such that the solution y of (1.1) can be expressed in the form*

$$y(t) = \psi(t) + \sum_{\nu=1}^{\nu_1} c_\nu t^{\alpha\nu} + \sum_{\nu=1}^{\nu_2} d_\nu t^{1+\alpha\nu}.$$

Direct computations lead to Theorems 2.2–2.4 [5].

Theorem 2.2. *If $y \in C^m[0, T]$ for some $m \in N$ and $0 < \alpha < m$, then*

$$D_*^\alpha y(t) = \sum_{l=0}^{m-\lceil\alpha\rceil-1} \frac{y^{(l+\lceil\alpha\rceil)}(0)}{\Gamma(\lceil\alpha\rceil - \alpha + l + 1)} t^{\lceil\alpha\rceil-\alpha+l} + g(t),$$

with some function $g \in C^{m-\lceil\alpha\rceil}[0, T]$. Moreover, the $(m - \lceil\alpha\rceil)$th derivative of g satisfies a Lipschitz condition of order $\lceil\alpha\rceil - \alpha$. Furthermore, $D_*^\alpha y \in C[0, T]$.

Theorem 2.3. (a) *Let $z \in C^1[0, T]$. Then,*

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} z(t) \mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \le \frac{1}{\alpha} \|z'\|_\infty t_{k+1}^\alpha h.$$

(b) *Let $z(t) = t^p$ for some $p \in (0, 1)$. Then,*

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} z(t) \mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \le C_{\alpha,p}^{\mathrm{Re}}\, t_{k+1}^{\alpha+p-1} h,$$

where $C_{\alpha,p}^{\mathrm{Re}}$ is a constant that depends only on α and p.

Theorem 2.4. (a) *If $z \in C^2[0,T]$ then there is a constant C_α^{Tr} depending only on α such that*

$$\left|\int_0^{t_{k+1}}(t_{k+1}-t)^{\alpha-1}z(t)\mathrm{d}t-\sum_{j=0}^{k+1}a_{j,k+1}z(t_j)\right|\le C_\alpha^{\mathrm{Tr}}\|z'\|_\infty t_{k+1}^\alpha h^2.$$

(b) *Let $z \in C^1[0,T]$ and assume that z' fulfils a Lipschitz condition of order μ for some $\mu \in (0,1)$. Then, there exist positive constants $B_{\alpha,\mu}^{\mathrm{Tr}}$ and $M(z,\mu)$ (depending only on α and μ, z and μ, respectively) such that*

$$\left|\int_0^{t_{k+1}}(t_{k+1}-t)^{\alpha-1}z(t)\mathrm{d}t-\sum_{j=0}^{k+1}a_{j,k+1}z(t_j)\right|\le B_{\alpha,\mu}^{\mathrm{Tr}}M(z,\mu)t_{k+1}^\alpha h^{1+\mu}.$$

(c) *Let $z(t)=t^p$ for some $p\in(0,2)$ and $\varrho := \min(2,p+1)$. Then,*

$$\left|\int_0^{t_{k+1}}(t_{k+1}-t)^{\alpha-1}z(t)\mathrm{d}t-\sum_{j=0}^{k+1}a_{j,k+1}z(t_j)\right|\le C_{\alpha,p}^{\mathrm{Tr}}t_{k+1}^{\alpha+p-\varrho}h^\varrho,$$

where $C_{\alpha,p}^{\mathrm{Tr}}$ is a constant that depends only on α and p.

On the basis of the above results, the main error estimates derived by Diethelm et al. are listed here.

Theorem 2.5 (*Diethelm–Ford–Freed (DFF) Theorem*). (a) *Let $0<\alpha$ and assume $D_*^\alpha y(t)\in C^2[0,T]$ for some suitable T. Then,*

$$\max_{0\le j\le N}|y(t_j)-y_j|=\begin{cases}O(h^2) & \text{if } \alpha\ge 1,\\ O(h^{1+\alpha}) & \text{if} \alpha<1.\end{cases}$$

(b) *Let $\alpha>1$ and assume that $y\in C^{1+\lceil\alpha\rceil}[0,T]$ for some suitable T. Then,*

$$\max_{0\le j\le N}|y(t_j)-y_j|=O(h^{1+\lceil\alpha\rceil-\alpha}).$$

(c) *Let $0<\alpha<1$ and assume that $y\in C^2[0,T]$ for some suitable T. Then, for $1\le j\le N$ one has*

$$|y(t_j)-y_j|\le Ct_j^{\alpha-1}\cdot\begin{cases}h^{1+\alpha} & \text{if } 0<\alpha<0.5,\\ h^{2-\alpha} & \text{if } 0.5\le\alpha<1,\end{cases}$$

where C is a constant independent of j and h. In particular,

$$\max_{0\le j\le N}|y(t_j)-y_j|=\begin{cases}O(h^{2\alpha}) & \text{if } 0<\alpha<0.5,\\ O(h) & \text{if } 0.5\le\alpha<1.\end{cases}$$

Moreover, for every $\epsilon\in(0,T)$ one has

$$\max_{t_j\in[\epsilon,T]}|y(t_j)-y_j|=\begin{cases}O(h^{1+\alpha}) & \text{if } 0<\alpha<0.5,\\ O(h^{2-\alpha}) & \text{if } 0.5\le\alpha<1.\end{cases}$$

(d) *Let $\alpha>1$. Then, if $f\in C^3(G)$, one gets*

$$\max_{0\le j\le N}|y(t_j)-y_j|=O(h^2).$$

3. Further studies on the fractional Adams method

In this section, we start to prove a theorem given below.

Theorem 3.1. *Let $\alpha>0$ and assume that $D_*^\alpha y(t)\in C^r[0,T]$ for $3\le r\in Z^+$ and some suitable T. Then,*

$$E[T,f]=y(T)-y_{T/h}=\sum_{j=1}^r h^{j+\alpha}\sum_{i=0}^{2T/h+1}c_{j,i,T/h}h^{i\alpha},$$

in which $c_{j,i,T/h}$ are coefficients which depend upon f and $D_*^\alpha y(t)$ (see Eq. (1.1)).

Proof. Firstly, using the Mean Value Theorem and simple calculations yields

$$\begin{aligned} y(t_{k+1}) - y^P_{k+1} &= \frac{1}{\Gamma(\alpha)}\left\{\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} f(t, y(t))\mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j)\right\} \\ &= \frac{1}{\Gamma(\alpha)}\left\{\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} D^\alpha_* y(t)\mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} D^\alpha_* y(t_j) + \sum_{j=0}^{k} b_{j,k+1} f_y(t_j, \eta_j)(y(t_j) - y_j)\right\}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} y(t_{k+1}) - y_{k+1} &= \frac{1}{\Gamma(\alpha)}\left\{\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} f(t, y(t))\mathrm{d}t + a_{k+1,k+1}(f(t_{k+1}, y(t_{k+1})) - f(t_{k+1}, y^P_{k+1}))\right\} \\ &= \frac{1}{\Gamma(\alpha)}\left\{\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} D^\alpha_* y(t)\mathrm{d}t - \sum_{j=0}^{k+1} a_{j,k+1} D^\alpha_* y(t_j)\right. \\ &\quad \left. + \sum_{j=0}^{k} a_{j,k+1} f_y(t_j, \eta_j)(y(t_j) - y_j) + a_{k+1,k+1} f_y(t_{k+1}, \eta_{k+1})(y(t_{k+1}) - y^P_{k+1})\right\}. \end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2) we get

$$\begin{aligned} y(t_{k+1}) - y_{k+1} &= \frac{1}{\Gamma(\alpha)}\left\{\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} D^\alpha_* y(t)\mathrm{d}t - \sum_{j=0}^{k+1} a_{j,k+1} D^\alpha_* y(t_j)\right. \\ &\quad + \sum_{j=0}^{k} a_{j,k+1} f_y(t_j, \eta_j)(y(t_j) - y_j) + a_{k+1,k+1} f_y(t_{k+1}, \eta_{k+1}) \\ &\quad \left. \cdot \frac{1}{\Gamma(\alpha)}\left\{\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} D^\alpha_* y(t)\mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} D^\alpha_* y(t_j) + \sum_{j=0}^{k} b_{j,k+1} f_y(t_j, \eta_j)(y(t_j) - y_j)\right\}\right\}. \end{aligned} \tag{3.3}$$

Now, we can rewrite (3.3) as follows:

$$\begin{aligned} y(t_{k+1}) - y_{k+1} &= \frac{1}{\Gamma(\alpha)}\left\{F_{k+1}[D^\alpha_* y(t)] + a_{k+1,k+1} f_y(t_{k+1}, \eta_{k+1})\frac{1}{\Gamma(\alpha)} E_{k+1}[D^\alpha_* y(t)]\right. \\ &\quad \left. + \sum_{j=0}^{k}\left[a_{j,k+1} + b_{j,k+1} a_{k+1,k+1} f_y(t_{k+1}, \eta_{k+1})\frac{1}{\Gamma(\alpha)}\right] f_y(t_j, \eta_j)(y(t_j) - y_j)\right\} \\ &= A_{k+1} + \sum_{j=0}^{k} B_{j,k+1}(y(t_j) - y_j), \end{aligned} \tag{3.4}$$

in which

$$E_{k+1}[D^\alpha_* y(t)] = \int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} D^\alpha_* y(t)\mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} D^\alpha_* y(t_j),$$

$$F_{k+1}[D^\alpha_* y(t)] = \int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} D^\alpha_* y(t)\mathrm{d}t - \sum_{j=0}^{k+1} a_{j,k+1} D^\alpha_* y(t_j),$$

$$A_{k+1} = \frac{1}{\Gamma(\alpha)}\left\{F_{k+1}[D^\alpha_* y(t)] + a_{k+1,k+1} f_y(t_{k+1}, \eta_{k+1})\frac{1}{\Gamma(\alpha)} E_{k+1}[D^\alpha_* y(t)]\right\},$$

and

$$B_{j,k+1} = \frac{1}{\Gamma(\alpha)}\left\{\left[a_{j,k+1} + b_{j,k+1} a_{k+1,k+1} f_y(t_{k+1}, \eta_{k+1})\frac{1}{\Gamma(\alpha)}\right] f_y(t_j, \eta_j)\right\}.$$

Next, we calculate $E_{k+1}[D^\alpha_* y(t)]$ and $F_{k+1}[D^\alpha_* y(t)]$. Set $g(t) = D^\alpha_* y(t)$. Since $D^\alpha_* y(t) \in C^r[0, T]$, we have

$$g(t) = g(0) + g'(0)t + \frac{g''(0)}{2!}t^2 + \cdots + \frac{g^{(r-1)}(0)}{(r-1)!}t^{r-1} + \frac{g^{(r)}(\xi)}{r!}t^r, \quad \xi \in (0, t).$$

By simple calculations, one gets

$$\begin{aligned} E_{k+1}[t^m] &= \int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} t^m \mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} t_j^m \\ &= \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1}-t)^{\alpha-1} (t^m - t_j^m) \mathrm{d}t \\ &= h^{m+\alpha} \sum_{j=0}^{k} \int_j^{j+1} (k+1-u)^{\alpha-1} (u^m - j^m) \mathrm{d}u \\ &= \overline{C}_{m,k} h^{m+\alpha}, \end{aligned}$$

using the substitution $t = hu$ and $\overline{C}_{m,k} = \sum_{j=0}^{k} \int_j^{j+1} (k+1-u)^{\alpha-1}(u^m - j^m)\mathrm{d}u$; thus,

$$\begin{aligned} E_{k+1}[g(t)] &= g'(0)\overline{C}_{1,k} h^{1+\alpha} + \frac{g''(0)}{2!}\overline{C}_{2,k} h^{2+\alpha} + \cdots + \frac{g^{(r-1)}(0)}{(r-1)!}\overline{C}_{r-1,k} h^{r-1+\alpha} + \frac{g^{(r)}(\xi)}{r!}\overline{C}_{r,k} h^{r+\alpha} \\ &= \sum_{j=1}^{r} \overline{\overline{C}}_{j,k} h^{j+\alpha}. \end{aligned} \tag{3.5}$$

Here, $\overline{\overline{C}}_{j,k} = \frac{g^{(j)}(0)}{j!}\overline{C}_{j,k} (j < r)$ and $\overline{\overline{C}}_{r,k} = \frac{g^{(r)}(\xi)}{r!}\overline{C}_{r,k}$.

$$\begin{aligned} F_{k+1}[t^m] &= \int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} t^m \mathrm{d}t - \sum_{j=0}^{k+1} a_{j,k+1} t_j^m \\ &= \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1}-t)^{\alpha-1} \left[t^m - t_j^m - \frac{1}{h}(t_{j+1}^m - t_j^m)(t-t_j) \right] \mathrm{d}t \\ &= h^{m+\alpha} \sum_{j=0}^{k} \int_j^{j+1} (k+1-u)^{\alpha-1} [u^m - j^m - ((j+1)^m - j^m)(u-j)] \mathrm{d}u \\ &= \overline{D}_{m,k} h^{m+\alpha}, \end{aligned}$$

where $t = hu$ and $\overline{D}_{m,k} = \sum_{j=0}^{k} \int_j^{j+1} (k+1-u)^{\alpha-1}[u^m - j^m - ((j+1)^m - j^m)(u-j)]\mathrm{d}u$; therefore,

$$F_{k+1}[g(t)] = \frac{g''(0)}{2!}\overline{D}_{2,k} h^{2+\alpha} + \cdots + \frac{g^{(r-1)}(0)}{(r-1)!}\overline{D}_{r-1,k} h^{r-1+\alpha} + \frac{g^{(r)}(\xi)}{r!}\overline{D}_{r,k} h^{r+\alpha} = \sum_{j=2}^{r} \overline{\overline{D}}_{j,k} h^{j+\alpha}. \tag{3.6}$$

Here, $\overline{\overline{D}}_{j,k} = \frac{g^{(j)}(0)}{j!}\overline{D}_{j,k} (j < r)$ and $\overline{\overline{D}}_{r,k} = \frac{g^{(r)}(\xi)}{r!}\overline{D}_{r,k}$.

From the expression for A_{k+1} (see (3.4)), (3.5) and (3.6), we obtain

$$\begin{aligned} A_{k+1} &= \frac{1}{\Gamma(\alpha)} \left\{ \sum_{j=2}^{r} \overline{\overline{D}}_{j,k} h^{j+\alpha} + a_{k+1,k+1} f_y(t_{k+1}, \eta_{k+1}) \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{r} \overline{\overline{C}}_{j,k} h^{j+\alpha} \right\} \\ &= \sum_{j=2}^{r} \frac{1}{\Gamma(\alpha)} \overline{\overline{D}}_{j,k} h^{j+\alpha} + \sum_{j=1}^{r} \frac{h^\alpha}{\Gamma(\alpha+2)} f_y(t_{k+1}, \eta_{k+1}) \frac{1}{\Gamma(\alpha)} \overline{\overline{C}}_{j,k} h^{j+\alpha} \\ &= \sum_{j=2}^{r} D_{j,k} h^{j+\alpha} + \sum_{j=1}^{r} C_{j,k} h^{j+2\alpha}, \end{aligned}$$

in which $C_{j,k} = \frac{1}{\Gamma(\alpha)\Gamma(\alpha+2)} f_y(t_{k+1}, \eta_{k+1}) \overline{\overline{C}}_{j,k}$ and $D_{j,k} = \frac{1}{\Gamma(\alpha)} \overline{\overline{D}}_{j,k}$.

Using the expression for $B_{j,k+1}$ (see (3.4)) gives

$$\begin{aligned} B_{j,k+1} &= \frac{1}{\Gamma(\alpha)} \left\{ \left[a_{j,k+1} + b_{j,k+1} a_{k+1,k+1} f_y(t_{k+1}, \eta_{k+1}) \frac{1}{\Gamma(\alpha)} \right] f_y(t_j, \eta_j) \right\} \\ &= e_{j,k+1} h^\alpha + f_{j,k+1} h^{2\alpha}, \end{aligned}$$

where

$$e_{j,k+1} = \frac{f_y(t_j, \eta_j)}{\Gamma(\alpha+2)} \begin{cases} (k^{\alpha+1} - (k-\alpha)(k+1)^\alpha) & \text{if } j = 0, \\ ((k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}) & \text{if } 1 \le j \le k, \end{cases}$$

and

$$f_{j,k+1} = \frac{f_y(t_{k+1}, \eta_{k+1}) f_y(t_j, \eta_j)}{\Gamma(\alpha+1)\Gamma(\alpha+2)}((k+1-j)^\alpha - (k-j)^\alpha).$$

Therefore,

$$\begin{aligned} y(t_{k+1}) - y_{k+1} &= A_{k+1} + \sum_{j=0}^{k} B_{j,k+1}(y(t_j) - y_j) \\ &= \sum_{j=2}^{r} D_{j,k} h^{j+\alpha} + \sum_{j=1}^{r} C_{j,k} h^{j+2\alpha} + \sum_{j=0}^{k} (e_{j,k+1} h^\alpha + f_{j,k+1} h^{2\alpha})(y(t_j) - y_j). \end{aligned} \tag{3.7}$$

Now, we prove the following equality (a compact form of (3.7)) by using mathematical induction:

$$y(t_{k+1}) - y_{k+1} = \sum_{j=1}^{r} h^{j+\alpha} \sum_{i=0}^{2k+1} c_{j,i,k} h^{i\alpha}. \tag{3.8}$$

The coefficients $c_{j,i,k}$ will be given later on.

When $k = 0$, from (3.7) we have

$$\begin{aligned} y(t_1) - y_1 &= \sum_{j=2}^{r} D_{j,0} h^{j+\alpha} + \sum_{j=1}^{r} C_{j,0} h^{j+2\alpha} \\ &= \sum_{j=1}^{r} h^{j+\alpha} \left\{ D_{j,0} + C_{j,0} h^{(2\times 0+1)\alpha} \right\} \\ &= \sum_{j=1}^{r} h^{j+\alpha} \sum_{i=0}^{2\times 0+1} c_{j,i,0} h^{i\alpha}. \end{aligned}$$

Here $D_{1,k} \equiv 0$, $(k = 0, 1, \ldots, N)$, $c_{j,0,0} = D_{j,0}$ and $c_{j,1,0} = C_{j,0}$ are used. So (3.8) holds for $k = 0$.

Assume that the formula (3.8) holds for $1 \le j \le k-1$; next we show that (3.8) is also satisfied for $j = k$. According to (3.7), one has

$$\begin{aligned} y(t_{k+1}) - y_{k+1} &= \sum_{l=2}^{r} D_{l,k} h^{l+\alpha} + \sum_{l=1}^{r} C_{l,k} h^{l+2\alpha} + \sum_{j=1}^{k} (e_{j,k+1} h^\alpha + f_{j,k+1} h^{2\alpha})(y(t_j) - y_j) \\ &= \sum_{l=2}^{r} D_{l,k} h^{l+\alpha} + \sum_{j=1}^{r} C_{j,k} h^{j+2\alpha} + \sum_{j=1}^{k} (e_{j,k+1} h^\alpha + f_{j,k+1} h^{2\alpha}) \left\{ \sum_{l=1}^{r} h^{l+\alpha} \sum_{i=0}^{2j-1} c_{l,i,j-1} h^{i\alpha} \right\} \\ &= \sum_{l=1}^{r} h^{l+\alpha} \left\{ D_{l,k} + C_{l,k} h^\alpha + \sum_{j=1}^{k} e_{j,k+1} \sum_{i=0}^{2j-1} c_{l,i,j-1} h^{(i+1)\alpha} + \sum_{j=1}^{k} f_{j,k+1} \sum_{i=0}^{2j-1} c_{l,i,j-1} h^{(i+2)\alpha} \right\}. \end{aligned}$$

The above formula can also be read as

$$\begin{aligned} y(t_{k+1}) - y_{k+1} = \sum_{l=1}^{r} h^{l+\alpha} \Bigg\{ & D_{l,k} + C_{l,k} h^\alpha + e_{1,k+1} \sum_{i=0}^{1} c_{l,i,0} h^{(i+1)\alpha} + e_{2,k+1} \sum_{i=0}^{3} c_{l,i,1} h^{(i+1)\alpha} + e_{3,k+1} \sum_{i=0}^{5} c_{l,i,2} h^{(i+1)\alpha} \\ & + \cdots + e_{k,k+1} \sum_{i=0}^{2k-1} c_{l,i,k-1} h^{(i+1)\alpha} + f_{1,k+1} \sum_{i=0}^{1} c_{l,i,0} h^{(i+2)\alpha} + f_{2,k+1} \sum_{i=0}^{3} c_{l,i,1} h^{(i+2)\alpha} \\ & + f_{3,k+1} \sum_{i=0}^{5} c_{l,i,2} h^{(i+2)\alpha} + f_{4,k+1} \sum_{i=0}^{7} c_{l,i,3} h^{(i+2)\alpha} + \cdots + f_{k,k+1} \sum_{i=0}^{2k-1} c_{l,i,k-1} h^{(i+2)\alpha} \Bigg\}. \end{aligned}$$

Then, we rewrite the above equation as

$$\begin{aligned}
y(t_{k+1}) - y_{k+1} = {} & \sum_{l=1}^{r} h^{l+\alpha} \Bigg\{ D_{l,k} + \left(C_{l,k} + \sum_{n=1}^{k} c_{l,0,n-1} e_{n,k+1} \right) h^{\alpha} + \left(\sum_{n=1}^{k} e_{n,k+1} c_{l,1,n-1} + \sum_{n=1}^{k} f_{n,k+1} c_{l,0,n-1} \right) h^{2\alpha} \\
& + \left(\sum_{n=2}^{k} e_{n,k+1} c_{l,2,n-1} + \sum_{n=1}^{k} f_{n,k+1} c_{l,1,n-1} \right) h^{3\alpha} + \left(\sum_{n=2}^{k} e_{n,k+1} c_{l,3,n-1} + \sum_{n=2}^{k} f_{n,k+1} c_{l,2,n-1} \right) h^{4\alpha} \\
& + \left(\sum_{n=3}^{k} e_{n,k+1} c_{l,4,n-1} + \sum_{n=2}^{k} f_{n,k+1} c_{l,3,n-1} \right) h^{5\alpha} + \cdots \\
& + \begin{cases} \left(\displaystyle\sum_{n=\frac{i+1}{2}}^{k} e_{n,k+1} c_{l,i-1,n-1} + \sum_{n=\frac{i-1}{2}}^{k} f_{n,k+1} c_{l,i-2,n-1} \right) h^{i\alpha} & \text{if } i \text{ odd} \\ \left(\displaystyle\sum_{n=\frac{i}{2}}^{k} e_{n,k+1} c_{l,i-1,n-1} + \sum_{n=\frac{i}{2}}^{k} f_{n,k+1} c_{l,i-2,n-1} \right) h^{i\alpha} & \text{if } i \text{ even} \end{cases} \\
& + \cdots + f_{k,k+1} c_{l,2k-1,k-1} h^{(2k+1)\alpha} \Bigg\} \\
= {} & \sum_{j=1}^{r} h^{j+\alpha} \sum_{i=0}^{2k+1} c_{j,i,k} h^{i\alpha}
\end{aligned} \tag{3.9}$$

in which $c_{j,0,k} = D_{j,k}$, $c_{j,1,k} = C_{j,k} + \sum_{n=1}^{k} c_{j,0,n-1} e_{n,k+1}$,

$$c_{j,i,k} = \begin{cases} \displaystyle\sum_{n=\frac{i+1}{2}}^{k} e_{n,k+1} c_{j,i-1,n-1} + \sum_{n=\frac{i-1}{2}}^{k} f_{n,k+1} c_{j,i-2,n-1} & \text{if } i = 3, 5, \ldots, 2k-1, \\ \displaystyle\sum_{n=\frac{i}{2}}^{k} e_{n,k+1} c_{j,i-1,n-1} + \sum_{n=\frac{i}{2}}^{k} f_{n,k+1} c_{j,i-2,n-1} & \text{if } i = 2, 4, \ldots, 2k, \end{cases}$$

and $c_{j,2k+1,k} = f_{k,k+1} c_{j,2k-1,k-1}$.

It immediately follows from (3.9) that (3.8) does hold for the case $j = k$. This completes the proof. □

From $c_{1,0,T/h} = D_{1,T/h} = 0$, we know that (3.8) includes $h^{2+\alpha}, h^{3+\alpha}, \ldots, h^{r+\alpha}$; $h^{1+2\alpha}, h^{1+3\alpha}, \ldots, h^{1+(1+2T/h)\alpha}$; …; $h^{r+2\alpha}, h^{r+3\alpha}, \ldots, h^{r+(1+2T/h)\alpha}$. So the expansion begins with an $h^{1+2\alpha}$ term for $\alpha \in (0, 1)$ whilst it begins with an $h^{2+\alpha}$ term for $\alpha > 1$.

In the following, we study the case with $\alpha \in (0, 1)$ and $f \in C^3(G)$ (see DFF Theorem (d) and compare them).

We first set

$$E_{k+1}[g(t)] = \left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} g(t) \mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} g(t_j) \right|,$$

and

$$F_{k+1}[g(t)] = \left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} g(t) \mathrm{d}t - \sum_{j=0}^{k+1} a_{j,k+1} g(t_j) \right|.$$

Theorem 3.2. *Assume that $0 < \alpha < 1$ and $f \in C^3(G)$; then we have*

$$\max_{0 \le j \le N} |y(t_j) - y_j| = \begin{cases} O(h^{2\alpha}) & \text{if } 0 < \alpha < 0.5, \\ O(h) & \text{if } 0.5 \le \alpha < 1; \end{cases}$$

and

$$\max_{t_j \in [\epsilon, T]} |y(t_j) - y_j| = \begin{cases} O(h^{1+\alpha}) & \text{if } 0 < \alpha < 0.5, \\ O(h^{2-\alpha}) & \text{if } 0.5 \le \alpha < 1, \end{cases}$$

for every $\epsilon > 0$.

Proof. Following Theorem 2.1(b), there exists a function $\psi \in C^2[0,T]$, and some $c_1, c_2, \ldots, c_{\nu_1} \in \mathbf{R}$ and $d_1, d_2, \ldots, d_{\nu_2} \in \mathbf{R}$ such that the solution y of (1.1) has the following form:

$$y(t) = \psi(t) + \sum_{\nu=1}^{\nu_1} c_\nu t^{\nu\alpha} + \sum_{\nu=1}^{\nu_2} d_\nu t^{1+\nu\alpha}$$

where $\nu_1 = \lceil 2/\alpha \rceil - 1$ and $\nu_2 = \lceil 1/\alpha \rceil - 1$.

Hence, by simple calculations we have

$$D_*^\alpha y(t) = D_*^\alpha \psi(t) + \sum_{\nu=1}^{\nu_1} \frac{\Gamma(\nu\alpha+1)}{\Gamma(\nu\alpha+1-\alpha)} c_\nu t^{\nu\alpha-\alpha} + \sum_{\nu=1}^{\nu_2} \frac{\Gamma(\nu\alpha+2)}{\Gamma(\nu\alpha+2-\alpha)} d_\nu t^{1+\nu\alpha-\alpha}. \tag{3.10}$$

Since $\psi(t) \in C^2[0,T]$, $0 < \alpha < 1$, following from Theorem 2.2, we have

$$D_*^\alpha \psi(t) = \frac{\psi'(0)}{\Gamma(2-\alpha)} t^{1-\alpha} + g(t), \tag{3.11}$$

with $g \in C^1[0,T]$ and function g' satisfies a Lipschitz condition of order $(1-\alpha) \in (0,1)$.

According to Theorem 2.4(b), there exist positive constants $B^{\mathrm{Tr}}_{\alpha,1-\alpha}$ and $M(g, 1-\alpha)$ such that

$$F_{k+1}[g(t)] \le B^{\mathrm{Tr}}_{\alpha,1-\alpha} M(g,1-\alpha) t_{k+1}^\alpha h^{2-\alpha}.$$

Now, applying DFF theorem (c), we get

$$F_{k+1}\left[\frac{\psi'(0)}{\Gamma(2-\alpha)} t^{1-\alpha}\right] \le C^{\mathrm{Tr}}_{\alpha,1-\alpha} t_{k+1}^{\alpha-1} h^{2-\alpha}.$$

Thus, from (3.11) we have

$$F_{k+1}[D_*^\alpha \psi(t)] \le 2\max\{B^{\mathrm{Tr}}_{\alpha,1-\alpha} M(g,1-\alpha) t_{k+1}^\alpha h^{2-\alpha}, C^{\mathrm{Tr}}_{\alpha,1-\alpha} t_{k+1}^{\alpha-1} h^{2-\alpha}\} \le C_1^{\mathrm{Tr}} t_{k+1}^{\alpha-1} h^{2-\alpha}. \tag{3.12}$$

Also by Theorem 2.4(c), one has

$$\begin{aligned} F_{k+1}\left[\sum_{\nu=1}^{\nu_1} \frac{\Gamma(\alpha\nu+1)}{\Gamma(\alpha\nu+1-\alpha)} c_\nu t^{\alpha\nu-\alpha}\right] &\le \sum_{\nu=2}^{\nu_1} \frac{\Gamma(\alpha\nu+1)}{\Gamma(\alpha\nu+1-\alpha)} |c_\nu| F_{k+1}[t^{\alpha\nu-\alpha}] \\ &\le \sum_{\nu=2}^{\nu_1} \frac{\Gamma(\alpha\nu+1)}{\Gamma(\alpha\nu+1-\alpha)} |c_\nu| C^{\mathrm{Tr}}_{\alpha,\alpha\nu-\alpha} t_{k+1}^{\alpha+\alpha\nu-\alpha-\sigma_\nu} h^{\sigma_\nu} \\ &\le C_2^{\mathrm{Tr}} t_{k+1}^{\alpha-1} h^{\alpha+1}, \end{aligned} \tag{3.13}$$

where $\sigma_\nu = \min(2, \alpha\nu - \alpha + 1)$. Similarly,

$$\begin{aligned} F_{k+1}\left[\sum_{\nu=1}^{\nu_2} \frac{\Gamma(\nu\alpha+2)}{\Gamma(\nu\alpha+2-\alpha)} d_\nu t^{1+\nu\alpha-\alpha}\right] &\le \sum_{\nu=1}^{\nu_2} \frac{\Gamma(\nu\alpha+2)}{\Gamma(\nu\alpha+2-\alpha)} |d_\nu| F_{k+1}[t^{1+\alpha\nu-\alpha}] \\ &\le \sum_{\nu=1}^{\nu_2} \frac{\Gamma(\nu\alpha+2)}{\Gamma(\nu\alpha+2-\alpha)} |d_\nu| C^{\mathrm{Tr}}_{\alpha,1+\alpha\nu-\alpha,\nu} t_{k+1}^{\alpha-1} h^2 \\ &\le C_3^{\mathrm{Tr}} t_{k+1}^{\alpha-1} h^2. \end{aligned} \tag{3.14}$$

From (3.12) to (3.14), it follows that

$$\begin{aligned} F_{k+1}[D_*^\alpha y(t)] &\le 3\max\{C_1^{\mathrm{Tr}} t_{k+1}^{\alpha-1} h^{2-\alpha}, C_2^{Tr} t_{k+1}^{\alpha-1} h^{\alpha+1}, C_3^{Tr} t_{k+1}^{\alpha-1} h^2\} \\ &\le C_2 t_{k+1}^{\alpha-1} \begin{cases} h^{\alpha+1} & \text{if } 0 < \alpha < 0.5, \\ h^{2-\alpha} & \text{if } 0.5 \le \alpha < 1. \end{cases} \end{aligned}$$

Next, we estimate $E_{k+1}[D_*^\alpha y(t)]$. By the same reasoning, one has

$$E_{k+1}[g(t)] \le \frac{1}{\alpha} \|g'\|_\infty t_{k+1}^\alpha h, \tag{3.15}$$

and

$$\begin{aligned} E_{k+1}\left[\sum_{\nu=1}^{\nu_2} \frac{\Gamma(\nu\alpha+2)}{\Gamma(\nu\alpha+2-\alpha)} d_\nu t^{1+\nu\alpha-\alpha}\right] &\le \sum_{\nu=1}^{\nu_2} \frac{\Gamma(\nu\alpha+2)}{\Gamma(\nu\alpha+2-\alpha)} |d_\nu| E_{k+1}[t^{1+\alpha\nu-\alpha}] \\ &\le C_1^{\mathrm{Re}} t_{k+1}^\alpha h. \end{aligned} \tag{3.16}$$

From Theorem 2.3(b), one gets

$$E_{k+1}\left[\frac{\psi'(0)}{\Gamma(2-\alpha)}t^{1-\alpha}\right] \le C_2^{\mathrm{Re}}h, \tag{3.17}$$

and

$$\begin{aligned} E_{k+1}\left[\sum_{\nu=1}^{\nu_1}\frac{\Gamma(\alpha\nu+1)}{\Gamma(\alpha\nu+1-\alpha)}c_\nu t^{\alpha\nu-\alpha}\right] &\le \sum_{\nu=1}^{\nu_1}\frac{\Gamma(\alpha\nu+1)}{\Gamma(\alpha\nu+1-\alpha)}|c_\nu|E_{k+1}[t^{\alpha\nu-\alpha}] \\ &\le \sum_{\nu=1}^{\nu_1}\frac{\Gamma(\alpha\nu+1)}{\Gamma(\alpha\nu+1-\alpha)}|c_\nu|C_{\alpha\nu-\alpha}^{\mathrm{Re}}t_{k+1}^{\alpha+\alpha\nu-\alpha-1}h \\ &\le C_3^{\mathrm{Re}}t_{k+1}^{\alpha-1}h. \end{aligned} \tag{3.18}$$

Combining (3.10) and (3.15)–(3.18) leads to

$$\begin{aligned} E_{k+1}[D_*^\alpha y(t)] &\le 4\max\left\{\frac{1}{\alpha}\|g'\|_\infty t_{k+1}^\alpha h, C_1^{\mathrm{Re}}t_{k+1}^\alpha h, C_2^{\mathrm{Re}}h, C_3^{\mathrm{Re}}t_{k+1}^{\alpha-1}h\right\} \\ &\le C_1 t_{k+1}^{\alpha-1}h. \end{aligned}$$

Thus, we have

$$E_{k+1}[D_*^\alpha y(t)] \le C_1 t_{k+1}^{\alpha-1}h,$$

and

$$F_{k+1}[D_*^\alpha y(t)] \le C_2 t_{k+1}^{\alpha-1}\begin{cases} h^{\alpha+1} & \text{if } 0<\alpha<0.5 \\ h^{2-\alpha} & \text{if } 0.5\le\alpha<1 \end{cases} = C_2 t_{k+1}^{\alpha-1}h^\kappa,$$

in which $\kappa = \min(1+\alpha, 2-\alpha)$.

Next, we show the following formula holds for sufficiently small h:

$$|y(t_j)-y_j| \le C t_j^{\alpha-1}h^q, \quad (q=\min(1+\alpha,2-\alpha)) \tag{3.19}$$

for all $j\in\{0,1,\dots,N\}$, where C is a suitable constant.

Here, we again use mathematical induction to show that (3.19) holds. In view of the given initial condition, the induction basis ($j=0$) is pre-assumed. Suppose that (3.19) is true for $j=0,1,\dots,k$ for $k\le N-1$. Now we prove that the inequality also holds for $j=k+1$. To do this, we first look at the error of the predictor y_{k+1}^P. By construction of the predictor we find that

$$\begin{aligned} |y(t_{k+1})-y_{k+1}^P| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_{k+1}}(t_{k+1}-t)^{\alpha-1}f(t,y(t)) - \sum_{j=0}^k b_{j,k+1}f(t_j,y_j)\right| \\ &\le \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_{k+1}}(t_{k+1}-t)^{\alpha-1}D_*^\alpha y(t)\mathrm{d}t - \sum_{j=0}^k b_{j,k+1}D_*^\alpha y(t_j)\right| + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^k b_{j,k+1}|f(t_j,y(t_j)-f(t_j,y_j))| \\ &\le \frac{C_1t_{k+1}^{\alpha-1}}{\Gamma(\alpha)}h + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^k b_{j,k+1}L|y(t_j)-y_j| \le \frac{C_1t_{k+1}^{\alpha-1}}{\Gamma(\alpha)}h + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^k b_{j,k+1}LCt_j^{\alpha-1}h^q. \end{aligned} \tag{3.20}$$

On the basis of the bound (3.20) for the predictor error we begin to determine the corrector error. For $j=k+1$,

$$\begin{aligned} |y(t_{k+1})-y_{k+1}| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_{k+1}}(t_{k+1}-t)^{\alpha-1}f(t,y(t)) - \sum_{j=0}^k a_{j,k+1}f(t_j,y_j) - a_{k+1,k+1}f(t_{k+1},y_{k+1}^P)\right| \\ &\le \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_{k+1}}(t_{k+1}-t)^{\alpha-1}D_*^\alpha y(t)\mathrm{d}t - \sum_{j=0}^{k+1} a_{j,k+1}D_*^\alpha y(t_j)\right| \\ &\quad + \sum_{j=0}^k a_{j,k+1}|f(t_j,y(t_j)-f(t_j,y_j))| + \frac{1}{\Gamma(\alpha)}a_{k+1,k+1}|f(t_{k+1},y(t_{k+1})-f(t_{k+1},y_{k+1}^P))| \\ &\le \frac{C_2t_{k+1}^{\alpha-1}}{\Gamma(\alpha)}h^\kappa + \frac{CL}{\Gamma(\alpha)}h^q\sum_{j=1}^k a_{j,k+1}t_j^{\alpha-1} + a_{k+1,k+1}\frac{L}{\Gamma(\alpha)}\left(\frac{C_1t_{k+1}^{\alpha-1}}{\Gamma(\alpha)}h + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^k b_{j,k+1}LCt_j^{\alpha-1}h^q\right) \end{aligned}$$

$$= \frac{C_2 t_{k+1}^{\alpha-1}}{\Gamma(\alpha)} h^{\kappa} + a_{k+1,k+1} \frac{C_1 L}{\Gamma^2(\alpha)} t_{k+1}^{\alpha-1} h$$
$$+ \frac{CL}{\Gamma(\alpha)} h^q \sum_{j=1}^{k} a_{j,k+1} t_j^{\alpha-1} + a_{k+1,k+1} \frac{CL^2}{\Gamma^2(\alpha)} h^q \sum_{j=1}^{k} b_{j,k+1} t_j^{\alpha-1}. \quad (3.21)$$

We now estimate terms of form $\sum_{j=1}^{k-1} a_{j,k+1} t_j^{\alpha-1}$ and $\sum_{j=1}^{k-1} b_{j,k+1} t_j^{\alpha-1}$. By the Mean Value Theorem again we have

$$\begin{aligned} 0 \le a_{j,k+1} &= \frac{h^\alpha}{\alpha(\alpha+1)}((k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}) \\ &\le ch^\alpha (k-j)^{\alpha-1}, \\ 0 \le b_{j,k+1} &= \frac{h^\alpha}{\alpha}((k+1-j)^\alpha - (k-j)^\alpha) \le h^\alpha (k-j)^{\alpha-1} \end{aligned}$$

for $1 \le j \le k-1$, where the constant c is independent of j and k, respectively, so the problem reduces to finding a bound for $S_k = h^{\alpha+\alpha-1} \sum_{j=1}^{k-1} j^{\alpha-1}(k-j)^{\alpha-1}$.

$$\begin{aligned} S_k &\le h^{\alpha+\alpha-1} \int_0^k x^{\alpha-1}(k-x)^{\alpha-1} \mathrm{d}x = h^{\alpha+\alpha-1} k^{\alpha+\alpha-1} \int_0^1 t^{\alpha-1}(1-t)^{\alpha-1} \mathrm{d}t \\ &= B(\alpha,\alpha) t_k^{\alpha+\alpha-1} \le 2B(\alpha,\alpha) t_{k+1}^{\alpha+\alpha-1} = B t_{k+1}^{\alpha+\alpha-1} \le BT^\alpha t_{k+1}^{\alpha-1}. \end{aligned} \quad (3.22)$$

It immediately follows from (3.21) and (3.22) that

$$\begin{aligned} |y(t_{k+1}) - y_{k+1}| &\le \frac{C_2 t_{k+1}^{\alpha-1}}{\Gamma(\alpha)} h^\kappa + a_{k+1,k+1} \frac{C_1 L}{\Gamma^2(\alpha)} t_{k+1}^{\alpha-1} h + \frac{CL}{\Gamma(\alpha)} h^q \sum_{j=1}^{k} a_{j,k+1} t_j^{\alpha-1} + a_{k+1,k+1} \frac{CL^2}{\Gamma^2(\alpha)} h^q \sum_{j=1}^{k} b_{j,k+1} t_j^{\alpha-1} \\ &\le \left\{ \frac{C_2}{\Gamma(\alpha)} h^{\kappa-q} + \frac{C_1 L}{\Gamma(\alpha)\Gamma(\alpha+2)} h^{1+\alpha-q} + \frac{CL}{\Gamma(\alpha)} \left(cBT^\alpha + 2\frac{2^\alpha h^\alpha}{\alpha(\alpha+1)} \right) \right. \\ &\quad \left. + \frac{CL^2 h^\alpha}{\Gamma(\alpha)\Gamma(\alpha+2)} \left(BT^\alpha + \frac{2^{\alpha+1} h^\alpha}{\alpha} \right) \right\} t_{k+1}^{\alpha-1} h^q \le C t_{k+1}^{\alpha-1} h^q, \end{aligned}$$

where $a_{k+1,k+1} = \frac{h^\alpha}{\alpha(\alpha+1)}$, $a_{k,k+1} \le \frac{2^{\alpha+1} h^\alpha}{\alpha(\alpha+1)}$, $b_{k,k+1} = \frac{h^\alpha}{\alpha}$, $t_{k+1} \le T$ and $(\frac{k}{k+1})^{\alpha-1} \le 2$ are used. Therefore, this completes (3.19).

Furthermore, we deduce

$$\max_{0 \le j \le N} |y(t_j) - y_j| = \begin{cases} O(h^{2\alpha}) & \text{if } 0 < \alpha < 0.5, \\ O(h) & \text{if } 0.5 \le \alpha < 1; \end{cases}$$

and for arbitrary $\epsilon > 0$ then with $t_j \in [\epsilon, T]$,

$$|y(t_j) - y_j| \le C\epsilon^{\alpha-1} \begin{cases} h^{\alpha+1} & \text{if } 0 < \alpha < 0.5, \\ h^{2-\alpha} & \text{if } 0.5 \le \alpha < 1; \end{cases}$$

or

$$\max_{t_j \in [\epsilon, T]} |y(t_j) - y_j| = \begin{cases} O(h^{1+\alpha}) & \text{if } 0 < \alpha < 0.5, \\ O(h^{2-\alpha}) & \text{if } 0.5 \le \alpha < 1. \end{cases}$$

The proof is thus finished. □

From Theorem 2.1(b), the condition of this theorem implies $y(t) \in C[0,T]$ but $y(t)$ does not lie in $C^1[0,T]$, let alone $y(t) \notin C^2[0,T]$. Compared to DFF Theorem (c), the condition here is weaker but the same results hold.

In the following, we give a lemma.

Lemma 3.3. *If* $\psi \in C^1[0,T]$, $0 < \alpha < 1$, *then*

$$E_{k+1}\left[\int_0^t (t-\tau)^{-\alpha} \psi'(\tau) \mathrm{d}\tau \right] \le \frac{2}{\alpha(1-\alpha)} \|\psi'\| t_{k+1}^\alpha h^{1-\alpha},$$

and

$$F_{k+1}\left[\int_0^t (t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right] \le \frac{A}{1-\alpha}\|\psi'\|_\infty t_{k+1}^{\alpha} h^{1-\alpha},$$

where A is a constant independent of k and h.

Proof. Using the expressions for E_{k+1}, we have

$$\begin{aligned} E_{k+1}\left[\int_0^t (t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right] &= \left|\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}\int_0^t (t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1}\int_0^{t_j}(t_j-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right| \\ &= \left|\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}\left\{\int_0^t (t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau - \int_0^{t_j}(t_j-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right\}(t_{k+1}-t)^{\alpha-1}\mathrm{d}t\right| \\ &\le \sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}\left|\int_0^t (t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau - \int_0^{t_j}(t_j-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right|(t_{k+1}-t)^{\alpha-1}\mathrm{d}t. \end{aligned}$$

$$\begin{aligned} &\left|\int_0^t (t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau - \int_0^{t_j}(t_j-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right| \\ &\quad\le \|\psi'\|_\infty\left(\int_0^{t_j}(t_j-\tau)^{-\alpha}\mathrm{d}\tau - \int_0^{t_j}(t-\tau)^{-\alpha}\mathrm{d}\tau + \int_{t_j}^{t}(t-\tau)^{-\alpha}\mathrm{d}\tau\right) \\ &\quad\le \frac{1}{1-\alpha}\|\psi'\|_\infty\left(t_j^{1-\alpha} + (t-t_j)^{1-\alpha} - t^{1-\alpha} + (t-t_j)^{1-\alpha}\right) \\ &\quad\le \frac{2}{1-\alpha}\|\psi'\|_\infty (t-t_j)^{1-\alpha} \le \frac{2}{1-\alpha}\|\psi'\|_\infty h^{1-\alpha}. \end{aligned}$$

So, one has

$$\begin{aligned} E_{k+1}\left[\int_0^t (t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right] &\le \frac{2}{1-\alpha}\|\psi'\|_\infty h^{1-\alpha}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-t)^{\alpha-1}\mathrm{d}t \\ &= \frac{2}{1-\alpha}\|\psi'\|_\infty h^{1-\alpha}\int_0^{t_{k+1}}(t_{k+1}-t)^{\alpha-1}\mathrm{d}t = \frac{2}{\alpha(1-\alpha)}\|\psi'\|_\infty t_{k+1}^{\alpha}h^{1-\alpha}. \end{aligned}$$

Applying the expression for F_{k+1} gives

$$\begin{aligned} &F_{k+1}\left[\int_0^t (t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right] = \left|\int_0^{t_{k+1}}(t_{k+1}-t)^{\alpha-1}\int_0^t (t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\mathrm{d}t - \sum_{j=0}^{k+1} a_{j,k+1}\int_0^{t_j}(t_j-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right| \\ &= \left|\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-t)^{\alpha-1}\left\{\int_0^t (t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right.\right. \\ &\quad \left.\left. - \int_0^{t_j}(t_j-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau - \frac{1}{h}\left[\int_0^{t_{j+1}}(t_{j+1}-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau - \int_0^{t_j}(t_j-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right](t-t_j)\right\}\mathrm{d}t\right| \\ &\le \sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-t)^{\alpha-1}\left|\int_0^t (t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right. \\ &\quad \left. - \int_0^{t_j}(t_j-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau - \frac{1}{h}\left[\int_0^{t_{j+1}}(t_{j+1}-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau - \int_0^{t_j}(t_j-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right](t-t_j)\right|\mathrm{d}t. \end{aligned}$$

By tedious calculations, one gets

$$\begin{aligned} &\left|\int_0^t (t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau - \int_0^{t_j}(t_j-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right. \\ &\quad \left. - \frac{1}{h}\left[\int_0^{t_{j+1}}(t_{j+1}-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau - \int_0^{t_j}(t_j-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right](t-t_j)\right| \\ &\le \left|\int_0^{t_j}\left[(t-\tau)^{-\alpha} - (t_j-\tau)^{-\alpha} - \frac{1}{h}((t_{j+1}-\tau)^{-\alpha} - (t_j-\tau)^{-\alpha})(t-t_j)\right]\psi'(\tau)\mathrm{d}\tau\right| \end{aligned}$$

$$
\begin{aligned}
&+\left|\int_{t_j}^{t}\left[(t-\tau)^{-\alpha}-\frac{1}{h}(t_{j+1}-\tau)^{-\alpha}(t-t_j)\right]\psi'(\tau)\mathrm{d}\tau\right|+\left|\frac{1}{h}\int_{t}^{t_{j+1}}(t_{j+1}-\tau)^{-\alpha}(t-t_j)\psi'(\tau)\mathrm{d}\tau\right|\\
&\le\frac{1}{1-\alpha}\|\psi'\|_\infty\left\{\left|-(t-t_j)^{1-\alpha}+t^{1-\alpha}-t_j^{1-\alpha}-\frac{1}{h}(t-t_j)[-(t_{j+1}-t_j)^{1-\alpha}+t_{j+1}^{1-\alpha}-t_j^{1-\alpha}]\right|\right.\\
&\left.+\left|(t-t_j)^{1-\alpha}-\frac{1}{h}(t-t_j)[-(t_{j+1}-t)^{1-\alpha}+(t_{j+1}-t_j)^{1-\alpha}]\right|+\frac{1}{h}(t_{j+1}-t)^{1-\alpha}(t-t_j)\right\}\\
&\le\frac{1}{1-\alpha}\|\psi'\|_\infty\left\{t^{1-\alpha}-t_j^{1-\alpha}-\frac{1}{h}(t_{j+1}^{1-\alpha}-t_j^{1-\alpha})(t-t_j)\right.\\
&\left.+((t-t_j)^{1-\alpha}-h^{-\alpha}(t-t_j))+(t-t_j)^{1-\alpha}-h^{-\alpha}(t-t_j)+\frac{2}{h}(t_{j+1}-t)^{1-\alpha}(t-t_j)\right\}\\
&=\frac{1}{1-\alpha}\|\psi'\|_\infty\left\{t^{1-\alpha}-t_j^{1-\alpha}-\frac{1}{h}(t_{j+1}^{1-\alpha}-t_j^{1-\alpha})(t-t_j)\right.\\
&\left.+2[h^{-1}(t_{j+1}-t)^{1-\alpha}(t-t_j)+(t-t_j)^{1-\alpha}-h^{-\alpha}(t-t_j)]\right\}\\
&\le\frac{1}{1-\alpha}\|\psi'\|_\infty\left\{t^{1-\alpha}-t_j^{1-\alpha}-\frac{1}{h}(t_{j+1}^{1-\alpha}-t_j^{1-\alpha})(t-t_j)+4h^{1-\alpha}\right\}.
\end{aligned}
$$

Utilizing the above inequalities and Theorem 2.4(c), we have

$$
\begin{aligned}
F_{k+1}\left[\int_0^t(t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right]&\le\frac{1}{1-\alpha}\|\psi'\|_\infty\left\{F_{k+1}[t^{1-\alpha}]+4h^{1-\alpha}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-t)^{\alpha-1}\mathrm{d}t\right\}\\
&\le\frac{1}{1-\alpha}\|\psi'\|_\infty\left\{C^{\mathrm{Tr}}_{\alpha,1-\alpha}t_{k+1}^{\alpha-1}h^{2-\alpha}+\frac{4}{\alpha}t_{k+1}^{\alpha}h^{1-\alpha}\right\}\le\frac{A}{1-\alpha}\|\psi'\|_\infty t_{k+1}^{\alpha}h^{1-\alpha}.
\end{aligned}
$$

The proof is completed. □

If we weaken the smooth condition of f in Theorem 3.2, we get the following results.

Theorem 3.4. *Let* $0<\alpha<1$. *Then, if* $f\in C^2(G)$,

$$\max_{t_j\in[\epsilon,T]}|y(t_j)-y_j|=O(h^{1-\alpha}),$$

for every $\epsilon>0$, *and*

$$\max_{0\le j\le N}|y(t_j)-y_j|=O(1).$$

Proof. Since $f\in C^2(G)$, using Theorem 2.1(a) we have

$$y(t)=\sum_{\nu=1}^{\nu_1}c_\nu t^{\alpha\nu}+\psi(t),$$

in which $\nu_1=\lceil 1/\alpha\rceil-1$, $\psi(t)\in C^1[0,T]$.

Now, using the Caputo fractional derivative definition yields

$$D_*^\alpha y(t)=\sum_{\nu=1}^{\nu_1}\frac{\Gamma(\alpha\nu+1)}{\Gamma(\alpha\nu+1-\alpha)}c_\nu t^{\alpha\nu-\alpha}+\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau. \tag{3.23}$$

Set $g(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau$. Next we estimate $E_{k+1}[g(t)]$ and $F_{k+1}[g(t)]$.

$$
\begin{aligned}
E_{k+1}[g(t)]&=\frac{1}{\Gamma(1-\alpha)}E_{k+1}\left[\int_0^t(t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right]\\
&\le\frac{1}{\Gamma(1-\alpha)}\frac{2}{\alpha(1-\alpha)}\|\psi'\|t_{k+1}^{\alpha}h^{1-\alpha}\\
&=\frac{2}{\alpha\Gamma(2-\alpha)}\|\psi'\|t_{k+1}^{\alpha}h^{1-\alpha}.
\end{aligned}\tag{3.24}
$$

Also,

$$\begin{aligned} F_{k+1}[g(t)] &= \frac{1}{\Gamma(1-\alpha)} F_{k+1}\left[\int_0^t (t-\tau)^{-\alpha}\psi'(\tau)\mathrm{d}\tau\right] \\ &\le \frac{A}{\Gamma(2-\alpha)}\|\psi'\|_\infty t_{k+1}^{\alpha} h^{1-\alpha} \\ &\le \frac{A}{\Gamma(2-\alpha)}\|\psi'\|_\infty t_{k+1}^{\alpha} h^{1-\alpha}. \end{aligned} \tag{3.25}$$

From Theorem 2.4(b) we have

$$\begin{aligned} E_{k+1}\left[\sum_{\nu=1}^{\nu_1} \frac{\Gamma(\alpha\nu+1)}{\Gamma(\alpha\nu+1-\alpha)} c_\nu t^{\alpha\nu-\alpha}\right] &\le \sum_{\nu=1}^{\nu_1} \frac{\Gamma(\alpha\nu+1)}{\Gamma(\alpha\nu+1-\alpha)} |c_\nu| E_{k+1}[t^{\alpha\nu-\alpha}] \\ &\le \sum_{\nu=1}^{\nu_1} \frac{\Gamma(\alpha\nu+1)}{\Gamma(\alpha\nu+1-\alpha)} |c_\nu| C^{\mathrm{Re}}_{\alpha,\alpha\nu-\alpha} t_{k+1}^{\alpha+\alpha\nu-\alpha-1} h \\ &\le C^{\mathrm{Re}} t_{k+1}^{\alpha-1} h, \end{aligned} \tag{3.26}$$

and from Theorem 2.4(c) we have

$$\begin{aligned} F_{k+1}\left[\sum_{\nu=1}^{\nu_1} \frac{\Gamma(\alpha\nu+1)}{\Gamma(\alpha\nu+1-\alpha)} c_\nu t^{\alpha\nu-\alpha}\right] &\le \sum_{\nu=2}^{\nu_1} \frac{\Gamma(\alpha\nu+1)}{\Gamma(\alpha\nu+1-\alpha)} |c_\nu| F_{k+1}[t^{\alpha\nu-\alpha}] \\ &\le \sum_{\nu=2}^{\nu_1} \frac{\Gamma(\alpha\nu+1)}{\Gamma(\alpha\nu+1-\alpha)} |c_\nu| C^{\mathrm{Tr}}_{\alpha,\alpha\nu-\alpha} t_{k+1}^{\alpha+\alpha\nu-\alpha-\sigma_\nu} h^{\sigma_\nu} \\ &\le C^{\mathrm{Tr}} t_{k+1}^{\alpha-1} h^{\alpha+1}, \end{aligned} \tag{3.27}$$

where $\sigma_\nu = \min(2, \alpha\nu - \alpha + 1)$.

Thus, from (3.23), (3.24) and (3.26) we get

$$\begin{aligned} E_{k+1}[D_*^\alpha y(t)] &\le 2\max\left\{\frac{2}{\alpha\Gamma(2-\alpha)}\|\psi'\| t_{k+1}^{\alpha} h^{1-\alpha}, C^{\mathrm{Re}} t_{k+1}^{\alpha-1} h\right\} \\ &\le C_1 t_{k+1}^{\alpha-1} h^{1-\alpha}, \end{aligned} \tag{3.28}$$

and from (3.23), (3.25) and (3.27) we also get

$$\begin{aligned} F_{k+1}[D_*^\alpha y(t)] &\le 2\max\left\{\frac{A}{\Gamma(2-\alpha)}\|\psi'\| t_{k+1}^{\alpha} h^{1-\alpha}, C^{\mathrm{Tr}} t_{k+1}^{\alpha-1} h^{\alpha+1}\right\} \\ &\le C_2 t_{k+1}^{\alpha-1} h^{1-\alpha}. \end{aligned} \tag{3.29}$$

Now, by using (3.28) and (3.29) and by almost the same reasoning as for Theorem 3.2, we obtain

$$|y(t_j) - y_j| \le C t_j^{\alpha-1} h^q, \quad (q = \min(1, 1-\alpha)).$$

So, if $\epsilon \in (0, T]$ and $t_j \in [\epsilon, T]$, then

$$|y(t_j) - y_j| \le C t_j^{\alpha-1} h^{1-\alpha} \le C\epsilon^{\alpha-1} h^q,$$

or

$$\max_{t_j\in[\epsilon,T]} |y(t_j) - y_j| = O(h^{1-\alpha}),$$

and

$$\max_{0\le j\le N} |y(t_j) - y_j| = O(1).$$

The proof is completed. □

According to Theorems 2.1, 3.2 and 3.4, the smooth condition of $\psi(t)$ plays an important role in error estimates. If this term is removed, what will happen? We will give an answer.

Theorem 3.5. *Suppose that the solution y of (1.1) is expressed in the form*

$$y(t)=\sum_{\nu=0}^{\nu_0}c_\nu t^{\alpha\nu}, \tag{3.30}$$

in which $\nu_0\in Z^+$, with the constants $c_1,c_2,\ldots,c_{\nu_0}\in\mathbf{R}$. Then, for $\epsilon\in(0,T]$,

$$\max_{t_j\in[\epsilon,T]}|y(t_j)-y_j|=\begin{cases}O(h^2) & \text{if } 1<\alpha,\\ O(h^{1+\alpha}) & \text{if } 0<\alpha<1,\end{cases}$$

and moreover,

$$\max_{0\le j\le N}|y(t_j)-y_j|=\begin{cases}O(h^2) & \text{if } 1<\alpha,\\ O(h^{2\alpha}) & \text{if } 0<\alpha<1.\end{cases}$$

Proof. Using (3.30) and the Caputo fractional derivative definition yields

$$D_*^\alpha y(t)=\sum_{\nu=1}^{\nu_0}\frac{\Gamma(\nu\alpha+1)}{\Gamma(\nu\alpha+1-\alpha)}c_\nu t^{\nu\alpha-\alpha}. \tag{3.31}$$

If $\alpha>1$, it follows from (3.31) and Theorems 2.3 and 2.4 that

$$E_{k+1}[D_*^\alpha y(t)]\le\sum_{\nu=1}^{\nu_0}\frac{\Gamma(\nu\alpha+1)}{\Gamma(\nu\alpha+1-\alpha)}|c_\nu|E_{k+1}[t^{\nu\alpha-\alpha}]\le C_1t_{k+1}^{\alpha}h, \tag{3.32}$$

and

$$F_{k+1}[D_*^\alpha y(t)]\le\sum_{\nu=1}^{\nu_0}\frac{\Gamma(\nu\alpha+1)}{\Gamma(\nu\alpha+1-\alpha)}|c_\nu|F_{k+1}[t^{\nu\alpha-\alpha}]\le C_2t_{k+1}^{2(\alpha-1)}h^2. \tag{3.33}$$

Now, by using (3.32) and (3.33) and Lemma 3 in [5], we get

$$|y(t_j)-y_j|\le Ch^q,\quad (q=\min(2,1+\alpha)).$$

So

$$\max_{0\le j\le N}|y(t_j)-y_j|=O(h^2).$$

If $0<\alpha<1$, by almost the same reasoning, we again have

$$E_{k+1}[D_*^\alpha y(t)]\le\sum_{\nu=1}^{\nu_0}\frac{\Gamma(\nu\alpha+1)}{\Gamma(\nu\alpha+1-\alpha)}|c_\nu|E_{k+1}[t^{\nu\alpha-\alpha}]\le C_3t_{k+1}^{\alpha-1}h, \tag{3.34}$$

and

$$F_{k+1}[D_*^\alpha y(t)]\le\sum_{\nu=1}^{\nu_0}\frac{\Gamma(\nu\alpha+1)}{\Gamma(\nu\alpha+1-\alpha)}|c_\nu|F_{k+1}[t^{\nu\alpha-\alpha}]\le C_4t_{k+1}^{\alpha-1}h^{\alpha+1}. \tag{3.35}$$

By using (3.34) and (3.35) and by almost the same proof as for Theorem 3.2, we obtain

$$|y(t_j)-y_j|\le C_5t^{\alpha-1}h^q,\quad (\text{here } q=1+\alpha).$$

So, for arbitrary $\epsilon>0$ and $t_j\in[\epsilon,T]$,

$$|y(t_j)-y_j|\le C_5\epsilon^{\alpha-1}h^{1+\alpha};$$

i.e.,

$$\max_{t_j\in[\epsilon,T]}|y(t_j)-y_j|=O(h^{1+\alpha});$$

and for $0\le j\le N$,

$$|y(t_j)-y_j|\le Ch^{2\alpha}.$$

This ends the proof. □

Table 1
Error for Eq. (4.1) with $0 < \alpha < 1$, taken at $t = 1$.

h	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
1/10	−1.38(−2)	−4.73(−3)	−2.71(−3)	−1.82(−3)	−1.31(−3)
1/20	−5.29(−3)	−1.62(−3)	−8.36(−4)	−5.15(−4)	−3.36(−4)
1/40	−2.21(−3)	−5.70(−4)	−2.67(−4)	−1.49(−4)	−8.74(−5)
1/80	−9.16(−4)	−2.05(−4)	−8.81(−5)	−4.44(−5)	−2.29(−5)
1/160	−3.79(−4)	−7.55(−5)	−2.95(−5)	−1.33(−5)	−6.07(−6)
1/320	−1.58(−4)	−2.82(−5)	−1.01(−5)	−4.04(−6)	−1.61(−6)
EOC	1.26	1.42	1.55	1.72	1.91

Compared to DFF Theorem (b) (the case with $\alpha > 1$ and $y \in C^{\lceil\alpha\rceil+1}[0, T]$), here the order of the estimate is much higher but $y \in C^{\lceil\alpha\rceil-1}[0, T]$, except for the special form of $y(t)$. The error estimates are actually the posterior estimates. A more general case relating to the posterior error estimates is given below.

Theorem 3.6. *Suppose that the solution y of (1.1) is expressed in the form*

$$y(t) = \sum_{\nu=0}^{\nu_0} c_{0,\nu} t^{\alpha\nu} + \sum_{\nu=1}^{\nu_1} c_{1,\nu} t^{1+\alpha\nu} + \sum_{\nu=1}^{\nu_2} c_{2,\nu} t^{2+\alpha\nu} + \cdots + \sum_{\nu=1}^{\nu_l} c_{l,\nu} t^{l+\alpha\nu}, \tag{3.36}$$

in which $\nu_0, \nu_1, \nu_2, \ldots, \nu_l, l \in Z^+$, all $c_{0,\nu}, c_{1,\nu}, \ldots, c_{l,\nu}$ are real numbers. Then, for $\epsilon \in (0, T]$,

$$\max_{t_j \in [\epsilon, T]} |y(t_j) - y_j| = \begin{cases} O(h^2) & \text{if } 1 < \alpha, \\ O(h^{1+\alpha}) & \text{if } 0 < \alpha < 1, \end{cases}$$

and moreover,

$$\max_{0 \le j \le N} |y(t_j) - y_j| = \begin{cases} O(h^2) & \text{if } 1 < \alpha, \\ O(h^{2\alpha}) & \text{if } 0 < \alpha < 1. \end{cases}$$

The proof of this theorem is the same as that of Theorem 3.5 and so is omitted here. It is worthy of note that the last l terms on the right hand side of Eq. (3.36) cannot begin with $\nu = 0$; otherwise, the above conclusion does not hold yet. For example, if $\alpha \in (0, 1)$ and the second term on the right hand side of (3.36) begins with $\nu = 0$, then the error estimate is the same as that of Theorem 3.2. Those cases can be considered in the same manner, so they are left out of this article.

4. A numerical example

In this section we present a numerical example to illustrate the error bounds derived above.

Example 1.

$$D_*^\alpha y(t) = -y(t) + \frac{1}{\Gamma(5-\alpha)} t^{4-\alpha}, \quad \alpha \in (0, 1), y(0) = 1. \tag{4.1}$$

Its exact solution is

$$y(t) = t^4 E_{\alpha,5}(-t^\alpha) + E_\alpha(-t^\alpha),$$

where

$$E_{\alpha,\beta}(z) = \sum_0^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad E_\alpha(z) = E_{\alpha,1}(z).$$

In Table 1 we show the errors of the Adams method at the point $t = 1$ for various step sizes and α. In each case, the leftmost column stands for the step length, while the following columns give the error of our scheme at $t = 1$. The last row states the experimental order of convergence (called "EOC" for brevity). As usual, the notation $-1.38(-2)$ indicates -1.38×10^{-2}, etc.

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