Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 342 (2008) 1160–1174

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On proper actions of Lie groups of dimension $n^2 + 1$ on n-dimensional complex manifolds

A.V. Isaev

Department of Mathematics, The Australian National University, Canberra, ACT 0200, Australia

Received 6 September 2007

Available online 28 January 2008

Submitted by Steven G. Krantz

Abstract

We explicitly classify all pairs (M, G), where M is a connected complex manifold of dimension $n \geqslant 2$ and G is a connected Lie group acting properly and effectively on M by holomorphic transformations and having dimension d_G satisfying $n^2 + 2 \leqslant d_G < n^2 + 2n$. We also consider the case $d_G = n^2 + 1$. In this case all actions split into three types according to the form of the linear isotropy subgroup. We give a complete explicit description of all pairs (M, G) for two of these types, as well as a large number of examples of actions of the third type. These results complement a theorem due to W. Kaup for the maximal group dimension $n^2 + 2n$ and generalize some of the author's earlier work on Kobayashi-hyperbolic manifolds with high-dimensional holomorphic automorphism group.

Keywords: Complex manifolds; Proper group actions

0. Introduction

This paper is a continuation of [16], where we gave an extensive introduction to the subject and motivations for our work. Below we briefly recap some of the introductory points from [16].

Let M be a connected C^∞-smooth manifold and $\mathrm{Diff}(M)$ the group of C^∞-smooth diffeomorphisms of M endowed with the compact-open topology. A topological group G is said to act continuously on M by diffeomorphisms, if a continuous homomorphism $\Phi : G \to \mathrm{Diff}(M)$ is specified. We only consider effective actions, that is, assume that the kernel of Φ is trivial. The action of G on M is called *proper*, if the map

$$\Psi : G \times M \to M \times M, \quad (g, p) \mapsto \big(\Phi(g)(p), p\big),$$

is proper, i.e., for every compact subset $C \subset M \times M$ its inverse image $\Psi^{-1}(C) \subset G \times M$ is compact as well. Proper actions are a natural generalization of actions of compact groups. In particular, one can assume that G is a Lie group acting smoothly and properly on the manifold M, and that it is realized as a closed subgroup of $\mathrm{Diff}(M)$ (see [2] for a brief survey on proper actions). Due to the results of [25,27], Lie groups acting properly and effectively on

E-mail address: alexander.isaev@maths.anu.edu.au.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.12.050

the manifold M by diffeomorphisms are precisely closed subgroups of the isometry groups of all possible smooth Riemannian metrics on M.

If G acts properly on M, then for every $p \in M$ its isotropy subgroup

$$G_p := \{g \in G\colon\ gp = p\}$$

is compact in G. Then by [4] the isotropy representation

$$\alpha_p : G_p \to GL\big(\mathbb{R}, T_p(M)\big), \quad g \mapsto dg(p)$$

is continuous and faithful, where $T_p(M)$ denotes the tangent space to M at p and $dg(p)$ is the differential of g at p. In particular, the linear isotropy subgroup

$$LG_p := \alpha_p(G_p)$$

is a compact subgroup of $GL(\mathbb{R}, T_p(M))$ isomorphic to G_p. In some coordinates in $T_p(M)$ the group LG_p becomes a subgroup of the orthogonal group $O_m(\mathbb{R})$, where $m := \dim M$. Hence $\dim G_p \leqslant \dim O_m(\mathbb{R}) = m(m-1)/2$. Furthermore, for every $p \in M$ its orbit

$$Gp := \{gp\colon\ g \in G\}$$

is a closed submanifold of M, and $\dim Gp \leqslant m$. Thus, setting $d_G := \dim G$, we obtain

$$d_G = \dim G_p + \dim Gp \leqslant m(m+1)/2.$$

It is a classical result (see [5,7,8]) that if G acts properly on a smooth manifold M of dimension $m \geqslant 2$ and $d_G = m(m+1)/2$, then M is isometric (with respect to some G-invariant metric) either to one of the standard complete simply-connected spaces of constant sectional curvature $\mathbb{R}^m$, S^m, $\mathbb{H}^m$ (where $\mathbb{H}^m$ is the hyperbolic space), or to $\mathbb{RP}^m$. Subgroups of lower dimensions turned out to be much more difficult to deal with, and many outstanding mathematicians were involved in determining such subgroups: Kobayashi, Nagano, H.-C. Wang, Yano, Egorov, to name a few. As a result of research activities that spanned over 20 years, a complete description of manifolds that admit proper actions of groups of dimension $(m-1)(m-1)/2+2$ or higher was produced for $m \geqslant 6$. There are many other results, especially for compact subgroups, but—to the best of our knowledge—no complete classifications exist beyond dimension $(m-1)(m-2)/2+2$ (see [22,23,28] and references therein for details).

We study proper group actions in the complex setting with the general aim to build a theory for group dimensions lower than $(m-1)(m-2)/2+2$, thus extending—in this setting—the classical results mentioned above. In our setting real Lie groups act by holomorphic transformations on complex manifolds. Thus, from now on, M will denote a complex manifold of complex dimension n (hence $m = 2n$) and G will be a subgroup of $\mathrm{Aut}(M)$, the group of all holomorphic automorphisms of M.

Proper actions by holomorphic transformations are found in abundance. A fundamental result due to Kaup (see [20]) states that every closed subgroup of $\mathrm{Aut}(M)$ that preserves a continuous distance on M acts properly on M. Thus, Lie groups acting properly and effectively on M by holomorphic transformations are precisely those closed subgroups of $\mathrm{Aut}(M)$ that preserve continuous distances on M. In particular, if M is a Kobayashi-hyperbolic manifold, then $\mathrm{Aut}(M)$ is a Lie group acting properly on M (see also [21]).

In the complex setting, in some coordinates in $T_p(M)$ the group LG_p becomes a subgroup of the unitary group U_n. Hence $\dim G_p \leqslant \dim U_n = n^2$, and therefore

$$d_G \leqslant n^2 + 2n.$$

We note that $n^2 + 2n < (m-1)(m-2)/2 + 2$ for $m = 2n$ and $n \geqslant 5$. Thus, the group dimension range that arises in the complex case, for $n \geqslant 5$ lies strictly below the dimension range considered in the classical real case and therefore is not covered by the existing results. Furthermore, overlaps with these results for $n = 2, 3, 4$ do not lead to any significant simplifications in the complex case.

If for complex manifolds M_j and subgroups $G_j \subset \mathrm{Aut}(M_j)$, $j = 1, 2$, there exists a biholomorphic map $F : M_1 \to M_2$ such that $F \circ G_1 \circ F^{-1} = G_2$, we say that the pairs (M_1, G_1) and (M_2, G_2) are equivalent. We will be characterizing pairs (M, G) up to this equivalence relation, where $G \subset \mathrm{Aut}(M)$ is connected and acts on M properly. The case $d_G = n^2 + 2n$ was considered by Kaup in [20]. In this case (M, G) is equivalent to one of the pairs

$(\mathbb{B}^n, \mathrm{Aut}(\mathbb{B}^n))$, $(\mathbb{C}^n, G(\mathbb{C}^n))$, $(\mathbb{CP}^n, G(\mathbb{CP}^n))$. Here $\mathbb{B}^n := \{z \in \mathbb{C}^n\colon |z| < 1\}$, $\mathrm{Aut}(\mathbb{B}^n) \simeq PSU_{n,1} := SU_{n,1}/(\text{center})$ is the group of all transformations

$$z \mapsto \frac{Az + b}{cz + d},$$

where

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU_{n,1};$$

$G(\mathbb{C}^n) \simeq U_n \ltimes \mathbb{C}^n$ is the group of all holomorphic automorphisms of $\mathbb{C}^n$ of the form

$$z \mapsto Uz + a, \tag{0.1}$$

where $U \in U_n$, $a \in \mathbb{C}^n$ (we usually write $G(\mathbb{C})$ instead of $G(\mathbb{C}^1)$); and $G(\mathbb{CP}^n) \simeq PSU_{n+1} := SU_{n+1}/(\text{center})$ is the group of all holomorphic automorphisms of $\mathbb{CP}^n$ of the form

$$\zeta \mapsto U\zeta, \tag{0.2}$$

where ζ is a point in $\mathbb{CP}^n$ written in homogeneous coordinates, and $U \in SU_{n+1}$ (this group is a maximal compact subgroup of the complex Lie group $\mathrm{Aut}(\mathbb{CP}^n) \simeq PSL_{n+1}(\mathbb{C}) := SL_{n+1}(\mathbb{C})/(\text{center})$). We remark that the groups $\mathrm{Aut}(\mathbb{B}^n)$, $G(\mathbb{C}^n)$, $G(\mathbb{CP}^n)$ are the full groups of holomorphic isometries of the Bergman metric on $\mathbb{B}^n$, the flat metric on $\mathbb{C}^n$, and the Fubini–Study metric on $\mathbb{CP}^n$, respectively, and that the above result due to Kaup can be obtained directly from E. Cartan's classification of Hermitian symmetric spaces (cf. [1, pp. 49–50]).

Next, in [16,18] a complete classification was obtained for $n^2 + 2 \leqslant d_G < n^2 + 2n$ (see also [17] for shorter proofs of the results of [16]). Furthermore, in [12–14,18] we considered the special case where M is a Kobayashi-hyperbolic manifold and $G = \mathrm{Aut}(M)$, and determined all manifolds with $n^2 - 1 \leqslant d_{\mathrm{Aut}(M)} < n^2 + 2n$, $n \geqslant 2$ (see [15] for a comprehensive exposition of these results). Our immediate goal is to generalize these results to arbitrary proper actions on not necessarily Kobayashi-hyperbolic manifolds.

In the present paper we assume that $d_G = n^2 + 1$. Note that this is the lowest group dimension for which proper actions are necessarily transitive (see [20]); indeed, for $d_G = n^2$ both G-homogeneous and non-G-homogeneous manifold occur (see [15]). For $d_G = n^2 + 1$ we have $\dim G_p = (n-1)^2$, and we start by describing connected subgroups of the unitary group U_n of dimension $(n-1)^2$ in Proposition 1.1 (see Section 1), thus determining the connected identity components of all possible linear isotropy subgroups. According to this description, every action falls into one of three types. In Sections 2 and 3 we deal with actions of types I and II, respectively, and obtain complete lists of the corresponding pairs (M, G) in Theorems 2.1 and 3.1. Our proofs are variants of those appeared in [17]. Actions of type III are more difficult to deal with. In Section 4 we give a large number of examples of such actions. It is our conjecture that these examples in fact cover all possible actions of type III (see Conjecture 4.1). We will deal with this conjecture in our future work.

1. Classification of linear isotropy subgroups

In this section we describe all connected closed subgroups of U_n of dimension $(n-1)^2$, for $n \geqslant 2$.

Proposition 1.1. *Let H be a connected closed subgroup of U_n of dimension $(n-1)^2$, $n \geqslant 2$. Then H is conjugate in U_n to one of the following subgroups*:

I. $e^{i\mathbb{R}}SO_3(\mathbb{R})$ (*here* $n = 3$);
II. $SU_{n-1} \times U_1$ *realized as the subgroup of all matrices*

$$\begin{pmatrix} A & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \tag{1.1}$$

where $A \in SU_{n-1}$ and $\theta \in \mathbb{R}$, for $n \geqslant 3$;

III. *the subgroup $H^n_{k_1,k_2}$ of all matrices*

$$\begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix}, \tag{1.2}$$

where k_1, k_2 are fixed integers such that $(k_1,k_2) = 1$, $k_1 > 0$, and $A \in U_{n-1}$, $a \in (\det A)^{\frac{k_2}{k_1}} := \exp(k_2/k_1 \operatorname{Ln}(\det A))$.

Remark 1.2. The groups $H^n_{k_1,k_2}$ are pairwise not conjugate to each other for $n \geqslant 3$, whereas $H^2_{k_1,k_2}$ and $H^2_{k_2,k_1}$ are conjugate provided $k_2 > 0$. Observe also that the group $H^n_{k_1,k_2}$ is a k_1-sheeted cover of U_{n-1} for every k_2 (note that for $k_2 = 0$ we have $k_1 = 1$).

Remark 1.3. Subgroups of classical groups were extensively studied and in some cases fully classified (see, e.g., [6,24]). It is possible that Proposition 1.1 can be derived from the existing classification results. Below we give an elementary and self-contained proof that does not rely on them.

Proof of Proposition 1.1. Since H is compact, it is completely reducible, i.e., $\mathbb{C}^n$ splits into the sum of H-invariant pairwise orthogonal complex subspaces, $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_m$, such that the restriction H_j of H to each V_j is irreducible. Let $n_j := \dim_{\mathbb{C}} V_j$ (hence $n_1 + \cdots + n_m = n$) and let U_{n_j} be the group of unitary transformations of V_j. Clearly, $H_j \subset U_{n_j}$, and therefore $\dim H \leqslant n_1^2 + \cdots + n_m^2$. On the other hand $\dim H = (n-1)^2$, which shows that $m \leqslant 2$.

Let $m = 2$. Then there exists a unitary change of coordinates in $\mathbb{C}^n$ such that all elements of H take the form (1.2), where $A \in U_{n-1}$ and $a \in U_1$. We note that the groups H_1, H_2 consist of all possible A and a, respectively.

If $\dim H_2 = 0$, then $H_2 = \{1\}$, and therefore $H_1 = U_{n-1}$. In this case we obtain the group $H^n_{1,0}$.

Assume that $\dim H_2 = 1$, i.e., $H_2 = U_1$. Then $(n-1)^2 - 1 \leqslant \dim H_1 \leqslant (n-1)^2$. Let $\dim H_1 = (n-1)^2 - 1$ first. The only connected subgroup of U_{n-1} of dimension $(n-1)^2 - 1$ is SU_{n-1}. Hence H is conjugate to the subgroup of matrices of the form (1.1) if $n \geqslant 3$ and to $H^2_{1,0}$ for $n = 2$. Now let $\dim H_1 = (n-1)^2$, i.e., $H_1 = U_{n-1}$. Consider the Lie algebra $\mathfrak{h}$ of H. Up to conjugation, it consists of matrices of the form

$$\begin{pmatrix} \mathfrak{A} & 0 \\ 0 & l(\mathfrak{A}) \end{pmatrix}, \tag{1.3}$$

where $\mathfrak{A} \in \mathfrak{u}_{n-1}$ and $l(\mathfrak{A}) \not\equiv 0$ is a linear function of the matrix elements of $\mathfrak{A}$ ranging in $i\mathbb{R}$. Clearly, $l(\mathfrak{A})$ must vanish on the derived algebra of $\mathfrak{u}_{n-1}$, that is, on $\mathfrak{su}_{n-1}$. Hence matrices (1.3) form a Lie algebra if and only if $l(\mathfrak{A}) = c \cdot \operatorname{trace} \mathfrak{A}$, where $c \in \mathbb{R} \setminus \{0\}$. Such an algebra can be the Lie algebra of a closed subgroup of $U_{n-1} \times U_1$ only if $c \in \mathbb{Q} \setminus \{0\}$. Hence H is conjugate to $H^n_{k_1,k_2}$ for some $k_1, k_2 \in \mathbb{Z}$, where one can always assume that $k_1 > 0$ and $(k_1,k_2) = 1$.

Now let $m = 1$. We shall proceed as in the proof of Lemma 2.1 in [18]. Let $\mathfrak{h}^{\mathbb{C}} := \mathfrak{h} + i\mathfrak{h} \subset \mathfrak{gl}_n$ be the complexification of $\mathfrak{h}$, where $\mathfrak{gl}_n := \mathfrak{gl}_n(\mathbb{C})$. The algebra $\mathfrak{h}^{\mathbb{C}}$ acts irreducibly on $\mathbb{C}^n$ and by a theorem of E. Cartan (see, e.g., [9]), $\mathfrak{h}^{\mathbb{C}}$ is either semisimple or the direct sum of the center $\mathfrak{c}$ of $\mathfrak{gl}_n$ and a semisimple ideal $\mathfrak{t}$. Clearly, the action of the ideal $\mathfrak{t}$ on $\mathbb{C}^n$ is irreducible.

Assume first that $\mathfrak{h}^{\mathbb{C}}$ is semisimple, and let $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_k$ be its decomposition into the direct sum of simple ideals. Then the natural irreducible n-dimensional representation of $\mathfrak{h}^{\mathbb{C}}$ (given by the embedding of $\mathfrak{h}^{\mathbb{C}}$ in $\mathfrak{gl}_n$) is the tensor product of some irreducible faithful representations of the $\mathfrak{h}_j$ (see, e.g., [9]). Let n_j be the dimension of the corresponding representation of $\mathfrak{h}_j$, $j = 1, \ldots, k$. Then $n_j \geqslant 2$, $\dim_{\mathbb{C}} \mathfrak{h}_j \leqslant n_j^2 - 1$, and $n = n_1 \cdot \ldots \cdot n_k$. The following observation is simple.

Claim. *If $n = n_1 \cdot \ldots \cdot n_k$, $k \geqslant 2$, $n_j \geqslant 2$ for $j = 1, \ldots, k$, then $\sum_{j=1}^k n_j^2 \leqslant n^2 - 2n$.*

Since $\dim_{\mathbb{C}} \mathfrak{h}^{\mathbb{C}} = (n-1)^2$, it follows from the above claim that $k = 1$, i.e., $\mathfrak{h}^{\mathbb{C}}$ is simple. The minimal dimensions of irreducible faithful representations of complex simple Lie algebras $\mathfrak{s}$ are well known (see, e.g., [26]). In Table 1 below V denotes representations of minimal dimension.

Table 1

$\mathfrak{s}$	$\dim V$	$\dim \mathfrak{s}$
$\mathfrak{sl}_k$, $k \geqslant 2$	k	$k^2 - 1$
$\mathfrak{o}_k$, $k \geqslant 7$	k	$k(k-1)/2$
$\mathfrak{sp}_{2k}$, $k \geqslant 2$	$2k$	$2k^2 + k$
$\mathfrak{e}_6$	27	78
$\mathfrak{e}_7$	56	133
$\mathfrak{e}_8$	248	248
$\mathfrak{f}_4$	26	52
$\mathfrak{g}_2$	7	14

Since $\dim_{\mathbb{C}} \mathfrak{h}^{\mathbb{C}} = (n-1)^2$, it follows that none of the above possibilities realize. Therefore, $\mathfrak{h}^{\mathbb{C}} = \mathfrak{c} \oplus \mathfrak{t}$, where $\dim \mathfrak{t} = n^2 - 2n$. Then, if $n = 2$, we obtain that H coincides with the center of U_2 which is impossible since its action on $\mathbb{C}^2$ is then not irreducible. Assuming that $n \geqslant 3$ and repeating the above argument for $\mathfrak{t}$ in place of $\mathfrak{h}^{\mathbb{C}}$, we see that $\mathfrak{t}$ can only be isomorphic to $\mathfrak{sl}_{n-1}$. But $\mathfrak{sl}_{n-1}$ does not have an irreducible n-dimensional representation unless $n = 3$.

Thus, $n = 3$ and $\mathfrak{h}^{\mathbb{C}} \simeq \mathbb{C} \oplus \mathfrak{sl}_2 \simeq \mathbb{C} \oplus \mathfrak{so}_3$. Further, we observe that every irreducible 3-dimensional representation of $\mathfrak{so}_3$ is equivalent to its defining representation. This implies that H is conjugate in $GL_3(\mathbb{C})$ to $e^{i\mathbb{R}}SO_3(\mathbb{R})$. Since $H \subset U_3$ it is straightforward to show that the conjugating element can be chosen to belong to U_3.

The proof of the proposition is complete. $\square$

Let M be a connected complex manifold of dimension $n \geqslant 2$, and suppose that a connected Lie group $G \subset \mathrm{Aut}(M)$ with $d_G = n^2 + 1$ acts properly on M. Fix $p \in M$, consider the linear isotropy subgroup LG_p, and choose coordinates in $T_p(M)$ so that $LG_p \subset U_n$. We say that the pair (M, G) (or the action of G on M) is of type I, II or III, if the connected identity component LG_p^0 of the group LG_p is conjugate in U_n to a subgroup listed in I, II or III of Proposition 1.1, respectively. Since M is G-homogeneous, this definition is independent of the choice of p.

We will now separately consider actions of each type.

2. Actions of type I

A classification of actions of type I follows immediately from the general theory of Hermitian symmetric spaces, as shown in the proof of the following theorem.

Theorem 2.1. *Let M be a connected complex manifold of dimension* 3 *and $G \subset \mathrm{Aut}(M)$ a connected Lie group with $d_G = 10$ that acts properly on M. If the pair (M, G) is of type* I, *then it is equivalent to one of the following*:

(i) $(\mathscr{S}, \mathrm{Aut}(\mathscr{S}))$, *where $\mathscr{S}$ is the Siegel space*
$$\mathscr{S} := \{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon\ Z\overline{Z} \ll \mathrm{id}\},$$
with
$$Z := \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$$
(*here* $\mathrm{Aut}(\mathscr{S})$ *is isomorphic to* $Sp_4(\mathbb{R})/\mathbb{Z}_2$);

(ii) $(Q_3, SO_5(\mathbb{R}))$, *where Q_3 is the complex quadric in $\mathbb{CP}^4$, and $SO_5(\mathbb{R})$ is realized as a maximal compact subgroup of* $\mathrm{Aut}(Q_3)^0 \simeq PSO_5(\mathbb{C})$;

(iii) $(\mathbb{C}^3, G_2(\mathbb{C}^3))$, *where $G_2(\mathbb{C}^3)$ is the group that consists of all maps from $G(\mathbb{C}^3)$ with $U \in e^{i\mathbb{R}}SO_3(\mathbb{R})$ (see* (0.1)*).*[1]

Proof. Fix a G-invariant Hermitian metric on M. Since LG_q for every $q \in M$ contains the element $-\mathrm{id}$, the manifold M equipped with this metric becomes a Hermitian symmetric space. The group LG_p^0 acts irreducibly on $T_p(M)$, and

[1] In [16] we introduced groups denoted by $G_1(\mathbb{C}^n)$, $G_2(\mathbb{C}^4)$ and $G_3(\mathbb{C}^4)$. Notation in the present paper is consistent with that in [16].

therefore M either is an irreducible Hermitian symmetric space, or is equivalent (holomorphically and isometrically) to $\mathbb{C}^3$ with the flat metric.

If M is an irreducible Hermitian symmetric space, it follows from the general theory of Riemannian symmetric spaces that G coincides with the connected identity component of the group of holomorphic isometries of M (see Theorem 1.1 in Chapter V of [11]). Now E. Cartan's classification of irreducible Hermitian symmetric spaces implies that (M, G) is equivalent to either $(\mathscr{S}, \mathrm{Aut}(\mathscr{S}))$ or $(Q_3, SO_5(\mathbb{R}))$ (see Chapter IX of [11]).

Let M be equivalent to $\mathbb{C}^3$ and let F be an equivalence map. The map F transforms G into a closed subgroup of $G(\mathbb{C}^3)$ (recall that $G(\mathbb{C}^3)$ is the full group of holomorphic isometries of $\mathbb{C}^3$ with respect to the flat metric). Let $p_0 \in M$ be such that $F(p_0) = 0$. Then F transforms $G^0_{p_0}$ into a closed subgroup H of $U_3 \subset G(\mathbb{C}^3)$ isomorphic to $e^{i\mathbb{R}}SO_3(\mathbb{R})$ and acting irreducibly on $T_0(\mathbb{C}^3)$. By Proposition 1.1, the subgroup H is conjugate in U_3 to the standard embedding of $e^{i\mathbb{R}}SO_3(\mathbb{R})$ in U_3, and hence there exists an equivalence map $\hat{F}$ between M and $\mathbb{C}^3$ that transforms $G^0_{p_0}$ into $e^{i\mathbb{R}}SO_3(\mathbb{R})$.

Let $\mathfrak{g}$ be the Lie algebra (isomorphic to the Lie algebra of G) of fundamental vector fields of the action of the group $\hat{G} := \hat{F} \circ G \circ \hat{F}^{-1}$ on $\mathbb{C}^3$, that is, $\mathfrak{g}$ consists of all holomorphic vector fields X on $\mathbb{C}^3$ for which there exists an element a of the Lie algebra of $\hat{G}$ such that for all $z \in \mathbb{C}^3$ we have

$$X(z) = \frac{d}{dt}\big[\exp(ta)(z)\big]\Big|_{t=0}.$$

Since $\hat{G} \subset G(\mathbb{C}^3)$, the algebra $\mathfrak{g}$ is generated by $\langle Z_0 \rangle \oplus \mathfrak{so}_3(\mathbb{R})$ and some affine holomorphic vector fields V_j, $j = 1, \dots, 6$, that do not vanish at the origin. Here

$$Z_0 := i \sum_{k=1}^{3} z_k \partial/\partial z_k,$$

and $\mathfrak{so}_3(\mathbb{R})$ is realized as the algebra of fundamental vector fields of the standard action of $SO_3(\mathbb{R})$ on $\mathbb{C}^3$. Considering $[Z_0, [V_j, Z_0]]$ instead of V_j, we can assume that V_j are constant vector fields for all j (cf. the proof of Satz 4.9 in [20]). It then follows that $\hat{G} = G_2(\mathbb{C}^3)$.

The proof is complete. □

3. Actions of type II

In this section we give a complete classification of actions of type II (cf. the proof of Theorem 4.2 in [17]).

Theorem 3.1. *Let M be a connected complex manifold of dimension $n \geqslant 3$ and $G \subset \mathrm{Aut}(M)$ a connected Lie group with $d_G = n^2 + 1$ that acts properly on M. If the pair (M, G) is of type II, then it is equivalent to $(\mathbb{C}^{n-1} \times M', G_1(\mathbb{C}^{n-1}) \times G')$, where M' is one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$, and G' is one of the groups $\mathrm{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$, respectively.*[2]

Proof. Fix $p \in M$. By Bochner's linearization theorem (see [4]) there exist an G_p-invariant neighborhood $\mathcal{V}$ of p in M, an LG_p-invariant neighborhood $\mathcal{U}$ of the origin in $T_p(M)$ and a biholomorphic map $F : \mathcal{V} \to \mathcal{U}$, with $F(p) = 0$, such that for every $g \in G_p$ the following holds in $\mathcal{V}$:

$$F \circ g = \alpha_p(g) \circ F,$$

where α_p is the isotropy representation at p. Let $\mathfrak{g}_M$ be the Lie algebra of fundamental vector fields on M of the action of G. Next, let $\mathfrak{g}_{\mathcal{V}}$ be the Lie algebra of the restrictions of the elements of $\mathfrak{g}_M$ to $\mathcal{V}$ and $\mathfrak{g}$ the Lie algebra of vector fields on $\mathcal{U}$ obtained by pushing forward the elements of $\mathfrak{g}_{\mathcal{V}}$ by means of F. Observe that $\mathfrak{g}_M$, $\mathfrak{g}_{\mathcal{V}}$, $\mathfrak{g}$ are naturally isomorphic, and we denote by $\varphi : \mathfrak{g}_M \to \mathfrak{g}$ the isomorphism induced by F.

Next, we fix coordinates in $T_p(M)$ in which $LG^0_p = SU_{n-1} \times U_1$. The algebra $\mathfrak{g}$ is generated by $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$ and some vector fields

[2] The group $G_1(\mathbb{C}^n)$ was introduced in [16] and consists of all maps from $G(\mathbb{C}^n)$ with $U \in SU_n$ (we usually write $G_1(\mathbb{C})$ instead of $G_1(\mathbb{C}^1)$).

$$V_j = \sum_{k=1}^{n} f_j^k \partial/\partial z_k,$$
$$W_j = \sum_{k=1}^{n} g_j^k \partial/\partial z_k,$$

where the functions f_j^k, g_j^k, $j, k = 1, \dots, n$, are holomorphic on $\mathcal{U}$ and satisfy the conditions

$$f_j^k(0) = \delta_j^k, \qquad g_j^k(0) = i\delta_j^k,$$

where δ_j^k is the Kronecker symbol. Here $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$ is realized as the algebra of vector fields on $\mathcal{U}$ of the form

$$\sum_{j=1}^{n-1} (a_{j\,1} z_1 + \dots + a_{j\,n-1} z_{n-1}) \partial/\partial z_j + iaz_n \partial/\partial z_n,$$

with

$$\begin{pmatrix} a_{1\,1} & \dots & a_{1\,n-1} \\ \vdots & \vdots & \vdots \\ a_{n-1\,1} & \dots & a_{n-1\,n-1} \end{pmatrix} \in \mathfrak{su}_{n-1},$$

and $a \in \mathbb{R}$.

Let

$$Z_n := iz_n \partial/\partial z_n$$

(observe that Z_n generates the $\mathfrak{u}_1$-component of $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$), and consider $[V_j, Z_n]$, $[W_j, Z_n]$ for $j = 1, \dots, n-1$. Since these commutators vanish at 0, they lie in $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$, which implies that the functions f_j^k, g_j^k are independent of z_n for $k = 1, \dots, n-1$ and that

$$f_j^n = \tilde{f}_j^n(z_1, \dots, z_{n-1}) z_n,$$
$$g_j^n = \tilde{g}_j^n(z_1, \dots, z_{n-1}) z_n,$$

for some holomorphic functions $\tilde{f}_j^n$, $\tilde{g}_j^n$.

For every pair of indices $1 \leqslant j, l \leqslant n-1$, $j \neq l$, the vector fields

$$X_{jl} := iz_j \partial/\partial z_j - iz_l \partial/\partial z_l,$$
$$Y_{jl} := z_l \partial/\partial z_j - z_j \partial/\partial z_l$$

lie in the $\mathfrak{su}_{n-1}$-component of $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$. We now compute the commutators $[V_j, X_{jl}]$, $[W_j, X_{jl}]$, $[V_j, Y_{jl}]$, $[V_l, Y_{jl}]$ and observe that $[V_j, X_{jl}] - W_j$, $[W_j, X_{jl}] + V_j$, $[V_j, Y_{jl}] + V_l$, $[V_l, Y_{jl}] - V_j$ vanish at the origin and hence lie in $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$. This yields

for $n \geqslant 4$, $j = 1, \dots, n-1$:

$$\tilde{f}_j^n = i\rho_j + \lambda z_j,$$
$$\tilde{g}_j^n = i\sigma_j - i\lambda z_j,$$

and for $n = 3$:

$$\tilde{f}_1^3 = i\rho_1 + \mu z_1 + \nu z_2,$$
$$\tilde{f}_2^3 = i\rho_2 - \nu z_1 + \mu z_2,$$
$$\tilde{g}_1^3 = i\sigma_1 - i\mu z_1 + i\nu z_2,$$
$$\tilde{g}_2^3 = i\sigma_2 - i\nu z_1 - i\mu z_2,$$

where $\rho_j, \sigma_j \in \mathbb{R}$, $\lambda, \mu, \nu \in \mathbb{C}$. We now define: $V_j' := V_j - \rho_j Z_n$, $W_j' := W_j - \sigma_j Z_n$ for $j = 1, \dots, n-1$.

Further, consider the commutators $[V_n, X_{jl}]$, $[W_n, X_{jl}]$, $[V_n, Y_{jl}]$, $[W_n, Y_{jl}]$. Each of these commutators vanishes at the origin and hence lies in $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$. This gives that f_n^n, g_n^n are independent of $z_1, \dots, z_{n-1}$ and that for $k = 1, \dots, n-1$ the following holds:

$$f_n^k = \alpha^k + \beta^k(z_n)z_k,$$
$$g_n^k = \gamma^k + \delta^k(z_n)z_k,$$

where α^k and γ^k are linear functions independent of z_k, z_n.

Next, computing the commutators $[V_n, Z_n]$ and $[W_n, Z_n]$, we see that $[V_n, Z_n] - W_n$ and $[W_n, Z_n] + V_n$ vanish at 0 and hence are elements of $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$. This gives

$$V_n = \sum_{k=1}^{n-1} \varepsilon^k z_k z_n \partial/\partial z_k + f_n^n \partial/\partial z_n \pmod{\mathfrak{su}_{n-1}},$$
$$W_n = -i\sum_{k=1}^{n-1} \varepsilon^k z_k z_n \partial/\partial z_k + g_n^n \partial/\partial z_n \pmod{\mathfrak{su}_{n-1}},$$

for some $\varepsilon^k \in \mathbb{C}$, $k = 1, \dots, n-1$, and we set

$$V_n' := \sum_{k=1}^{n-1} \varepsilon^k z_k z_n \partial/\partial z_k + f_n^n \partial/\partial z_n,$$
$$W_n' := -i\sum_{k=1}^{n-1} \varepsilon^k z_k z_n \partial/\partial z_k + g_n^n \partial/\partial z_n.$$

Consider now for each $1 \leqslant j \leqslant n-1$ the commutator $[V_j', V_n']$. Its linear part $\mathcal{L}_j$ is easy to find:

for $n \geqslant 4$, $j = 1, \dots, n-1$:

$$\mathcal{L}_j = \varepsilon^j z_n \partial/\partial z_j - \lambda z_j \partial/\partial z_n,$$

and for $n = 3$:

$$\mathcal{L}_1 = \varepsilon^1 z_3 \partial/\partial z_1 - (\mu z_1 + \nu z_2)\partial/\partial z_3,$$
$$\mathcal{L}_2 = \varepsilon^2 z_3 \partial/\partial z_2 - (-\nu z_1 + \mu z_2)\partial/\partial z_3.$$

Clearly, every commutator $[V_j', V_n']$ vanishes at 0. Hence it is an element of $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$ and thus coincides with $\mathcal{L}_j$. However, for $n \geqslant 4$ the vector field $\mathcal{L}_j$ can be an element of $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$ only if $\varepsilon^j = \lambda = 0$. For $n = 3$ the vector fields $\mathcal{L}_1$ and $\mathcal{L}_2$ can be elements of $\mathfrak{su}_2 \oplus \mathfrak{u}_1$ only if $\varepsilon^1 = \varepsilon^2 = \mu = \nu = 0$. Therefore, V_j', W_j', for $j = 1, \dots, n-1$, are independent of z_n and V_n', W_n' are independent of $z_1, \dots, z_{n-1}$.

Thus, we have $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_1$ is the ideal generated by $\mathfrak{su}_{n-1}$ and V_j', W_j', for $j = 1, \dots, n-1$, and $\mathfrak{g}_2$ is the ideal generated by $\mathfrak{u}_1$ and V_n', W_n'.

Let G_j be the connected normal (possibly non-closed) subgroup of G with Lie algebra $\tilde{\mathfrak{g}}_j := \varphi^{-1}(\mathfrak{g}_j) \subset \mathfrak{g}_M$ for $j = 1, 2$. Clearly, for each j the subgroup G_j contains $\alpha_p^{-1}(L_{j\,p}) \subset G_p^0$, where $L_{1\,p} \simeq SU_{n-1}$ and $L_{2\,p} \simeq U_1$ are the subgroups of LG_p^0 given by $\alpha = 0$ and $A = \mathrm{id}$ in formula (1.1), respectively. Consider the orbit $G_j p$, $j = 1, 2$. Clearly, for each j there exists a neighborhood $\mathcal{W}_j$ of the identity in G_j such that

$$\mathcal{W}_1 p = F^{-1}\big(\mathcal{U}' \cap \{z_n = 0\}\big),$$
$$\mathcal{W}_2 p = F^{-1}\big(\mathcal{U}' \cap \{z_1 = 0, \dots, z_{n-1} = 0\}\big),$$

for some neighborhood $\mathcal{U}' \subset \mathcal{U}$ of the origin in $T_p(M)$. Thus, each $G_j p$ is a complex (possibly non-closed) submanifold of M, and the ideal $\tilde{\mathfrak{g}}_j$ consists exactly of those vector fields from $\mathfrak{g}_M$ that are tangent to $G_j p$ at some point (and hence at all points).

Furthermore, for the isotropy subgroup $G_{j\,p}$ of the point p with respect to the G_j-action we have $G^0_{j\,p}=\alpha_p^{-1}(L_{j\,p})$, $j=1,2$. Since $L_{j\,p}$ acts transitively on real directions in $T_p(G_j p)$ for $j=1,2$, by [3,10] we obtain that $G_1 p$ is holomorphically equivalent to one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$ and $G_2 p$ is holomorphically equivalent to one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$.

We will now show that each G_j is closed in G. We assume that $j=1$; for $j=2$ the proof is similar. Let $\mathfrak{U}$ be a connected neighborhood of 0 in $\mathfrak{g}_M$ where the exponential map into G is a diffeomorphism, and let $\mathfrak{V}:=\exp(\mathfrak{U})$. To prove that G_1 is closed in G it is sufficient to show that for some neighborhood $\mathfrak{W}$ of $e\in G$, $\mathfrak{W}\subset\mathfrak{V}$, we have $G_1\cap\mathfrak{W}=\exp(\tilde{\mathfrak{g}}_1\cap\mathfrak{U})\cap\mathfrak{W}$. Assuming the opposite we obtain a sequence $\{g_j\}$ of elements of G_1 converging to e in G such that for every j we have $g_j=\exp(a_j)$ with $a_j\in\mathfrak{U}\setminus\tilde{\mathfrak{g}}_1$. Observe now that there exists a connected neighborhood $\mathcal{V}'$ of p in M foliated by complex submanifolds holomorphically equivalent to $\mathbb{B}^{n-1}$ in such a way that the leaf passing through p lies in $G_1 p$. Specifically, we take $\mathcal{V}':=F^{-1}(\mathcal{U}')$ for a suitable neighborhood $\mathcal{U}'\subset\mathcal{U}$ of the origin in $T_p(M)$, and the leaves of the foliation are then given as $F^{-1}(\mathcal{U}'\cap\{z_n=\text{const}\})$. For every $s\in\mathcal{V}'$ we denote by N_s the leaf of the foliation passing through s. Observe that for every $s\in\mathcal{V}'$ vector fields from $\tilde{\mathfrak{g}}_1$ are tangent to N_s at every point. Let $p_j:=g_j p$. If j is sufficiently large, we have $p_j\in\mathcal{V}'$. We will now show that $N_{p_j}\neq N_p$ for large j.

Let $\mathfrak{U}''\subset\mathfrak{U}'\subset\mathfrak{U}$ be connected neighborhoods of 0 in $\mathfrak{g}_M$ such that:

(a) $\exp(\mathfrak{U}'')\cdot\exp(\mathfrak{U}'')\subset\exp(\mathfrak{U}')$;
(b) $\exp(\mathfrak{U}'')\cdot\exp(\mathfrak{U}')\subset\exp(\mathfrak{U})$;
(c) $\mathfrak{U}'=-\mathfrak{U}'$;
(d) $G_{1\,p}\cap\exp(\mathfrak{U}')\subset\exp(\tilde{\mathfrak{g}}_1\cap\mathfrak{U}')$.

We also assume that $\mathcal{V}'$ is chosen so that $N_p\subset\exp(\tilde{\mathfrak{g}}_1\cap\mathfrak{U}'')p$. Suppose that $p_j\in N_p$. Then $p_j=sp$ for some $s\in\exp(\tilde{\mathfrak{g}}_1\cap\mathfrak{U}'')$ and hence $t:=g_j^{-1}s$ is an element of $G_{1\,p}$. For large j we have $g_j^{-1}\in\exp(\mathfrak{U}'')$. Condition (a) now implies that $t\in\exp(\mathfrak{U}')$ and hence by (c), (d) we have $t^{-1}\in\exp(\tilde{\mathfrak{g}}_1\cap\mathfrak{U}')$. Therefore, by (b) we obtain $g_j\in\exp(\tilde{\mathfrak{g}}_1\cap\mathfrak{U})$ which contradicts our choice of g_j. Thus, for large j the leaves N_{p_j} are distinct from N_p. Furthermore, they accumulate to $N_p\subset G_1 p$. At the same time, since vector fields from $\tilde{\mathfrak{g}}_1$ are tangent to every N_{p_j}, we have $N_{p_j}\subset G_1 p$ for all j, and thus the orbit $G_1 p$ accumulates to itself. Below we will show that this is in fact impossible thus obtaining a contradiction. Clearly, we only need to consider the case when $G_1 p$ is equivalent to one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$.

Since $G^0_{1\,p}$ acts on $G_1 p$ effectively, by the result of [10], the orbit $G_1 p$ is holomorphically equivalent to one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$ by means of a map that takes p into the origin and transforms $G^0_{1\,p}$ into $SU_{n-1}\subset G(\mathbb{C}^{n-1})$. Consider the set $S:=G_1 p\cap G_2 p$. The orbit $G_1 p$ accumulates to itself, and therefore S contains a point other than p. Note that S does not contain any curve. Since $G^0_{1\,p}$ preserves each of $G_1 p$, $G_2 p$, it preserves S. However, the $G^0_{1\,p}$-orbit of every point in $G_1 p$ other than p is a hypersurface in $G_1 p$ diffeomorphic to the sphere S^{2n-3}. This contradiction shows that in fact S consists of p alone, and hence G_1 is closed in G.

Thus, we have proved that G_j is closed in G for $j=1,2$. Hence G_j acts on M properly and $G_j p$ is a closed submanifold of M for each j. Recall that $G_1 p$ is equivalent to one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$ and $G_2 p$ is equivalent to one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$, and denote by F_1, F_2 the respective equivalence maps. Let $K_j\subset G_j$ be the ineffectivity kernel of the G_j-action on $G_j p$ for $j=1,2$. Clearly, $K_j\subset G_{j\,p}$ and, since $G^0_{j\,p}$ acts on $G_j p$ effectively, K_j is a discrete normal subgroup of G_j for each j (in particular, K_j lies in the center of G_j for $j=1,2$). Since $d_{G_1}=n^2-2=(n-1)^2+2(n-1)-1$, Theorem 1.1 in [16] yields that $G_1 p$ is in fact equivalent to $\mathbb{C}^{n-1}$ and that F_1 can be chosen to transform G_1/K_1 into $G_1(\mathbb{C}^{n-1})$. Further, since $d_{G_2}=3$, the map F_2 can be chosen to transform G_2/K_2 into one of $\text{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$, respectively. Here G_j/K_j is viewed as a subgroup of $\text{Aut}(G_j p)$ for each j.

We will now show that the subgroup K_j is in fact trivial for $j=1,2$. Let first $j=1$. Since G_1/K_1 is isomorphic to the simply-connected group $G_1(\mathbb{C}^{n-1})\simeq SU_{n-1}\ltimes\mathbb{C}^{n-1}$ and since G_1 covers G_1/K_1 with fiber K_1, it follows that K_1 is trivial. Let $j=2$. The action of $G^0_{2\,p}$ on $G_2 p$ is effective, and thus we have $K_2\setminus\{e\}\subset G_{2\,p}\setminus G^0_{2\,p}$. Suppose that G_2/K_2 is isomorphic to either $\text{Aut}(\mathbb{B}^1)$ or $G(\mathbb{C})$. Every maximal compact subgroup of each of these groups is 1-dimensional, hence so is every maximal compact subgroup of G_2. Since $G^0_{2\,p}$ is 1-dimensional, it is maximal compact in G_2. Therefore $G_{2\,p}$ is connected, which implies that K_2 is trivial. Suppose next that G_2/K_2 is isomorphic to $G(\mathbb{CP}^1) \simeq PSU_2$. If K_2 is non-trivial, then $G_2 \simeq SU_2$ and $K_2 \simeq \mathbb{Z}_2$. Then $G^0_{2\,p}$ is conjugate in G_2 (upon the identification of G_2 with SU_2) to the subgroup of matrices of the form

$$\begin{pmatrix} 1/b & 0 \\ 0 & b \end{pmatrix},$$

where $|b| = 1$ (see, e.g., Lemma 2.1 of [19]). Since this subgroup contains the center of SU_2, the subgroup $G^0_{2\,p}$ contains the center of G_2. In particular, $K_2 \subset G^0_{2\,p}$ which contradicts the non-triviality of K_2. Thus, G_1 is isomorphic to $G_1(\mathbb{C}^{n-1})$ and G_2 is isomorphic to one of $\mathrm{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$.

We remark here that since M is G-homogeneous and G_j is normal in G, the discussion above remains valid for any point $q \in M$ in place of p; in particular, all G_j-orbits are pairwise holomorphically equivalent, $j = 1, 2$.

Next, since $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, the group G is a locally direct product of G_1 and G_2. We claim that $\mathscr{T} := G_1 \cap G_2$ is trivial. Indeed, $\mathscr{T}$ is a discrete normal subgroup of each of G_1, G_2. However, every discrete normal subgroup of each of $G_1(\mathbb{C}^{n-1})$, $\mathrm{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$ is trivial, since the center of each of these groups is trivial. Hence $\mathscr{T}$ is trivial and therefore $G = G_1 \times G_2$.

We will now observe that for every $q_1, q_2 \in M$ the orbits $G_1 q_1$ and $G_2 q_2$ intersect at exactly one point. Let $g \in G$ be an element such that $g q_2 = q_1$. It can be uniquely represented in the form $g = g_1 g_2$ with $g_j \in G_j$ for $j = 1, 2$, and therefore we have $g_2 q_2 = g_1^{-1} q_1$. Hence the intersection $G_1 q_1 \cap G_2 q_2$ is non-empty. Next, the fact that for every $q \in M$ the intersection $G_1 q \cap G_2 q$ consists of q alone follows by the argument used at the end of the proof of the closedness of G_1, G_2.

Let, as before, F_1 be a biholomorphic map from $G_1 p$ onto $\mathbb{C}^{n-1}$ that transforms G_1 into $G_1(\mathbb{C}^{n-1})$, and F_2 a biholomorphic map from $G_2 p$ onto M', where M' is one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$, that transforms G_2 into G', where G' is one of $\mathrm{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$, respectively. We will now construct a biholomorphic map $\mathcal{F}$ from M onto $\mathbb{C}^{n-1} \times M'$. For $q \in M$ consider $G_2 q$ and let r be the unique point of intersection of $G_1 p$ and $G_2 q$. Let $g \in G_1$ be an element such that $r = gp$. Then we set $\mathcal{F}(q) := (F_1(r), F_2(g^{-1} q))$. Clearly, $\mathcal{F}$ is a well-defined diffeomorphism from M onto $\mathbb{C}^{n-1} \times M'$. Since the foliation of M by G_j-orbits is holomorphic for each j, the map $\mathcal{F}$ is in fact holomorphic. By construction, $\mathcal{F}$ transforms G into $G_1(\mathbb{C}^{n-1}) \times G'$.

The proof is complete. □

4. Actions of type III

In this section we give a large number of examples of actions of type III (see also [17]). Some of the examples can be naturally combined into classes and some of the actions form parametric families. In what follows $n \geqslant 2$.

(i). Here both the manifolds and the groups are represented as direct products.

(ia). $M = M' \times \mathbb{C}$, where M' is one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$, and $G = G' \times G_1(\mathbb{C})$, where G' is one of the groups $\mathrm{Aut}(\mathbb{B}^{n-1})$, $G(\mathbb{C}^{n-1})$, $G(\mathbb{CP}^{n-1})$, respectively.

(ib). $M = M' \times \mathbb{C}^*$, where M' is as in (ia), and $G = G' \times \mathrm{Aut}(\mathbb{C}^*)^0$, where G' is as in (ia).

(ic). $M = M' \times \mathbb{T}$, where M' is as in (ia) and $\mathbb{T}$ is an elliptic curve; $G = G' \times \mathrm{Aut}(\mathbb{T})^0$, where G' is as in (ia).

(id). $M = M' \times \mathcal{P}_>$, where M' is as in (ia) and $\mathcal{P}_> := \{\xi \in \mathbb{C}\colon \operatorname{Re} \xi > 0\}$; $G = G' \times G(\mathcal{P}_>)$, where G' as in (ia) and $G(\mathcal{P}_>)$ is the group of all maps of the form

$$\xi \mapsto \lambda \xi + ia,$$

with $a \in \mathbb{R}$, $\lambda > 0$.

(ii). Parts (iib) and (iic) of this example are obtained by passing to quotients in part (iia).

(iia). $M=\mathbb{B}^{n-1}\times\mathbb{C}$, and G consists of all maps of the form

$$z'\mapsto\frac{Az'+b}{cz'+d},$$
$$z_n\mapsto z_n+\ln(cz'+d)+a,$$

where

$$\begin{pmatrix} A & b\\ c & d\end{pmatrix}\in SU_{n-1,1},$$

$z':=(z_1,\dots,z_{n-1})$ and $a\in\mathbb{C}$. In fact, for $T\in\mathbb{C}$ one can consider the following family of groups acting on $\mathbb{B}^{n-1}\times\mathbb{C}$

$$z'\mapsto\frac{Az'+b}{cz'+d},$$
$$z_n\mapsto z_n+T\ln(cz'+d)+a, \tag{4.1}$$

where A,a,b,c,d are as above. Example (ia) for $M'=\mathbb{B}^{n-1}$ is included in this family for $T=0$. If $T\neq 0$, then conjugating group (4.1) in $\mathrm{Aut}(\mathbb{B}^{n-1}\times\mathbb{C})$ by the automorphism

$$z'\mapsto z',$$
$$z_n\mapsto z_n/T, \tag{4.2}$$

we can assume that $T=1$.

(iib). $M=\mathbb{B}^{n-1}\times\mathbb{C}^*$, and for a fixed $T\in\mathbb{C}^*$ the group G consists of all maps of the form

$$z'\mapsto\frac{Az'+b}{cz'+d},$$
$$z_n\mapsto\chi(cz'+d)^T z_n, \tag{4.3}$$

where A,b,c,d are as in (iia) and $\chi\in\mathbb{C}^*$. Example (ib) for $M'=\mathbb{B}^{n-1}$ can be included in this family for $T=0$. This family is obtained from (4.1) by passing to a quotient in the last variable.

(iic). $M=\mathbb{B}^{n-1}\times\mathbb{T}$, where $\mathbb{T}$ is an elliptic curve, and for a fixed $T\in\mathbb{C}^*$ the group G consists of all maps of the form

$$z'\mapsto\frac{Az'+b}{cz'+d},$$
$$[z_n]\mapsto\bigl[\chi(cz'+d)^T z_n\bigr], \tag{4.4}$$

where A,b,c,d,χ are as in (iib), $\mathbb{T}$ is obtained from $\mathbb{C}^*$ by taking the quotient with respect to the equivalence relation $z_n\sim dz_n$, for some $d\in\mathbb{C}^*$, $|d|\neq 1$, and $[z_n]\in\mathbb{T}$ is the equivalence class of a point $z_n\in\mathbb{C}^*$. Example (ic) for $M'=\mathbb{B}^{n-1}$ can be included in this family for $T=0$. Clearly, after passing to the quotient, (4.3) turns into (4.4).

(iii). Part (iiib) of this example is obtained by passing to a quotient in part (iiia).

(iiia). $M=\mathbb{C}^n$, and G consists of all maps of the form

$$z'\mapsto e^{\mathrm{Re}\,b}Uz'+a,$$
$$z_n\mapsto z_n+b,$$

where $U\in U_{n-1}$, $a\in\mathbb{C}^{n-1}$, $b\in\mathbb{C}$. In fact, for $T\in\mathbb{C}$ one can consider the following family of groups acting on $\mathbb{C}^n$

$$z'\mapsto e^{\mathrm{Re}(Tb)}Uz'+a,$$
$$z_n\mapsto z_n+b, \tag{4.5}$$

where U, a, b are as above. Example (ia) for $M' = \mathbb{C}^{n-1}$ is included in this family for $T = 0$. If $T \neq 0$, then conjugating group (4.5) in $\mathrm{Aut}(\mathbb{C}^n)$ by the automorphism

$$z' \mapsto z',$$
$$z_n \mapsto T z_n,$$

we can assume that $T = 1$.

(iiib). $M = \mathbb{C}^{n-1} \times \mathbb{C}^*$, and for a fixed $T \in \mathbb{R}^*$ the group G consists of all maps of the form

$$z' \mapsto e^{T\,\mathrm{Re}\,b} U z' + a,$$
$$z_n \mapsto e^b z_n,$$

where U, a, b are as in (iiia). Example (ib) for $M' = \mathbb{C}^{n-1}$ can be included in this family for $T = 0$. This family is obtained from (4.5) for $T \in \mathbb{R}^*$ by passing to a quotient in the last variable.

(iv). Parts (ivb) and (ivc) of this example are obtained by passing to quotients in part (iva).

(iva). $M = \mathbb{C}^n$, and G consists of all maps of the form

$$z' \mapsto U z' + a,$$
$$z_n \mapsto z_n + \langle U z', a\rangle + b,$$

where $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $b \in \mathbb{C}$, and $\langle \cdot, \cdot\rangle$ is the inner product in $\mathbb{C}^{n-1}$. In fact, for $T \in \mathbb{C}$ one can consider the following family of groups acting on $\mathbb{C}^n$

$$z' \mapsto U z' + a,$$
$$z_n \mapsto z_n + T\langle U z', a\rangle + b, \tag{4.6}$$

where U, a, b are as above. Example (ia) for $M' = \mathbb{C}^{n-1}$ is included in this family for $T = 0$. If $T \neq 0$, then conjugating group (4.6) in $\mathrm{Aut}(\mathbb{C}^n)$ by automorphism (4.2), we can assume that $T = 1$.

(ivb). $M = \mathbb{C}^{n-1} \times \mathbb{C}^*$, and for a fixed $0 \leqslant \tau < 2\pi$ the group G consists of all maps of the form

$$z' \mapsto U z' + a,$$
$$z_n \mapsto \chi \exp\bigl(e^{i\tau}\langle U z', a\rangle\bigr) z_n, \tag{4.7}$$

where U, a are as in (iva) and $\chi \in \mathbb{C}^*$. In fact, for $T \in \mathbb{C}$ one can consider the following family of groups acting on $\mathbb{C}^{n-1} \times \mathbb{C}^*$

$$z' \mapsto U z' + a,$$
$$z_n \mapsto \chi \exp\bigl(T\langle U z', a\rangle\bigr) z_n, \tag{4.8}$$

where U, a, χ are as above. Example (ib) for $M' = \mathbb{C}^{n-1}$ is included in this family for $T = 0$. For $T \neq 0$ this family is obtained from (4.6) by passing to a quotient in the last variable. Furthermore, conjugating group (4.8) for $T \neq 0$ in $\mathrm{Aut}(\mathbb{C}^{n-1} \times \mathbb{C}^*)$ by the automorphism

$$z' \mapsto \sqrt{|T|} z',$$
$$z_n \mapsto z_n,$$

we obtain the group defined in (4.7) for $\tau = \arg T$.

(ivc). $M = \mathbb{C}^{n-1} \times \mathbb{T}$, where $\mathbb{T}$ is an elliptic curve, and for a fixed $0 \leqslant \tau < 2\pi$ the group G consists of all maps of the form

$$\begin{aligned} z' &\mapsto Uz' + a, \\ [z_n] &\mapsto \bigl[\chi \exp\bigl(e^{i\tau}\langle Uz', a\rangle\bigr) z_n\bigr], \end{aligned} \tag{4.9}$$

where U, a, χ are as in (ivb), $\mathbb{T}$ is obtained from $\mathbb{C}^*$ by taking the quotient with respect to the equivalence relation $z_n \sim dz_n$, for some $d \in \mathbb{C}^*$, $|d| \neq 1$, and $[z_n] \in \mathbb{T}$ is the equivalence class of a point $z_n \in \mathbb{C}^*$. In fact, for $T \in \mathbb{C}$ one can consider the following family of groups acting on $\mathbb{C}^{n-1} \times \mathbb{T}$

$$\begin{aligned} z' &\mapsto Uz' + a, \\ [z_n] &\mapsto \bigl[\chi \exp\bigl(T\langle Uz', a\rangle\bigr) z_n\bigr], \end{aligned} \tag{4.10}$$

where U, a, χ are as above. Example (ic) for $M' = \mathbb{C}^{n-1}$ is included in this family for $T = 0$. For $T \neq 0$ this family is obtained from (4.8) by passing to the quotient described above. Furthermore, conjugating group (4.10) for $T \neq 0$ in $\mathrm{Aut}(\mathbb{C}^{n-1} \times \mathbb{T})$ by the automorphism

$$\begin{aligned} z' &\mapsto \sqrt{|T|}z', \\ \xi &\mapsto \xi, \end{aligned}$$

where $\xi \in \mathbb{T}$, we obtain the group defined in (4.9) for $\tau = \arg T$.

(v). $M = \mathbb{C}^{n-1} \times \mathcal{P}_>$, and for a fixed $T \in \mathbb{R}^*$ the group G consists of all maps of the form

$$\begin{aligned} z' &\mapsto \lambda^T Uz' + a, \\ z_n &\mapsto \lambda z_n + ib, \end{aligned}$$

where $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $b \in \mathbb{R}$, $\lambda > 0$. Example (id) for $M' = \mathbb{C}^{n-1}$ can be included in this family for $T = 0$.

(vi). $M = \mathbb{C}^n$, and for fixed $k_1, k_2 \in \mathbb{Z}$, $(k_1, k_2) = 1$, $k_1 > 0$, $k_2 \neq 0$, the group G consists of all maps of the form (0.1) with $U \in H^n_{k_1,k_2}$ (see (1.2)). Example (ia) for $M' = \mathbb{C}^{n-1}$ can be included in this family for $k_2 = 0$.

(vii). Part (viib) of this example is obtained by passing to a quotient in part (viia).

(viia). $M = \mathbb{C}^{n*}/\mathbb{Z}_l$, where $\mathbb{C}^{n*} := \mathbb{C}^n \setminus \{0\}$, $l \in \mathbb{N}$, and the group G consists of all maps of the form

$$\{z\} \mapsto \{\lambda Uz\},$$

where $U \in U_n$, $\lambda > 0$, and $\{z\} \in \mathbb{C}^{n*}/\mathbb{Z}_l$ is the equivalence class of a point $z \in \mathbb{C}^{n*}$.

(viib). $M = M_d/\mathbb{Z}_l$, where M_d is the Hopf manifold $\mathbb{C}^{n*}/\{z \sim dz\}$, for $d \in \mathbb{C}^*$, $|d| \neq 1$, and $l \in \mathbb{N}$; the group G consists of all maps of the form

$$\bigl\{[z]\bigr\} \mapsto \bigl\{[\lambda Uz]\bigr\},$$

where U, λ are as in (viia), $[z] \in M_d$ denotes the equivalence class of a point $z \in \mathbb{C}^{n*}$, and $\{[z]\} \in M_d/\mathbb{Z}_l$ denotes the equivalence class of $[z] \in M_d$.

(viii). In this example the manifolds are the open orbits of the action of a group of affine transformations on $\mathbb{C}^n$. Let $G_{\mathcal{P}}$ be the group of all maps of the form

$$\begin{aligned} z' &\mapsto \lambda Uz' + a, \\ z_n &\mapsto \lambda^2 z_n + 2\lambda\langle Uz', a\rangle + |a|^2 + ib, \end{aligned}$$

where $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $b \in \mathbb{R}$, $\lambda > 0$.

(viiia). $M = \mathcal{P}^n_>$, $G = G_{\mathcal{P}}$, where

$$\mathcal{P}^n_> := \bigl\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}\colon\ \mathrm{Re}\, z_n > |z'|^2\bigr\}.$$

Observe that $\mathcal{P}^n_>$ is holomorphically equivalent to $\mathbb{B}^n$.

(viiib). $M = \mathcal{P}^n_<$, $G = G_{\mathcal{P}}$, where

$$\mathcal{P}^n_< := \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}\colon \operatorname{Re} z_n < |z'|^2\}.$$

Observe that $\mathcal{P}^n_<$ is holomorphically equivalent to $\mathbb{CP}^n \setminus (\overline{\mathbb{B}^n} \cup L)$, where L is a complex hyperplane tangent to $\partial\mathbb{B}^n$ at some point.

(ix). Here $n = 2$, $M = \mathbb{B}^1 \times \mathbb{C}$, and G consists of all maps of the form

$$z_1 \mapsto \frac{az_1 + b}{\bar{b}z_1 + \bar{a}},$$
$$z_2 \mapsto \frac{z_2 + cz_1 + \bar{c}}{\bar{b}z_1 + \bar{a}},$$

where $a, b \in \mathbb{C}$, $|a|^2 - |b|^2 = 1$, $c \in \mathbb{C}$.

(x). Here $n = 3$, $M = \mathbb{CP}^3$, and G consists of all maps of the form (0.2) for $n = 3$ with $U \in Sp_2$ (where Sp_2 is the compact real form of $Sp_4(\mathbb{C})$). It is isomorphic to $Sp_2/\mathbb{Z}_2$.

(xi). Let $n = 3$ and $(z : w)$ be homogeneous coordinates in $\mathbb{CP}^3$ with $z = (z_1 : z_2)$, $w = (w_1 : w_2)$. Set $M = \mathbb{CP}^3 \setminus \{w = 0\}$ and let G be the group of all maps of the form

$$z \mapsto Uz + Aw,$$
$$w \mapsto Vw,$$

where $U, V \in SU_2$, and

$$A = \begin{pmatrix} a & i\bar{b} \\ b & -i\bar{a} \end{pmatrix},$$

for some $a, b \in \mathbb{C}$.

(xii). Here $n = 3$, $M = \mathbb{C}^3$, and G consists of all maps of the form

$$z' \mapsto Uz' + a,$$
$$z_3 \mapsto \det U z_3 + \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Uz'\right] \cdot a + b,$$

where $z' := (z_1, z_2)$, $U \in U_2$, $a \in \mathbb{C}^2$, $b \in \mathbb{C}$, and $\cdot$ is the dot product in $\mathbb{C}^2$.

We conclude the paper with the following conjecture.

Conjecture 4.1. *Let M be a connected complex manifold of dimension $n \geqslant 2$ and $G \subset \mathrm{Aut}(M)$ a connected Lie group with $d_G = n^2 + 1$ that acts properly on M. If the pair (M, G) is of type* III*, then it is equivalent to one of the pairs listed in* (i)–(xii) *above.*

Acknowledgments

Part of this work was done while the author was visiting the Ruhr-Universität Bochum in January–February 2007 and the Max-Plank Institut für Mathematik in Bonn in April–May 2007. We would like to thank G. Fels for making a large number of useful comments and interest in our work.

References

[1] D.N. Akhiezer, Lie Group Actions in Complex Analysis, Aspects Math., Vieweg, Braunschweig, Wiesbaden, 1995.
[2] H. Biller, Proper actions on cohomology manifolds, Trans. Amer. Math. Soc. 355 (2002) 407–432.
[3] J. Bland, T. Duchamp, M. Kalka, A characterization of $\mathbb{CP}^n$ by its automorphism group, in: Complex Analysis, University Park, PA, 1986, in: Lecture Notes in Math., vol. 1268, Springer-Verlag, 1987, pp. 60–65.
[4] S. Bochner, Compact groups of differentiable transformations, Ann. of Math. 46 (1945) 372–381.
[5] E. Cartan, Leçons sur la Géométrie des Espaces de Riemann, Gauthier–Villars, Paris, 1928.
[6] E.B. Dynkin, Maximal subgroups of the classical groups, Amer. Math. Soc. Transl. Ser. 2 6 (1957) 245–378 (translated from Russian).
[7] L.P. Eisenhart, Continuous Groups of Transformations, Princeton Univ. Press, 1933.
[8] G. Fubini, Sugli spazii che ammettono un gruppo continou di movementi, Ann. Mat. Pura Appl. (3) 8 (1903) 39–81.
[9] M. Goto, F. Grosshans, Semisimple Lie Algebras, Lect. Notes Pure Appl. Math., vol. 38, Marcel Dekker, Inc., New York, 1978.
[10] R.E. Greene, S.G. Krantz, Characterization of complex manifolds by the isotropy subgroups of their automorphism groups, Indiana Univ. Math. J. 34 (1985) 865–879.
[11] S. Helgason, Differential Geometry and Symmetric Spaces, Academic Press, 1962.
[12] A.V. Isaev, Hyperbolic manifolds of dimension n with automorphism group of dimension $n^2 - 1$, J. Geom. Anal. 15 (2005) 239–259.
[13] A.V. Isaev, Hyperbolic 2-dimensional manifolds with 3-dimensional automorphism group, preprint, available from http://front.math.ucdavis.edu/math.CV/0509030.
[14] A.V. Isaev, Hyperbolic n-dimensional manifolds with automorphism group of dimension n^2, Geom. Funct. Anal. 17 (2007) 192–219.
[15] A.V. Isaev, Lectures on the Automorphism Groups of Kobayashi-Hyperbolic Manifolds, Lecture Notes in Math., vol. 1902, Springer-Verlag, 2007.
[16] A.V. Isaev, Proper actions of high-dimensional groups on complex manifolds, J. Geom. Anal. 17 (2007) 649–667.
[17] A.V. Isaev, Complex manifolds admitting proper actions of high-dimensional groups, J. Lie Theory 18 (2008) 141–160.
[18] A.V. Isaev, S.G. Krantz, On the automorphism groups of hyperbolic manifolds, J. Reine Angew. Math. 534 (2001) 187–194.
[19] A.V. Isaev, N.G. Kruzhilin, Effective actions of the unitary group on complex manifolds, Canad. J. Math. 54 (2002) 1254–1279.
[20] W. Kaup, Reelle Transformationsgruppen und invariante Metriken auf komplexen Räumen, Invent. Math. 3 (1967) 43–70.
[21] S. Kobayashi, Hyperbolic Manifolds and Holomorphic Mappings, Marcel Dekker, New York, 1970.
[22] S. Kobayashi, Transformation Groups in Differential Geometry, Springer-Verlag, 1972.
[23] S. Kobayashi, T. Nagano, Riemannian manifolds with abundant isometries, in: Differential Geometry, Kinokuniya, Tokyo, 1972, pp. 195–219 (in honor of Kentaro Yano).
[24] B.P. Komrakov, Maximal subalgebras of real Lie algebras and a problem of Sophus Lie, Soviet Math. Dokl. 41 (1990) 269–273 (translated from Russian).
[25] S.B. Myers, N. Steenrod, The group of isometries of a Riemannian manifold, Ann. of Math. 40 (1939) 400–416.
[26] A. Onishchik, E. Vinberg, Lie Groups and Algebraic Groups, Springer-Verlag, 1990.
[27] R.S. Palais, On the existence of slices for actions of non-compact Lie groups, Ann. of Math. 73 (1961) 295–323.
[28] K. Yano, The Theory of Lie Derivatives and Its Applications, North-Holland Publishing Co., P. Noordhoff Ltd., Interscience Publishers Inc., 1957.