NOTE

Symmetrically Homoclinic Orbits for Symmetric Hamiltonian Systems

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In this paper, we study the existence of symmetric homoclinic orbits for first order and second order Hamiltonian systems with some symmetric Hamiltonian functions.

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In recent years, many authors [1–3, 5–30, 34, 36–46] have used the variational methods to study the existence and the multiplicity of homoclinic orbits for Hamiltonian systems. In this paper, we will study the existence of a symmetric homoclinic orbit for the first order symmetric Hamiltonian system and the existence of infinitely many odd homoclinic orbits for classical Hamiltonian systems with even potentials.

We are given a $C^2$ map $H: \mathbb{R}^{2N} \rightarrow \mathbb{R}$, and we consider the associated system of ordinary differential equations

$$\begin{align*}
\dot{x}(t) &= JH'(x) \\
x(\pm \infty) &= 0,
\end{align*}$$

(1.1)

where $J$ denotes the $2N \times 2N$ matrix

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

with $J^* = J^{-1} = -J$.
We obtain the following results:

**THEOREM 1.1.** Assume $H$ satisfies

(H1) $H \in C^2(R^{2N}, R)$
(H2) $H(-p, q) = H(p, q)$, $\forall p, q \in R^N$
(H3) $H_y(0, 0) = 0$
(H4) $H''(0) = 0$
(H5) $\exists \alpha > 2$ such that $\forall x \in R^{2N}$, $\alpha H(x) \leq H'(x)x$
(H6) $\exists k_1 > 0$ such that $\forall x \in R^{2N}$, $H(x) \geq k_1|x|^{\alpha}$
(H7) $\exists k_2 > 0$ such that $\forall x \in R^{2N}$, $|H'(x)| \leq k_2|x|^{-1}$.

Then (1.1) has at least one homoclinic orbit $x = (p, q)$ to the origin which satisfies $p(-t) = -p(t)$ and $q(-t) = q(t)$.

**Remark 1.** In all published papers, there is a quadratic term for the Hamiltonian function. Here we remove this term.

**Remark 2.** (H5) implies $H(x) = 0(|x|^2)$ as $|x| \to 0$. (H4) can be canceled out.

**THEOREM 1.2.** Assume $V$ satisfies

(V1) $V \in C^2(R^n, R)$;
(V2) $V(-x) = V(x)$, $\forall x \in R^n$;
(V3) there is a $\mu > 2$ such that $0 = \mu V(x) \leq x \cdot V'(x)$, $\forall x \in R^n \setminus \{0\}$;
(V4) $V''(0) = 0$.

Then there are infinitely many odd homoclinic orbits for the second order Hamiltonian system:

$$\ddot{x} + V'(x) = 0$$

$$x(\pm \infty) = \dot{x}(\pm \infty) = 0. \quad \text{(1.2)}$$

2. **THE PROOF OF THEOREM 1.1**

Let $W = W^{1,2}(R, R^{2N})$ be the Sobolev space of $R^{2N}$-valued functions defined on $R$:

$$E = \{ x = (p, q) \in W | \ p(-t) = -p(t), \ q(-t) = q(t), \ \forall \ t \in R \}. \quad \text{(2.1)}$$

The functional corresponding to the system (1.1) is defined by

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{2}(-\dot{x}, x) \ dt - \int_{-\infty}^{\infty} H(x) \ dt \quad \forall x \in E. \quad \text{(2.2)}$$
Following the ideas of [35, 31–33], we have

**Lemma 2.1.** Suppose (H1) and (H4) hold. Then \( f \in C^1(E, R) \), and \( x = (p, q) \in E \) is a critical point of \( f \) restricted on \( E \) if and only if it is a \( C^1(R, R^n) \)-solution of \( (1.1) \) such that \( p \) is odd and \( q \) even in \( t \).

**Proof.** (i) By (H1) and (H4), similar to the proof of Coti Zelati and Rabinowitz [26], \( f \in C^1(E, R) \).

(ii) Suppose \( x \in E \) is a critical point of \( f \) on \( E \). Then there holds

\[
\int_{-\infty}^{\infty} \left( -J\dot{x} \cdot y - H'(x) \cdot y \right) \, dt = 0, \quad \forall y \in E. \tag{2.3}
\]

By (H1), \( H' \in C^1(W^{1,2}, W^{1,2}) \). (H2) and (H3) imply \( H'(0) = 0 \). By \( x \in W^{1,2}(R, R^{2n}) \) and the regularity theorem on composition mappings, we have \( u \equiv H'(x(\cdot)) \in W^{1,2}(R, R^{2n}) \) and \( u \in E \); that is, \( u = (u_1, u_2) \) satisfies

\[
u_1(-t) = -u_1(t) \quad \text{and} \quad u_2(-t) = u_2(t). \tag{2.4}
\]

We consider the boundary value problem of the linear system,

\[
\dot{z}(t) = Ju, \quad z(\pm \infty) = 0, \tag{2.5}
\]

which possesses a unique solution \( Z(t) \in C^1(R, R^{2n}) \) and is given by

\[
Z(t) = J \cdot \int_{-\infty}^{t} u(s) \, ds, \quad \forall t \in R. \tag{2.6}
\]

By (2.4) and (2.6) we know that

\[
Z(t) = (Z_1(t), Z_2(t)) = \left( -\int_{-\infty}^{t} u_2(s) \, ds, \int_{-\infty}^{t} u_1(s) \, ds \right)
\]

satisfies

\[
Z_1(-t) = -Z_1(t), \quad Z_2(-t) = Z_2(t). \tag{2.7}
\]

By (2.6) and \( u \in W^{1,2}(R, R^{2n}) \) we know \( Z \in W^{2,2}(R, R^{2n}) \).

From (2.5) we obtain that for \( \forall y \in E \) there holds

\[
\int_{-\infty}^{\infty} \left( -J\dot{Z} \cdot y - H'(x) \cdot y \right) \, dt = 0. \tag{2.8}
\]

Combining with (2.3) yields

\[
\int_{-\infty}^{\infty} J(\dot{x} - \dot{Z}) \cdot y \, dt = 0, \quad \forall y \in E. \tag{2.9}
\]
By (2.3) and (2.6) we have \( x \in W^{2,2}(R, R^{2n}) \), \( Z \in W^{2,2}(R, R^{2n}) \). So
\[
\tilde{y} = J(\tilde{x} - \tilde{Z}) \in W^{1,2}(R, R^{2n}).
\] (2.10)

Set \( x = (x_1, x_2) \), \( Z = (Z_1, Z_2) \); then
\[
\tilde{y} = J(\tilde{x} - \tilde{Z}) = (\tilde{Z}_2 - \tilde{x}_2, \tilde{x}_1 - \tilde{Z}_1) = (\tilde{y}_1, \tilde{y}_2).
\]
Then
\[
\tilde{y}_1(-t) = -\tilde{y}_1(t), \quad \tilde{y}_2(-t) = \tilde{y}_2(t). \quad (2.11)
\]

Hence \( \tilde{y} \in E \).

In (2.9), we can set \( y = \tilde{y} \) to obtain
\[
\int_{-\infty}^{\infty} |\tilde{x} - \tilde{Z}|^2 \, dt = 0. \quad (2.12)
\]

Hence
\[
x(t) - Z(t) \equiv \text{constant}, \quad \forall t \in R. \quad (2.13)
\]
By \( x(\pm \infty) = Z(\pm \infty) = 0 \), we know
\[
x(t) - Z(t) = 0.
\]
Thus \( x(t) = Z(t) \in C^1(R, R^n) \) and is a solution of (1.1) by (2.5). Now the proof of Theorem 1.1 is similar to that of Hofer and Wysocki [29].

3. THE PROOF OF THEOREM 1.2

Let \( W = W^{1,2}(R, R^n) \), which has the usual norm \( (\int_{-\infty}^{\infty} (|\dot{q}|^2 + |q|^2))^{1/2} \)
which is equivalent to the norm
\[
\|q\| = \left( \int_{-\infty}^{\infty} |\dot{q}|^2 \, dt + |q(0)|^2 \right)^{1/2}. \quad (3.1)
\]

The functional corresponding to the system (1.2) \( f(x) \) is defined by
\[
f(x) = \int_{-\infty}^{\infty} \left[ \frac{1}{2} |\dot{x}|^2 - V(x) \right] \, dt, \quad \forall x \in W. \quad (3.2)
\]

Let
\[
\tilde{E} = \{ x \in W | x(-t) = -x(t), \forall t \in R \}. \quad (3.3)
\]
Then $\tilde{E}$ is a closed subspace of $W$ and, therefore, is a Hilbert space. By $x(-t) = -x(t)$ we have $x(0) = 0$. Hence we have

$$||x|| = \left( \int_{-\infty}^{\infty} |\dot{x}|^2 dt \right)^{1/2}, \quad \forall x \in \tilde{E}. \quad (3.4)$$

Following the ideas of [31–33, 35], we have

**Lemma 3.1.** Suppose (V1), (V2), and (V4) hold. Then $f \in C^1(\tilde{E}, R)$, and $x \in \tilde{E}$ is a critical point of $f$ restricted on $\tilde{E}$ if and only if it is an odd $C^2(R, R^n)$-solution of (1.2).

**Proof.** (i) By (V1), (V4), and [26], we know $f \in C^1(\tilde{E}, R)$.

(ii) Suppose $x \in \tilde{E}$ is a critical point of $f$ on $\tilde{E}$. Then there holds

$$\int_{-\infty}^{\infty} (\dot{x}y - V'(x) \cdot y) dt = 0, \quad \forall y \in \tilde{E}. \quad (3.5)$$

By (V1), we have $w = V'(x(\cdot), t) \in C(R, R^n)$. Furthermore, by (V1), $V'' \in C^1(R^n \times R, R)$ and $x \in W^{1,2}(R, R^n)$. By (V2), we have $V'(0) = 0$. So by the regular theorem about the composition mapping we have $w \in W^{1,2}(R, R^n)$.

The boundary value problem of the linear system

$$\ddot{q} + w = 0$$

$$q(\pm \infty) = \dot{q}(\pm \infty) = 0 \quad (3.6)$$

possesses a unique solution $Q \in C^2(R, R^n)$ and

$$\int_{S_1}^{S} \dot{Q}(\tau) d\tau = \int_{S_1}^{S} w(\tau) d\tau, \quad \forall S, S_1 \in R \quad (3.7)$$

$$\dot{Q}(S) - \dot{Q}(S_1) = -\int_{S_1}^{S} w(\tau) d\tau, \quad \forall S, S_1 \in R. \quad (3.8)$$

Because $\lim_{t_1 \to -\infty} Q(S_1) = 0$, so $\int_{-\infty}^{S} w(\tau) d\tau$ exists and

$$-\int_{-\infty}^{S} w(\tau) d\tau = \dot{Q}(S) \quad (3.9)$$

$$-\int_{t_1}^{t} \left( \int_{-\infty}^{S} W(\tau) d\tau \right) ds = Q(t) - Q(t_1), \quad \forall t_1, t \in R. \quad (3.10)$$
Because \( \lim_{t \to -\infty} Q(t) = 0 \), so \( -\int_{-\infty}^{t} \int_{-\infty}^{S} w(\tau) \, d\tau \, ds \) exists and
\[
Q(t) = -\int_{-\infty}^{t} \int_{-\infty}^{S} w(\tau) \, d\tau \, ds. \tag{3.11}
\]
So \( Q \in C^{2}(R, R^{n}) \).
Since \( w \) is odd, so is \( Q \). By \( Q(\pm \infty) = \hat{Q}(\pm \infty) = 0 \), we know \( Q \in \hat{E} \).
From (3.6) we obtain that for \( \forall y \in \hat{E} \) there holds
\[
\int_{-\infty}^{\infty} (\hat{Q} \dot{y} - V'(x) \cdot y) \, dt = 0. \tag{3.12}
\]
Combining with (3.5) yields
\[
\int_{-\infty}^{\infty} (\dot{x} - \hat{Q}) \cdot \dot{y} \, dt = 0, \quad \forall y \in \hat{E}. \tag{3.13}
\]
Letting \( y = x - Q \), by the fact \( x(0) = Q(0) = 0 \) we obtain
\[
|\dot{x}(t) - Q(t)| \leq \int_{0}^{t} \left| \dot{x}(s) - \hat{Q}(s) \right| \, ds \leq \sqrt{t} \| \dot{x} - \hat{Q} \|_{L^{2}} = 0, \quad \forall t \in R. \tag{3.14}
\]
Thus \( x = Q \in C^{2}(R, R^{n}) \) and is a solution of (1.2) by (3.6).
Now the proof of Theorem 1.2 follows from Lemma 3.1 and the arguments of Coti Zelati-Rabinowitz [26].

REFERENCES


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