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Lagrangian Duality for Preinvex Set-Valued Functions

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In this paper, generalizing the concept of cone convexity, we have defined cone preinvexity for set-valued functions and given an example in support of this generalization. A Farkas–Minkowski type theorem has been proved for these functions. A Lagrangian type dual has been defined for a fractional programming problem involving preinvex set-valued functions and duality results are established.

1. INTRODUCTION AND PRELIMINARIES

Problems involving set-valued functions have a wide range of applications in economics [8], nonlinear programming [17], differential inclusion [4], and many more. Tanino and Sawaragi [12] and Corley [5] developed the duality theories for multiobjective programming problems which implicitly involve set-valued functions. Later, Corley [6] considered maximization of concave set-valued functions, in possibly infinite dimensions, and established an existence result and developed Lagrangian duality theory for a programming problem involving such functions. More recently, Lin [9] generalized the Moreau–Rockafeller type theorem and the Farkas–Minkowski type theorem for convex set-valued functions and obtained necessary and sufficient optimality conditions for the existence of a Geoffrion

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efficient solution for a set-valued problem. He further defined Mond–Weir and Wolfe type duals and established duality results.

The notion of preinvexity for scalar-valued functions was introduced into literature by Weir and Jeyakumar [16] and Weir [15] by relaxing the convexity assumption on the domain set of the functions. The advantage with preinvex functions is that their nonnegative linear combination is also preinvex.

Several contributions have been made in the past (see [2, 3, 7, 10, 11] and the references therein) in developing the duality theory for nonlinear fractional programming problems by using a parameterization technique given by Bector [1].

Motivated by these ideas, in the present paper, we have extended the class of cone-convex set-valued functions to the class of cone-preinvex set-valued functions. A fractional programming problem involving set-valued functions has been considered. A weakly efficient solution of this problem has been related to the weakly efficient solution of a multiobjective set-valued programming problem obtained by using a parameterization technique. Also a Lagrangian type dual has been defined and duality results are established.

Let X and Y be topological vector spaces.

A set-valued function F from X into Y is a map that associates a unique subset of Y with each point of X. Equivalently, F can be viewed as a function from X into the power set of Y, i.e., $F: X \to 2^Y$.

The domain of $F: X \to 2^Y$ is given by

$$D(F) = \{x \in X \mid F(x) \neq \emptyset\}.$$

For $E \subseteq X$, $F: E \to 2^Y$, denote, $F(E) = \bigcup_{x \in E} F(x)$.

A subset Ω of Y is said to be a cone if $\lambda\xi \in \Omega$ for every $\xi \in \Omega$ and $\lambda \geqq 0$.

A convex cone is one for which $\lambda_1 \xi_1 + \lambda_2 \xi_3 \in \Omega$ for every $\xi_1, \xi_2 \in \Omega$ and $\lambda_1, \lambda_2 \geqq 0$.

A pointed cone is one for which $\Omega \cap (-\Omega) = \{0\}$, where 0 is the zero element of Y.

Let Ω be a pointed convex cone with int $\Omega \neq \emptyset$. Then we define three cone orders with respect to Ω as

$$\begin{aligned} \xi_1 \leqq_\Omega \xi_2 \quad & \text{iff } \xi_2 - \xi_1 \in \Omega, \\ \xi_1 \leq_\Omega \xi_2 \quad & \text{iff } \xi_2 - \xi_1 \in \Omega \setminus \{0\}, \\ \xi_1 <_\Omega \xi_2 \quad & \text{iff } \xi_2 - \xi_1 \in \operatorname{int} \Omega. \end{aligned}$$

The set of all the weak Ω-minimal points and weak Ω-maximal points of a set A in Y are defined as

$$\text{w-Min}_{\Omega} A = \{ y_0 \in A \mid \text{there exist no } y \in A \text{ for which } y <_{\Omega} y_0 \},$$
$$\text{w-Max}_{\Omega} A = \{ y_0 \in A \mid \text{there eixst no } y \in A \text{ for which } y_0 <_{\Omega} y \}.$$

If $y_0 \in A$ is a weak minima of A with respect to cone Ω then it is denoted by $y_0 \in \text{w-Min}_{\Omega} A$.

The polar cone Ω^* of Ω is defined as

$$\Omega^* = \{ y^* \in Y^* \mid \langle y^*, y \rangle \geqq 0 \text{ for all } y \in \Omega \}.$$

The following result is due to Wang and Li [14].

LEMMA 1.1. *If $\Omega \in Y$ is a pointed convex cone with* int $\Omega \neq \emptyset$, *then*

(i) $\Omega + \text{int}\,\Omega \subset \text{int}\,\Omega$

(ii) $\langle y^*, y \rangle > 0$ *for any* $y^* \in \Omega^* \setminus \{0\}$ *and* $y \in \text{int}\,\Omega$.

If $A, B \subseteq R^p$, $\alpha \in R$, *then we define*

$$A + B = \{ a + b \mid a \in A, b \in B \}$$
$$\alpha A = \{ \alpha a \mid a \in A \}$$

and if $B \subseteq \text{int}\, R^p_+$ *then define*

$$\frac{A}{B} = \Big\{ a/b = \big(a_1/b_1, a_2/b_2, \dots, a_p/b_p \big) \mid$$
$$a = (a_1, a_2, \dots, a_p) \in A,\ b = \big(b_1, b_2, \dots, b_p \big) \in B \Big\},$$

where R^p_+ *denotes the nonnegative orthant of* R^p.

DEFINITION 1.1 [13]. Let $E \subset X$ be a convex set and $F : E \to 2^Y$ be a set-valued function and Ω be a pointed convex cone in Y. Then F is said to be Ω-convex on E if for every $x_1, x_2 \in E$, $t \in [0,1]$,

$$tF(x_1) + (1-t)F(x_2) \subset F\big(tx_1 + (1-t)x_2\big) + \Omega.$$

2. PREINVEX SET-VALUED FUNCTION

In this section, we define a new class of set-valued function, called a preinvex set-valued function, as a generalization of a convex set-valued function.

DEFINITION 2.1. Let E be a subset of $X, F : E \to 2^Y$ and let Ω be a pointed convex cone in Y. F is said to be Ω-preinvex on E if there exists a function η defined on $X \times X$ and with values in X such that for any $x_1, x_2 \in E$, $t \in [0, 1]$,

$$tF(x_1) + (1 - t)F(x_2) \subset F\big(x_2 + t\eta(x_1, x_2)\big) + \Omega.$$

It is implicit in the above definition that for $x_1, x_2 \in E$ and $t \in [0, 1]$, $x_2 + t\eta(x_1, x_2) \in E$. We call such a set E to be an invex set with respect to η.

This definition generalizes the class of Ω-convex set-valued functions, as in the case where F is a Ω-convex function on E; then by taking $x_1 - x_2 = \eta(x_1, x_2)$ for all $x_1, x_2 \in E$, F becomes Ω-preinvex. However, the converse need not be true, that is, a Ω-preinvex set-valued function need not be Ω-convex.

EXAMPLE 2.1. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $\Omega = \mathbb{R}^2_+$, and let $F : X \to 2^Y$ be defined as

$$F(x) = \big\{(\alpha_1, \alpha_2) \in Y \,|\, \alpha_1 + \alpha_2 \geqq -|x|\big\}.$$

Then F is not Ω-convex on X, as for $x_1 = -1 \in X$, $x_2 = 2 \in X$, $t = 5/6$, we have

$$tx_1 + (1 - t)x_2 = -(1/2)$$

$$F(x_1) = \big\{(\alpha_1, \alpha_2) \in Y \,|\, \alpha_1 + \alpha_2 \geqq -1\big\},$$

$$F(x_2) = \big\{(\alpha_1, \alpha_2) \in Y \,|\, \alpha_1 + \alpha_2 \geqq -2\big\},$$

and

$$tF(x_1) + (1 - t)F(x_2) \not\subset F\big(tx_1 + (1 - t)x_2\big) + \Omega$$

because, for $(-1, 0) \in F(x_1)$, $(0, -2) \in F(x_2)$, we have

$$t(-1, 0) + (1 - t)(0, -2) = (-5/6, -1/3) \in tF(x_1) + (1 - t)F(x_2)$$

but $(-5/6, -1/3) \notin F(-1/2) + \Omega$.

However, F is Ω-preinvex on X with respect to a function η defined on $X \times X$ as

$$\begin{aligned} \eta(x_1, x_2) &= x_1 - x_2 \qquad \text{if either } x_1 > 0, x_2 > 0 \text{ or } x_1 < 0, x_2 < 0 \\ &= x_2 - x_1 \qquad \text{if either } x_1 > 0, x_2 < 0 \text{ or } x_1 < 0, x_2 > 0. \end{aligned}$$

The following theorem characterizes the generalized Farkas–Minkowski type theorem for preinvex set-valued functions.

THEOREM 2.1. *Let E be an invex subset of X (with respect to a function $\eta : X \times X \to X$). If the set-valued function $F : E \to 2^Y$ is Ω-preinvex and $G : E \to 2^Z$ is $\wedge$-preinvex (with respect to same function η), where Ω and $\wedge$ are pointed convex cones in topological vector spaces Y and Z, respectively, then exactly one of the following statements is true*:

(i) *there exists $x \in E$ such that*

$$F(x) \cap (-\operatorname{int} \Omega) \neq \varnothing$$
$$G(x) \cap (-\operatorname{int} \wedge) \neq \varnothing$$

(ii) *there exists $(y^*, z^*) \neq (0,0)$ in $\Omega^* \times \wedge^*$ such that for every $x \in E$,*

$$\langle y^*, F(x) \rangle + \langle z^*, G(x) \rangle \geqq 0.$$

Proof. We will first show that (i) and (ii) cannot hold simultaneously. Let if possible (i) and (ii) both be true. Then by (i), there exists some $\hat{x} \in E$ such that

$$F(\hat{x}) \cap (-\operatorname{int} \Omega) \neq \varnothing$$
$$G(\hat{x}) \cap (-\operatorname{int} \wedge) \neq \varnothing,$$

i.e.,

$$\hat{y} <_{\Omega} 0 \qquad \text{for some } \hat{y} \in F(\hat{x})$$
$$\hat{z} <_{\wedge} 0 \qquad \text{for some } \hat{z} \in G(\hat{x}).$$

Now for any $(y^*, z^*) \neq (0,0)$ in $\Omega^* \times \wedge^*$, we have

$$\langle y^*, \hat{y} \rangle + \langle z^*, \hat{z} \rangle < 0$$
$$\text{for some } \hat{y} \in F(\hat{x}),\ \hat{z} \in G(\hat{x}), \text{ and some } \hat{x} \in E.$$

But this contradicts the assumption that (ii) holds for every $x \in E$.

Next, we will show that not (i) $\Rightarrow$ (ii).

Let (i) be not true.

On the lines of Lin [9], it is sufficient to show that the set A defined as

$$A = \{(y, z) \in Y \times Z \mid \text{there exists } x \in E \text{ such that for some } u \in F(x) \text{ and } v \in G(x),\ u <_{\Omega} y \text{ and } v <_{\wedge} z\}$$

is convex in $Y \times Z$ and $(0,0) \notin A$.

Let $(y_1, z_1) \in A$ and $(y_2, z_2) \in A$. Then there exists $x_1 \in E$, $x_2 \in E$ such that for some $u_1 \in F(x_1)$, $v_1 \in G(x_1)$, $u_2 \in F(x_2)$, $v_2 \in G(x_2)$

$$\left.\begin{array}{lcl} u_1 <_\Omega y_1 & \text{and} & v_1 <_\wedge z_1 \\ u_2 <_\Omega y_2 & \text{and} & v_2 <_\wedge z_2. \end{array}\right\} \tag{2.1}$$

Since $u_1 \in F(x_1)$, $u_2 \in F(x_2)$, $v_1 \in G(x_1)$, $v_2 \in G(x_2)$ therefore for $t \in [0,1]$,

$$\left.\begin{array}{l} tu_1 + (1-t)u_2 \in tF(x_1) + (1-t)F(x_2) \\ tv_1 + (1-t)v_2 \in tG(x_1) + (1-t)G(x_2). \end{array}\right\} \tag{2.2}$$

By preinvexity of F and G with respect to same η, we have

$$tF(x_1) + (1-t)F(x_2) \subset F\big(x_2 + t\eta(x_1, x_2)\big) + \Omega$$

$$tG(x_1) + (1-t)G(x_2) \subset G\big(x_2 + t\eta(x_1, x_2)\big) + \wedge.$$

Hence by virtue of (2.2), there exists $\bar{u} \in F(x_2 + t\eta(x_1, x_2))\xi \in \Omega$, and $\bar{v} \in G(x_2 + t\eta(x_1, x_2))$, $\tau \in \wedge$ such that

$$tu_1 + (1-t)u_2 = \bar{u} + \xi$$

$$tv_1 + (1-t)v_2 = \bar{v} + \tau,$$

i.e.,

$$\bar{u} = tu_1 + (1-t)u_2 - \xi \leqq_\Omega tu_1 + (1-t)u_2 <_\Omega ty_1 + (1-t)y_2$$

$$\bar{v} = tv_1 + (1-t)v_2 - \tau \leqq_\wedge tv_1 + (1-t)v_2 <_\wedge tz_1 + (1-t)z_2$$

$$\big(\text{by using } (2.1)\big).$$

This shows that there exists $\bar{x} = x_2 + t\eta(x_1, x_2) \in E$, as E is an invex set with respect to η, such that

$$\bar{u} <_\Omega ty_1 + (1-t)y_2$$

$$\bar{v} <_\wedge tz_1 + (1-t)z_2$$

for some $\bar{u} \in F(\bar{x})$, $\bar{v} \in G(\bar{x})$.

Hence,

$$t(y_1, z_1) + (1-t)(y_2, z_2) \in A, \qquad t \in [0,1].$$

COROLLARY 2.2. *If in Theorem* 2.1, *we assume further that there exists* $\hat{x} \in E$ *such that* $G(\hat{x}) \cap (-\text{int } \wedge) \neq \emptyset$ *then* $y^* \neq 0$.

The proof follows along similar lines as that of Corollary 3.4 in [9].

3. LAGRANGIAN DUALITY

We are concerned with the following problem.

Let R^p and R^m be ordered by pointed convex cones Ω and Λ, respectively, with int $\Omega \neq \emptyset$, int $\Lambda \neq \emptyset$.

Let $E_0 \subset R^n$, $F : E_0 \to 2^{R^p}$, $G : E_0 \to 2^{R^p}$, and $H : E_0 \to 2^{R^m}$ be set-valued maps defined on E_0. Then the problem is

$$\text{W-Min}_{\Omega} \frac{F(x)}{G(x)} = \left[\frac{F_1(x)}{G_1(x)}, \dots, \frac{F_p(x)}{G_p(x)} \right] \tag{FP}$$

$$\text{subject to } H(x) \cap (-\Lambda) \neq \emptyset, \qquad x \in E_0.$$

We assume that for each $x \in E_0$, and for each $i = 1, 2, \dots, p$,

$$F_i(x) \subset R_+ \qquad \text{and} \qquad G_i(x) \subset \text{int } R_+.$$

Let $E = \{x \in E_0 \mid H(x) \cap (-\Lambda) \neq \emptyset\}$ be the feasible set of (FP).

In this section, all the relations, until otherwise stated, are with respect to the cone order Ω.

DEFINITION 3.1. A point $x^* \in E$ is said to be a weakly efficient solution (or an efficient solution) for (FP) if there exists $y^* \in F(x^*)$ and $z^* \in G(x^*)$ such that

$$\frac{y^*}{z^*} \in \text{W-Min} \bigcup_{x \in E} \frac{F(x)}{G(x)}$$

$$\left(\text{or respectively, } \frac{y^*}{z^*} \in \text{Min} \bigcup_{x \in E} \frac{F(x)}{G(x)} \right).$$

On using the parametric approach, we consider the following optimization problem

$$\text{W-Min}\left[F_1(x) - \lambda_1 G_1(x), \dots, F_p(x) - \lambda_p G_p(x) \right]$$

$$\text{subject to } x \in E, \tag{P$_\lambda$}$$

$$\lambda = (\lambda_1, \dots, \lambda_p) \in R^p_+.$$

The following lemma for set-valued functions can easily be proved on the lines of the corresponding result proved for real valued functions (in terms of cones) by Chandra *et al.* [3].

LEMMA 3.1. *Let $x^* \in E$. Then x^* is weakly efficient for* (FP) *with $y^*/z^* \in F(x^*)/G(x^*)$ as a weakly efficient value of* (FP) *if and only if x^* is weakly efficient for* $(\text{P})_{\lambda^*}$ *where $\lambda^* = y^*/z^*$ and $0 \in F(x^*) - \lambda^* G(x^*)$ as a weakly efficient value of* $(\text{P})_{\lambda^*}$.

Let $L = B^+(R^m, R^p)$ be the set of all bounded continuous linear functions $s: R^m \to R^p$ such that $s(\Lambda) \subset \Omega$ (i.e., s is nonnegative with respect to cones Ω and Λ).

The weak dual problem associated with (FP) is defined as

$$\text{W-Max} \bigcup_{s \in L} \left[\text{W-Min} \left(\frac{F}{G} + \frac{s(H)}{G} \right)(E) \right].$$

Letting

$$\begin{aligned} \psi(s) &= \text{W-Min} \left(\frac{F}{G} + \frac{s(H)}{G} \right)(E) \\ &= \text{W-Min} \bigcup_{x \in E} \left(\frac{F}{G} + \frac{s(H)}{G} \right)(x) \end{aligned}$$

the dual problem can be rewritten as

$$\text{W-Max} \bigcup_{s \in L} \psi(s), \tag{DFP}$$

i.e., we have to determine the weak maximal elements of $\bigcup_{s \in L} \psi(s)$ with respect to the cone Ω.

THEOREM 3.2 (Weak Duality Theorem). *Suppose x_0 is feasible for* (FP) *and s_0 is feasible for* (DFP). *Then for any $y_0 \in F(x_0)$. $z_0 \in G(x_0)$, and $v_0 \in \psi(s_0)$,*

$$\frac{y_0}{z_0} \nless v_0.$$

Proof. Suppose, to the contrary, that there exists some $y_0 \in F(x_0)$, $z_0 \in G(x_0)$, and $v_0 \in \psi(s_0)$ for which

$$\frac{y_0}{z_0} < v_0$$

i.e.,

$$v_0 - \frac{y_0}{z_0} \in \text{int}\, \Omega. \tag{3.1}$$

Since, $v_0 \in \psi(s_0) = \text{W-Min} \bigcup_{x \in E} (F/G + s(H)/G)(x)$, hence for any $x_1 \in E$ with $y_1 \in F(x_1)$, $z_1 \in G(x_1)$, and $w_1 \in H(x_1)$, we have

$$\frac{y_1}{z_1} + \frac{s_0(w_1)}{z_1} \nless v_0. \tag{3.2}$$

But as $x_0 \in E$, so in particular, for $y_0 \in F(x_0)$, $z_0 \in G(x_0)$, and $w_0 \in H(x_0) \cap (-\Lambda)$ (this set is non-empty because of the feasibility of x_0 for (FP)), we have from (3.2),

$$\frac{y_0}{z_0} + \frac{s_0(w_0)}{z_0} \not< v_0. \tag{3.3}$$

Now,

$$w_0 \in H(x_0) \cap (-\Lambda) \Rightarrow w_0 \in -\Lambda$$

which implies

$$s_0(w_0) \leqq 0 \qquad (\text{since } s_0 \in L).$$

Also $z_0 > 0$ and Ω is a convex cone hence

$$\frac{s_0(w_0)}{z_0} \leqq 0,$$

i.e.,

$$-\frac{s_0(w_0)}{z_0} \in \Omega. \tag{3.4}$$

From (3.1) and (3.4), we get

$$v_0 - \left(\frac{y_0}{z_0} + \frac{s_0(w_0)}{z_0} \right) \in \operatorname{int} \Omega$$

on account of Lemma 1.1.

Hence,

$$\frac{y_0}{z_0} + \frac{s_0(w_0)}{z_0} < v_0$$

which contradicts (3.3). The result follows.

THEOREM 3.3 (Strong Duality Theorem). *Suppose that Ω and Λ are pointed convex cones in R^p and R^m, respectively, with* $\operatorname{int} \Omega \neq \emptyset$, $\operatorname{int} \Lambda \neq \emptyset$. *Let E_0 be an invex subset of R^n with respect to function $\eta : R^n \times R^n \to R^n$. Let F, $-G$ be Ω-preinvex and H be Λ-preinvex on E_0 with respect to same η and further assume that there exist $\hat{x} \in E_0$ such that $H(\hat{x}) \cap (-\operatorname{int} \Lambda) \neq \emptyset$. If x_0 is a weakly efficient solution for* (FP) *then there exists $y_0 \in F(x_0)$, $z_0 \in G(x_0)$ such that*

$$\frac{y_0}{z_0} \in \text{W-Max} \bigcup_{s \in L} \psi(s).$$

Proof. Since x_0 is a weakly efficient solution for (FP) hence there exists $y_0 \in F(x_0)$, $z_0 \in G(x_0)$ such that

$$\frac{y_0}{z_0} \in \text{W-Min} \bigcup_{x \in E} \frac{F}{G}(x).$$

Therefore by Lemma 3.1, x_0 is a weakly efficient solution for $(\mathrm{P})_{\lambda_0}$ where $\lambda_0 = y_0/z_0$ with $0 \in F(x_0) - \lambda_0 G(x_0)$ as a weakly efficient value of $(\mathrm{P})_{\lambda_0}$.

This shows that the system

$$\left.\begin{aligned} [F(x) - \lambda_0 G(x)] \cap (-\operatorname{int} \Omega) \neq \varnothing \\ H(x) \cap (-\operatorname{int} \Lambda) \neq \varnothing \end{aligned}\right\} \tag{3.5}$$

has no solution.

Since $F, -G$ are Ω-preinvex with respect to η and $\lambda_0 \geqq 0$ hence $F - \lambda_0 G$ is Ω-preinvex with respect to η on E_0. Therefore by Theorem 2.1 and Corollary 2.2, there exists $(u^*, v^*) \in \Omega^* \times \Lambda^*$ such that for all $x \in E_0$

$$\langle u^*, F(x) - \lambda_0 G(x) \rangle + \langle v^*, H(x) \rangle \geqq 0 \tag{3.6}$$

$$(u^*, v^*) \neq (0,0), \qquad u^* \neq 0. \tag{3.7}$$

Let $x = x_0$ in (3.6). We get

$$\langle u^*, F(x_0) - \lambda_0 G(x_0) \rangle + \langle v^*, H(x_0) \rangle \geqq 0. \tag{3.8}$$

Since $y_0 \in F(x_0)$ and $z_0 \in G(x_0)$, hence, we have

$$\langle u^*, y_0 - \lambda_0 z_0 \rangle + \langle v^*, H(x_0) \rangle \geqq 0$$

giving

$$\langle v^*, H(x_0) \rangle \geqq 0 \qquad \left(\text{by using } \lambda_0 = \frac{y_0}{z_0}\right). \tag{3.9}$$

Also x_0 is feasible for (FP) therefore

$$H(x_0) \cap (-\Lambda) \neq \varnothing.$$

Choose $w_0 \in H(x_0) \cap (-\Lambda)$; then

$$\langle v^*, w_0 \rangle \geqq 0 \qquad (\text{by using } (3.9)). \tag{3.10}$$

Moreover $w_0 \in -\Lambda$ and $v^* \in \Lambda^*$, therefore

$$\langle v^*, w_0 \rangle \leqq 0. \tag{3.11}$$

Combining (3.10) and (3.11), we get

$$\langle v^*, w_0 \rangle = 0. \tag{3.12}$$

Since $u^* \in \Omega^*$, $u^* \neq 0$, choose $\xi \in \Omega$ such that $\langle u^*, \xi \rangle = 1$. Define $s_0 : R^m \to R^p$ as $s_0(w) = \langle v^*, w \rangle \xi$. Then clearly $s_0 \in L$.

Further,

$$s_0(w_0) = 0 \qquad \text{and} \qquad \langle u^*, s_0(w) \rangle = \langle v^*, w \rangle. \tag{3.13}$$

We will show that $y_0/z_0 \in \psi(s_0)$. To the contrary, let if possible, $y_0/z_0 \notin \psi(s_0)$.

This implies that there exists $\hat{x} \in E$ such that for some $\hat{y} \in F(\hat{x})$, $\hat{z} \in G(\hat{x})$, and $\hat{w} \in H(\hat{x})$,

$$\frac{\hat{y}}{\hat{z}} + \frac{s_0(\hat{w})}{\hat{z}} < \frac{y_0}{z_0}$$

giving

$$\frac{y_0}{z_0} - \left(\frac{\hat{y}}{\hat{z}} + \frac{s_0(\hat{w})}{\hat{z}} \right) \in \operatorname{int} \Omega,$$

hence,

$$\left\langle u^*, \frac{y_0}{z_0} - \left(\frac{\hat{y}}{\hat{z}} + \frac{s_0(\hat{w})}{\hat{z}} \right) \right\rangle > 0 \qquad \text{as } u^* \in \Omega^*, u^* \neq 0.$$

Therefore, we have

$$\langle u^*, \hat{y} - \lambda_0 \hat{z} \rangle + \langle u^* \cdot s_0(\hat{w}) \rangle < 0 \qquad \left(\text{using the fact that } \lambda_0 = \frac{y_0}{z_0} \right)$$

which in view of (3.13) gives

$$\langle u^*, \hat{y} - \lambda_0 \hat{z} \rangle + \langle v^*, \hat{w} \rangle < 0.$$

But this contradicts (3.8).

Hence $y_0/z_0 \in \psi(s_0)$.

Next, we have to show that

$$\frac{y_0}{z_0} \in \text{W-Max} \bigcup_{s \in L} \psi(s).$$

Assume to the contrary. Then there exists $s_1 \in L$ such that for some $v_1 \in \psi(s_1)$, we have

$$\frac{y_0}{z_0} < v_1 \Rightarrow v_1 - \frac{y_0}{z_0} \in \operatorname{int} \Omega. \tag{3.14}$$

Now, x_0 is feasible for (FP) hence $H(x_0) \cap (-\Lambda) \neq \varnothing$. Therefore for $w_0 \in H(x_0) \cap (-\Lambda)$, we have $s_1(w_0) \leqq 0$. Since $z_0 > 0$ and Λ is a convex cone, therefore

$$\frac{s_1(w_0)}{z_0} \leqq 0$$

i.e.,

$$-\frac{s_1(w_0)}{z_0} \in \Omega. \tag{3.15}$$

From (3.14) and (3.15), we get

$$v_1 - \left(\frac{y_0}{z_0} + \frac{s_1(w_0)}{z_0} \right) \in \operatorname{int} \Omega$$

on account of Lemma 1.1.

Hence, we have

$$\frac{y_0}{z_0} + \frac{s_1(w_0)}{z_0} < v_1. \tag{3.16}$$

But (3.16) is a contradiction to the fact that

$$\frac{y_0}{z_0} + \frac{s_1(w_0)}{z_0} \in \bigcup_{x \in E} \left[\frac{F}{G} + \frac{s_1(H)}{G} \right](x)$$

and

$$v_1 \in \psi(s_1) = \text{W-Min} \bigcup_{x \in E} \left(\frac{F}{G} + \frac{s_1(H)}{G} \right)(x).$$

Therefore s_0 is a weakly efficient solution for (DFP) with $y_0/z_0 \in \psi(s_0)$ as a weakly efficient value of (DFP).

THEOREM 3.4. *Suppose the assumptions of the previous theorem* (*Strong Duality Theorem*) *are satisfied. Let x_0 be a weakly efficient solution for* (FP) *and $v_0 \in$ W-Max $\bigcup_{s \in L} \psi(s)$. Then for every $y_0 \in F(x_0)$, $z_0 \in G(x_0)$, $y_0/z_0 \nless v_0$. Moreover there exists $\hat{y}_0 \in F(x_0)$ and $\hat{z}_0 \in G(x_0)$ for which $v_0 \nless \hat{y}_0/\hat{z}_0$.*

Proof. Since x_0 is a weakly efficient solution for (FP) hence it is feasible for (FP) and $v_0 \in$ W-Max $\bigcup_{s \in L} \psi(s)$. Therefore there exists $s_0 \in L$ such that $v_0 \in \psi(s_0)$.

Hence by Theorem 3.2 (Weak Duality Theorem) for every $y_0 \in F(x_0)$ and $z_0 \in G(x_0)$,

$$\frac{y_0}{z_0} \nless v_0.$$

On using Theorem 3.3 (Strong Duality Theorem), we get an existence of $\hat{y}_0 \in F(x_0)$, $\hat{z}_0 \in G(x_0)$, and $s_0 \in L$ such that

$$\frac{\hat{y}_0}{\hat{z}_0} \in \psi(s_0).$$

But $v_0 \in$ W-Max $\bigcup_{s \in L} \psi(s)$, hence $v_0 \nless \hat{y}_0 / \hat{z}_0$.

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