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Existence and stability of periodic solution of a Lotka–Volterra predator–prey model with state dependent impulsive effects[☆]

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ABSTRACT

According to biological and chemical control strategy for pest, we investigate the dynamic behavior of a Lotka–Volterra predator–prey state-dependent impulsive system by releasing natural enemies and spraying pesticide at different thresholds. By using Poincaré map and the properties of the Lambert *W* function, we prove that the sufficient conditions for the existence and stability of semi-trivial solution and positive periodic solution. Numerical simulations are carried out to illustrate the feasibility of our main results.

1. Introduction

The dynamic analysis between predators and their preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. There are many mathematical models for predator–prey behavior. One of the most basic and important models is the Lotka–Volterra type systems. In the last decades, considerable work on the permanence, the extinction and the global asymptotic stability of autonomous or nonautonomous Lotka–Volterra type systems have been studied extensively, for example [2,3,7,12,30] and the references therein. In addition to these, the books by Takeuchi [3], Gopalsamy [15] and Kuang [17] are good sources for dynamical behavior of Lotka–Volterra systems.

In the natural world, however, the ecological system is often affected by environmental changes and other human activities, such as vaccination, chemotherapeutic treatment of disease, chemostat, birth pulse, control and optimization, etc. These discrete nature of human actions or environmental changes lead to population densities changing very rapidly in a short space of time. These short-time perturbations are often assumed to be in the form of impulses in the modeling process. Various population models – characterized by the fact that a sudden change in their state and process under depends on their earlier history at each moment of time – can be expressed by impulsive differential equations (IDES), which are found in almost every domain of applied science. The theory and applications of IDES are emerging as an important area of investigation, since it is far richer than the corresponding theory of nonimpulsive differential equations.

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In recent years, there has been a significant development in theory of IDES, especially in the area in which impulses are at fixed times; these investigations are mainly focused on the basic theories. For example [10] established the conditions for the existence and stability of periodic solution of population dynamic with birth pulses, [9,16,19,22,27–29,32] studies the dynamical behavior of epidemic model with pulse vaccination, and [18,20,21,24–26,31,33,34] researched the population models with impulsive effects and devoted to the criteria for the existence, stability, orbital stability of periodic solutions.

In addition, impulsive state feedback control strategy is widely used in real life problems. In some ecological systems, however, we note that the control measures (by catching, poisoning or releasing the natural enemy, etc) are taken only when the amount of species reaches a threshold, rather than the usual impulsive fixed time control strategy. Some recent studies work on IDES with state-dependent impulsive effects; see [5,11,13,14,23]. These work assume the impulsive effects (poisoning the prey and releasing the predator) have the same threshold, i.e. at the same time. But they ignored the side effects of pesticide on natural enemies, and they assumed the time of spraying pesticide and releasing natural enemies is the same. It assumption is unreasonable.

In this paper, according to biological and chemical control strategy for pest, we construct a Lotka–Volterra predator–prey state-dependent impulsive system by releasing natural enemies and spraying pesticide at different thresholds. The system can be written as

$$\begin{cases} \left.\begin{aligned} \frac{dx(t)}{dt} &= x(t)\left[b_1 - a_{11}x(t) - a_{12}y(t)\right] \\ \frac{dy(t)}{dt} &= y(t)\left[-b_2 + a_{21}x(t)\right] \end{aligned}\right\} & x \neq h_1, h_2, \\ \left.\begin{aligned} \Delta x(t) &= 0 \\ \Delta y(t) &= y(t^+) - y(t) = \alpha \end{aligned}\right\} & x = h_1, \\ \left.\begin{aligned} \Delta x(t) &= x(t^+) - x(t) = -px(t) \\ \Delta y(t) &= y(t^+) - y(t) = -qy(t) \end{aligned}\right\} & x = h_2, \end{cases} \tag{1.1}$$

where x and y represent the population densities at time t; b_1, b_2, a_{12} and a_{21} are positive constants, $a_{11} \geq 0$, $\alpha \geq 0$, $p, q \in (0, 1), h_1 > 0, h_2 > 0$ and $(1-p)h_2 < h_1$. When the amount of the prey reaches the threshold h_1 at time t_{h_1}, releasing natural enemies (the predator) and amount of predator abruptly turn to $y(t_{h_2}) + \alpha$. Further, when the amount of the prey reaches the threshold h_2 at time t_{h_2}, spraying pesticide and amount of prey and predator abruptly turn to $(1-p)x(t_{h_2})$ and $(1-q)y(t_{h_2})$, respectively.

This paper is organized as follows. In the next section, as preliminaries we present some basic definitions, two Poincaré maps, Lambert W function and a important lemma. In Section 3, we state and prove a general criterion for the semi-trivial periodic solution of system (1.1). The sufficient conditions for the existence and stability of positive periodic solutions of system (1.1) are obtained in Section 4. In the last section, some specific examples are given to illustrate our results.

2. Preliminaries

In system (1.1), when parameter $p = q = \alpha = 0$ we obtain the following system without impulsive effect

$$\begin{cases} \dfrac{dx(t)}{dt} = x(t)\left[b_1 - a_{11}x(t) - a_{12}y(t)\right], \\ \dfrac{dy(t)}{dt} = y(t)\left[-b_2 + a_{21}x(t)\right]. \end{cases} \tag{2.1}$$

The dynamic behaviors for system (2.1) are studied by considerable investigators. Throughout this paper, we assume that system (2.1) has a unique positive equilibrium.

By the biological background of system (1.1), we only consider system (1.1) in the biological meaning region $D = \{(x, y) : x \geq 0, y \geq 0\}$. Obviously, the global existence and uniqueness of solutions of system (1.1) are guaranteed by the smoothness properties of f, which denotes the mapping defined by right-side of system (1.1) – for more details see [1,4].

Set $R = (-\infty, \infty)$. Firstly, we give the notion of the distance between a point and a set. It is defined as follows. Let $S \in R^2 = \{(x, y) : x \in R, y \in R\}$ be an arbitrary set and $P \in R^2$ be an arbitrary point. Then the distance between the point P and the set S is denoted by

$$d(P, S) = \inf_{P_0 \in S} |P - P_0|.$$

Let $z(t) = (x(t), y(t))$ be any solution of (1.1). Next, we define the positive orbit through the point $z_0 \in R^2_+ = \{(x, y) : x \geq 0, y \geq 0\}$ for $t \geq t_0$ as

$$O^+(z_0, t_0) = \{z \in R^2_+ : z = z(t), t \geq t_0, z(t_0) = z_0\}.$$

We introduce the following definitions to simplify the statements that will follow later.

Definition 1 (*Orbital Stability*). $z^*(t)$ is said to be orbitally stable if, given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for any other solution, $z(t)$, of system (1.1) satisfying $|z^*(t_0) - z(t_0)| < \delta$, then $d(z(t), O^+(z_0, t_0)) < \varepsilon$ for $t > t_0$.

Definition 2 (*Asymptotic Orbital Stability*). $z^*(t)$ is said to be asymptotically orbitally stable if it is orbitally stable and for any other solution, $z(t)$, of system (1.1), there exists a constant $\eta > 0$ such that, if $|z^*(t_0) - z(t_0)| < \eta$, then $\lim_{t\to\infty} d(z(t), O^+(z_0, t_0)) = 0$.

To discuss the dynamics of system (1.1), we define three cross-sections to the vector field (1.1) by $\sum^p = \{(x, y) : x = (1-p)h_2, y \geq 0\}$, $\sum^{h_1} = \{(x, y) : x = h_1, y \geq 0\}$ and $\sum^{h_2} = \{(x, y) : x = h_2, y \geq 0\}$. Now, we construction two Poincaré maps of $\sum^{h_2}$. First, we define the Poincaré maps of $\sum^{h_2}$ with $\alpha = 0$. Suppose system (1.1) has a positive T periodic solution $z(t) = (\varphi(t), \psi(t))$ with the initial condition $z_0 = z(0) = ((1-p)h_2, y_0)$, where $y_0 > 0$. Then, the periodic trajectory $O^+(z_0, 0)$ starts from the $A^+((1-p)h_2, y_0)$ on $\sum^p$ and intersects $\sum^{h_2}$ at the point $A(h_2, y_1)$. At the point A, the trajectory of (1.1) is subjected by impulsive effects to jumps to the point A^+ again. Thus

$$\varphi(0) = (1-p)h_2, \qquad \psi(0) = y_0, \qquad \varphi(T) = h_2, \quad \text{and} \quad \psi(T) = y_1 = \frac{y_0}{1-q}.$$

Now, we consider another solution $\widetilde{z}(t) = (\widetilde{\varphi}(t), \widetilde{\psi}(t))$ of small-amplitude perturbation of the periodic solution $z(t)$ with initial condition $\widetilde{z}_0 = \widetilde{z}(0) = ((1-p)h_2, \widetilde{y}_0)$. Suppose the trajectory $O^+(\widetilde{z}_0, 0)$ which starts form $A_0((1-p)h_2, \widetilde{y}_0)$ first intersects $\sum^{h_2}$ at the point $A_1(h_2, \widetilde{y}_1)$ when $t = T + \delta t$ and then jumps to the point $A_1^+((1-p)h_2, \widetilde{y}_2)$ on $\sum^p$. Then, we have

$$\widetilde{\varphi}(0) = (1-p)h_2, \qquad \widetilde{\psi}(0) = \widetilde{y}_0, \quad \widetilde{\varphi}(T + \delta t) = h_2 \quad \text{and} \quad \widetilde{\psi}(T + \delta t) = \widetilde{y}_1.$$

Let $u(t) = \widetilde{\varphi}(t) - \varphi(t)$ and $v(t) = \widetilde{\psi}(t) - \psi(t)$, then $u_0 = u(0) = \widetilde{\varphi}(0) - \varphi(0) = 0$ and $v_0 = v(0) = \widetilde{\psi}(0) - \psi(0)$. Let $v_1 = \widetilde{y}_2 - y_0$ and $v_0^* = \widetilde{y}_1 - y_1$. It is known that for $0 < t < T$, the variables $u(t)$ and $v(t)$ are described by the relation

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + o(u_0^2 + v_0^2) = \Phi(t) \begin{pmatrix} 0 \\ v_0 \end{pmatrix} + o \begin{pmatrix} 0 \\ v_0^2 \end{pmatrix}, \tag{2.2}$$

where the fundamental solution matrix $\Phi(t)$ satisfies the matrix equation

$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = \begin{pmatrix} b_1 - 2a_{11}\varphi(t) - a_{12}\psi(t) & -a_{12}\varphi(t) \\ a_{21}\psi(t) & -b_2 + a_{21}\varphi(t) \end{pmatrix} \Phi(t) \tag{2.3}$$

with $\Phi(0) = I$ (the unit matrix). Set $g_1(t) = \varphi(t)[b_1 - a_{11}\varphi(t) - a_{12}\psi(t)]$ and $g_2(t) = \psi(t)[-b_2 + a_{21}\varphi(t)]$. We can express the perturbed trajectory in a first-order Taylor expansion

$$\begin{cases} \widetilde{\varphi}(T + \delta t) \approx \varphi(T) + u(T) + g_1(T)\delta t, \\ \widetilde{\psi}(T + \delta t) \approx \psi(T) + v(T) + g_2(T)\delta t. \end{cases}$$

From $\widetilde{\varphi}(T + \delta t) = \varphi(T) = h$, we have

$$\delta t = -\frac{u(T)}{g_1(T)} \quad \text{and} \quad v_0^* = \widetilde{y}_1 - y_1 = v(T) - \frac{g_2(T)u(T)}{g_1(T)}.$$

In view of $\widetilde{y}_2 = (1-q)\widetilde{y}_1$ and $\widetilde{y}_2 - y_0 = (1-q)(\widetilde{y}_1 - y_1)$, thus $v_1 = (1-q)v_0^*$. So, we defined the Poincaré map of $\sum^p$ as follows

$$v_1 = P_1(q, v_0) = (1-q)\left[v(T) - \frac{g_2(T)u(T)}{g_1(T)}\right], \tag{2.4}$$

where $u(T)$ and $v(T)$ are calculated according to (2.2).

Now, we consider the another Poincaré map with $\alpha \in (0, \alpha^*)$, where α^* is a positive constant. Suppose that the trajectory $O^+(C_n, t_n)$ starts from the point $C_n(h_2, y_n)$ on $\sum^{h_2}$, then it jumps to the point $A_{n+1}((1-p)h_2, (1-q)y_n)$ on $\sum^p$ due to the impulsive effects $\Delta x(t) = -px(t)$ and $\Delta y(t) = -qy(t)$, and then reaches the point $B_{n+1}(h_1, \widetilde{y}_{n+1})$ on the section $\sum^{h_1}$. Further, the point $B_{n+1}(h_1, \widetilde{y}_{n+1})$ jumps to the point $B_{n+1}^+(h_1, \widetilde{y}_{n+1} + \alpha)$ on $\sum^{h_1}$ due to the impulsive effects $\Delta x(t) = 0$ and $\Delta y(t) = \alpha$, and then reaches the point $C_{n+1}(h_2, y_{n+1})$ on $\sum^{h_2}$, where y_{n+1} is decided by the parameters q, α and y_n. Therefore, we defined the Poincaré map of $\sum^{h_2}$ as follows

$$y_{n+1} = P_2(q, \alpha, y_n). \tag{2.5}$$

Let $z(t) = (x(t), y(t))$ be a solution of system (1.1) with initial conditions $z_0 = z(t_0) = ((1-p)h_2, y_0) \in R_+^2$. This trajectory $O^+(z_0, t_0)$ starts from the point $E_0((1-p)h_2, y_0)$ first intersects $\sum^{h_1}$ at the point $F_0(h_1, \widehat{y}_0)$, next jumps to the point $F_0^+(h_1, \widehat{y}_0 + \alpha)$ on $\sum^{h_1}$ due to the impulsive effects, and then reaches the point $G_1(h_2, \widetilde{y}_0)$ on $\sum^{h_2}$. At the sate G_1, the trajectory of (1.1) is subjected by impulsive effects to jumps to the point $E_1((1-p)h_2, y_1)$ on $\sum^p$ again. Repeating the above process, we have two impulsive points' sequences $\{E_k((1-p)h_2, y_k)\}$ and $\{G_k(h, \widetilde{y}_k)\}(k = 0, 1, 2, \ldots)$. We notice that the coordinates satisfy the relation $y_k = (1-q)\widetilde{y}_{k-1}(k = 1, 2, \ldots)$.

Definition 3. A trajectory $O^+(z_0, t_0)$ of system (1.1) is said to be order-k periodic if there exist positive integer $k \geq 1$ such that k is the smallest integer for $y_0 = y_k$.

Definition 4. A solution $z(t) = (x(t), y(t))$ of system (1.1) is said a semi-trivial solution if its a component is zero and another is nonzero.

Definition 5 (*Corless et al. [6]*). The Lambert W function is defined to be a multiple valued inverse of the function $f : Z \mapsto ze^z$ satisfying

$$W(z)\exp(W(z)) = z.$$

The Lambert W function $W(z)$ has two branches for $z \geq -1/e$, here we define the inverse function of $W(z)$ restricted to the interval $[-1, \infty)$ to be $W_0(z)$ and the inverse function of $W(z)$ restricted to the interval $(-\infty, -1]$ to be $W_{-1}(z)$. It is clear that the branch $W_0(z)$ satisfies $-1 < W_0(z) < 0$ for $z \in (-\exp(-1), 0)$ and its derivative satisfies

$$W_0'(z) = \frac{W_0(z)}{z(1 + W_0(z))}. \tag{2.6}$$

This follows from the Lagrange inversion theorem (see e.g. [8]), which gives the series expansion below for $W_0(z)$

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n.$$

For more details of the concepts and properties of the Lambert W function, see Corless et al. [6] and Waldvogel [7].

Lemma 1. *if $z > 0$, then $1 - z + \ln z \leq 0$, where the equals sign holds for $z = 1$.*

3. Existence and stability of semi-trivial periodic solution with $\alpha = 0$

Let $y(t) = 0$ for $t \in [0, \infty)$, then from system (1.1) we have

$$\begin{cases} \dfrac{dx(t)}{dt} = [b_1 - a_{11}x(t)]x(t), & x \neq h_2, \\ \Delta x = x(t^+) - x(t) = -px, & x = h_2. \end{cases}$$

Set $x_0 = x(0) = (1 - p)h_2$, then the solution of equation $dx(t)/dt = [b_1 - a_{11}x(t)]x(t)$ is $x(t) = [b_1\exp(a_{11}t)]/[a_{11}\exp(b_1 t) + c]$, where $c = [b_1 - (1 - p)a_{11}h_2]/(1 - p)h_2$. Let $T = b_1^{-1}\ln\{[b_1 - (1 - p)a_{11}h_2]/(1 - p)(b_1 - a_{11}h_2)\}$, then $x(T) = h_2$ and $x(T^+) = x_0$. This means that system (1.1) with $\alpha = 0$ has the following semi-trivial periodic solution for $(k - 1)T < t \leq kT (k = 1, 2, \ldots)$,

$$\begin{cases} \varphi(t) = \dfrac{(1 - p)b_1h_2\exp[b_1(t - (k - 1)T)]}{b_1 + (1 - p)a_{11}h_2\exp[b_1(t - (k - 1)T)] - (1 - p)a_{11}h_2}, \\ \psi(t) = 0. \end{cases} \tag{3.1}$$

On the stability of this semi-trivial periodic solution, we have the following result.

Theorem 1. *Suppose that the following condition holds,*

$$0 < \lambda = (1 - q)(1 - p)^{\frac{b_2}{b_1}} \left[\frac{b_1 - (1 - p)a_{11}h_2}{b_1 - a_{11}h_2}\right]^{\frac{a_{21}}{a_{11}} - \frac{b_2}{b_1}} < 1. \tag{3.2}$$

Then, (3.1) be a stable semi-trivial periodic solution of system (1.1) with $\alpha = 0$.

Proof. In fact, from $\psi(t) = 0$ and (2.3), we have

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} b_1 - 2a_{11}\varphi(t) & -a_{12}\varphi(t) \\ 0 & -b_2 + a_{21}\varphi(t) \end{pmatrix} \Phi(t), \qquad M(0) = I. \tag{3.3}$$

Let

$$\Phi(t) = \begin{pmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{pmatrix}.$$

Then, by (3.3) we can obtain that

$$\begin{cases} \dfrac{d\phi_{11}(t)}{dt} = [b_1 - 2a_{11}\varphi(t)]\phi_{11}(t) - a_{12}\varphi(t)\phi_{21}(t), & \phi_{11}(0) = 1, \\ \dfrac{d\phi_{12}(t)}{dt} = [b_1 - 2a_{11}\varphi(t)]\phi_{12}(t) - a_{12}\varphi(t)\phi_{22}(t), & \phi_{12}(0) = 0, \\ \dfrac{d\phi_{21}(t)}{dt} = [-b_2 + a_{21}\varphi(t)]\phi_{21}(t), & \phi_{21}(0) = 0, \\ \dfrac{d\phi_{22}(t)}{dt} = [-b_2 + a_{21}\varphi(t)]\phi_{22}(t), & \phi_{22}(0) = 1, \end{cases} \tag{3.4}$$

for $0 < t < T$.

Let $\widetilde{z}(t) = (\widetilde{\varphi}(t), \widetilde{\psi}(t))$ be any positive solution of system (1.1) with the initial condition $\widetilde{z}(0) = ((1-p)h_2, y_0)(y_0 > 0)$. Note that $u_0 = 0$ and $g_2(T) = 0$, from (2.2) and (2.4), we have

$$\begin{aligned} v_1 &= (1-q)\left[v(T) - \frac{g_2(T)u(T)}{g_1(T)}\right] = (1-q)v(T) \\ &= (1-q)[\phi_{21}(T)u_0 + \phi_{22}(T)v_0] = (1-q)\phi_{22}(T)v_0. \end{aligned}$$

On the other hand, from the fourth equation of (3.4), we obtain that

$$\begin{aligned} \phi_{22}(t) &= c_1 \exp\left\{\int \left[-b_2 + \frac{(1-p)a_{21}b_1h_2\exp(b_1t)}{b_1 + (1-p)a_{11}h_2\exp(b_1t) - (1-p)a_{11}h_2}\right] dt\right\} \\ &= c_1[b_1 - (1-p)a_{11}h_2 + (1-p)a_{11}h_2\exp(b_1t)]^{\frac{a_{21}}{a_{11}}}\exp(-b_2t), \end{aligned}$$

where $c_1 = b_1^{-(a_{21}/a_{11})}$. Further, from $T = b_1^{-1}\ln\{[b_1 - (1-p)a_{11}h_2]/[(1-p)(b_1 - a_{11}h_2)]\}$, we have that

$$\phi_{22}(T) = b_1^{-\frac{a_{21}}{a_{11}}}\left[\frac{b_1(b_1 - (1-p)a_{11}h_2)}{b_1 - a_{11}h_2}\right]^{\frac{a_{21}}{a_{11}}}\left[\frac{b_1 - (1-p)a_{11}h_2}{(1-p)(b_1 - a_{11}h_2)}\right]^{-\frac{b_2}{b_1}}.$$

Therefore,

$$\begin{aligned} v_1 &= (1-q)\phi_{22}(T)v_0 \\ &= (1-q)v_0 b_1^{-\frac{a_{21}}{a_{11}}}\left[\frac{b_1(b_1 - (1-p)a_{11}h_2)}{b_1 - a_{11}h_2}\right]^{\frac{a_{21}}{a_{11}}}\left[\frac{b_1 - (1-p)a_{11}h_2}{(1-p)(b_1 - a_{11}h_2)}\right]^{-\frac{b_2}{b_1}}. \end{aligned}$$

Note that $y_0 = 0$ is a fixed point of $P_1(q, y_k)$ and

$$D_{v_0}P_1(q, 0) = (1-q)\,(1-p)^{\frac{b_2}{b_1}}\left[\frac{b_1 - (1-p)a_{11}h_2}{b_1 - a_{11}h_2}\right]^{\frac{a_{21}}{a_{11}} - \frac{b_2}{b_1}}.$$

If (3.2) holds, then $0 < D_{v_0}P_1(q, 0) < 1$. Thus, (3.1) be a stable semi-trivial periodic solution of system (1.1) with $\alpha = 0$. This completes the proof. □

4. Existence and stability of positive periodic solutions

In this subsection, we give the sufficient conditions for the existence and stability of positive periodic solution of system (1.1) with $a_{11} = 0$, in the cases of $h_2 \le (b_2/a_{21})$ and $h_2 > (b_2/a_{21}) > h_1$. For any two points (x_1, y_1) and (x_2, y_2) on a trajectory of continuous system (2.1) with $a_{11} = 0$, we have

$$b_1 \ln\left(\frac{y_2}{y_1}\right) - a_{12}(y_2 - y_1) = a_{21}(x_2 - x_1) - b_2 \ln\left(\frac{x_2}{x_1}\right). \tag{4.1}$$

4.1. The case of $h_2 \le (b_2/a_{21})$

On the existence of positive periodic solution of system (1.1) with $a_{11} = 0$, we have the following theorem.

Theorem 2. *For any $p, q \in (0, 1)$, if*

$$0 < \alpha < \frac{b_1}{a_{12}}W_0\left(-\exp\left(\frac{L^* - b_1}{b_1}\right)\right) + \frac{b_1}{a_{12}} := \alpha^0$$

holds, where $L^* = a_{21}(h_1 - (1-p)h_2) - b_2\ln[h_1/(1-p)h_2]$. Then system (1.1) has a positive order-1 periodic solution.

Proof. Suppose that the trajectory $O^+(E^*, t_0)$ of system (1.1) starts from the initial point $E^*((1-p)h_2, b_1/a_{12})$ intersects $\sum^{h_1}$ at the point $F^*(h_1, y^*)$. Then, by (4.1), we can determine y^* from the following relation

$$b_1 \ln\left(\frac{a_{12}y^*}{b_1}\right) - a_{12}\left(y^* - \frac{b_1}{a_{12}}\right) = a_{21}(h_1 - (1-p)h_2) - b_2 \ln\left(\frac{h_1}{(1-p)h_2}\right) = L^*,$$

$$-\frac{a_{12}y^*}{b_1}\exp\left(-\frac{a_{12}y^*}{b_1}\right) = -\exp\left(\frac{L^* - b_1}{b_1}\right).$$

From Lemma 1, it is easy to prove that $L^* < 0$. So, we only need to consider the branch $W_0(z)$ of the Lambert W function. That is, we obtain that

$$y^* = -\frac{b_1}{a_{12}} W_0\left(-\exp\left(\frac{L^* - b_1}{b_1}\right)\right).$$

Thus, for any $0 < \alpha < \alpha^0 = b_1/a_{12} - y^*$, the trajectory $O^+(S, t_0)$ from the point $S_1((1-p)h_2, y)$ $(y \in (0,\ b_1/a_{12}))$ on $\sum^p$ which will intersects with $\sum^{h_1}$ and $\sum^{h_2}$ infinite times due to the impulsive effects.

Let the point $A_1((1-p)h_2, \beta_1)$ on $\sum^p$, where β_1 is small enough. From the geometrical construction of the phase space of system (2.1), suppose that the trajectory $O^+(A_1, t_0)$ of system (1.1) starts from the initial point A_1 intersects $\sum^{h_1}$ at the point $B_1(h_1, \theta_1)$ at the time $t = t_1$, and then jumps to the point $B_1^+(h_1, \theta_1 + \alpha)$ due to the impulsive $\Delta y(t) = \alpha$. Further, the trajectory $O^+(A_1, t_0)$ intersects the section $\sum^{h_2}$ at the point $C_1(h_2, \gamma_1)$ when $t = t_2$. At the state C_1, the trajectory $O^+(A_1, t_0)$ is subjected by impulsive effects to jumps to the point $A_2((1-p)h_2, \beta_2)$ on $\sum^p$, where $\beta_2 = (1-q)\gamma_1$. Further, the trajectory $O^+(A_1, t_0)$ intersects $\sum^{h_1}$ and $\sum^{h_2}$ at the points $B_2(h_1, \theta_2)$, $B_2^+(h_1, \theta_2 + \alpha)$ and $C_2(h_2, \gamma_2)$, respectively. Let ε be a small enough positive constant such that $\theta_1 + \alpha < (b_1 - \varepsilon)/a_{12}$, then integrating both side of the first equation of system (1.1) from the orbit $\widehat{B_1^+C_1}$ we have

$$\begin{aligned} t_2 - t_1 = \int_{t_1}^{t_2} \mathrm{d}t &= \int_{h_1}^{h_2} \frac{\mathrm{d}x}{x(b_1 - a_{12}y)} \\ &< \int_{h_1}^{h_2} \frac{\mathrm{d}x}{x\left(b_1 - a_{12}\frac{b_1-\varepsilon}{a_{12}}\right)} \\ &< \ln\left(\frac{h_2}{h_1}\right)^{a_0}, \end{aligned} \tag{4.2}$$

where $a_0 = 1/\varepsilon$.

Further, integrating both side of the second equation of system (1.1) from the orbit $\widehat{B_1^+C_1}$ we have

$$\begin{aligned} \ln\frac{\gamma_1}{\theta_1 + \alpha} = \int_{\theta_1+\alpha}^{\gamma_1} \frac{\mathrm{d}y}{y} &= \int_{t_1}^{t_2} (-b_2 + a_{21}x)\,\mathrm{d}t \\ &\geq \int_{t_1}^{t_2} (-b_2 + a_{21}h_1)\,\mathrm{d}t \\ &\geq (-b_2 + a_{21}h_1)(t_2 - t_1). \end{aligned}$$

From (4.2), this implies that

$$\gamma_1 \geq (\theta_1 + \alpha)\left(\frac{h_2}{h_1}\right)^{a_0(-b_2+a_{21}h_1)}.$$

If β_1 is small enough and $\beta_1 < (1-q)\alpha(h_2/h_1)^{a_0(-b_2+a_{21}h_1)}$, then the point A_2 is above the point A_1. Therefore, the point C_2 is above the point C_1. Hence, from (2.5) we have $\gamma_2 = P_2(q, \alpha, \gamma_1)$ and

$$\gamma_1 - P_2(q, \alpha, \gamma_1) = \gamma_1 - \gamma_2 < 0. \tag{4.3}$$

On the other hand, for any $0 < \alpha < \alpha_0$, suppose that the trajectory $O^+(E_1, t_0)$ starts from the initial point $E_1((1-p)h_2, b_1/a_{12})$ intersects $\sum^{h_1}$ and $\sum^{h_2}$ at the points $F_1(h_1, \eta_1)$, $F_1^+(h_1, \eta_1+\alpha)$ and $G_1(h_2, \lambda_1)$, respectively. At the state G_1, the trajectory $O^+(E_1, t_0)$ jumps to the point $E_2((1-p)h_2, (1-q)\lambda_1)$ on $\sum^p$ and then intersects the sections $\sum^{h_1}$ and $\sum^{h_2}$ at the points $F_2(h_1, \eta_2)$, $F_2^+(h_1, \eta_2 + \alpha)$ and $G_2(h_2, \lambda_2)$ again. In view of the geometrical construction of the phase space of system (1.1), we obtain that the point G_2 is under the point G_1 for any $p, q \in (0,\ 1)$ and $\alpha \in (0,\ \alpha_0)$. Therefore, from (2.5) we have

$$\lambda_1 - P_2(q, \alpha, \lambda_1) = \lambda_1 - \lambda_2 > 0. \tag{4.4}$$

By (4.3) and (4.4), it follows that the Poincaré map (2.5) has a fixed point, that is the system (1.1) has a positive order-1 periodic solution. The proof of Theorem 2 is therefore complete. □

Remark 1. Considering the geometrical construction of the phase space of the system (2.1), if $\alpha > \alpha^0$, the trajectory which starts from the point $S((1-p)h_2, y)(y < b_1/a_{12})$ may intersects $\sum^{h_2}$ finite times. So, $0 < \alpha < \alpha^0$ is a sufficient condition for system (1.1) has a positive order-1 periodic solution.

Next, we are state and prove our result on the stability of positive order-1 periodic solutions of system (1.1).

Theorem 3. *Suppose that conditions of Theorem 2 hold. Let $(\phi(t), \psi(t))$ be a positive order-1 periodic solution of system (1.1) which starts from the point (h_2, η_0). If the condition*

$$\mu = \left| \frac{w_1(\eta_0)[b_1 - a_{12}(w_1(\eta_0) + \alpha)](b_1 - (1-q)a_{12}w_1(\eta_0))}{(b_1 - a_{12}w_1(\eta_0))(w_1(\eta_0) + \alpha)(b_1 - a_{12}\eta_0)} \right| < 1 \tag{4.5}$$

holds, where $w_1(\eta_0)$ is given in (4.9). Then $(\varphi(t), \psi(t))$ is locally orbitally asymptotically stable and which has asymptotic phase property.

Proof. Base on the geometrical construction of the phase space of system (2.1), we know that $\eta_0 < b_1/a_{12}$. Further, for any the point $C_k(h_2, y_k)$ on $\sum^{h_2}$ and $0 < y_k < b_1/a_{12}$, the trajectory of system (1.1) initiates from the point C_k intersects $\sum^{p}$, $\sum^{h_1}$ and $\sum^{h_2}$ at the points $A_{k+1}((1-p)h_2, (1-q)y_k)$, $B_{k+1}(h_1, y^*_{k+1})$, $B^+_{k+1}(h_1, y^*_{k+1} + \alpha)$ and $C_{k+1}(h_2, y_{k+1})$, respectively. Then we have $0 < y_{k+1} = P_2(q, \alpha, y_k) < b_1/a_{12}$. From (4.1), the relation between A_{k+1} and B_{k+1} is

$$b_1 \ln\left(\frac{y^*_{k+1}}{(1-q)y_k}\right) - a_{12}(y^*_{k+1} - (1-q)y_k) = a_{21}(h_1 - (1-p)h_2) - b_2 \ln\left(\frac{h_1}{(1-p)h_2}\right) := L_1,$$
$$-\frac{a_{12}y^*_{k+1}}{b_1}\exp\left(-\frac{a_{12}y^*_{k+1}}{b_1}\right) = -\frac{(1-q)a_{12}y_k}{b_1}\exp\left(\frac{L_1 - (1-q)a_{12}y_k}{b_1}\right).$$

From Lemma 1, we note that $L_1 < 0$ for any $p \in (0, 1)$. Since $0 < y_k, y^*_{k+1} < (b_1 - a_{12}h_2)/a_{12}$, here we only need to consider the branch $W_0(z)$ of the Lambert W function. That is, we obtain that

$$y^*_{k+1} = -\frac{b_1}{a_{12}} W_0\left(-\frac{(1-q)a_{12}y_k}{b_1}\exp\left(\frac{L_1 - (1-q)a_{12}y_k}{b_1}\right)\right) := w_1(y_k). \tag{4.6}$$

Next, we calculate the relation of the points B^+_{k+1} and C_{k+1}. By (4.1) and (4.6), it follows that

$$b_1 \ln\left(\frac{y_{k+1}}{y^*_{k+1} + \alpha}\right) - a_{12}(y_{k+1} - (y^*_{k+1} + \alpha)) = a_{21}(h_2 - h_1) - b_2 \ln\left(\frac{h_2}{h_1}\right) := L_2,$$
$$-\frac{a_{12}y_{k+1}}{b_1}\exp\left(-\frac{a_{12}y_{k+1}}{b_1}\right) = -\frac{a_{12}(w_1(y_k) + \alpha)}{b_1}\exp\left(\frac{L_2 - a_{12}(w_1(y_k) + \alpha)}{b_1}\right).$$

Similar the above, in view to $L_2 < 0$ and $0 < y_k, y_{k+1} < (b_1 - a_{12}h_2)/a_{12}$, we have

$$y_{k+1} = -\frac{b_1}{a_{12}} W_0\left(-\frac{a_{12}(w_1(y_k) + \alpha)}{b_1}\exp\left(\frac{L_2 - a_{12}(w_1(y_k) + \alpha)}{b_1}\right)\right) := P_2(q, \alpha, y_k). \tag{4.7}$$

From (2.6), the derivative of the Poincaré map (4.7) with respect to y_k, it follows that

$$\frac{\partial P_2(q, \alpha, y_k)}{\partial y_k} = -\frac{b_1 W_0(Z_1)}{a_{12}Z_1(1 + W_0(Z_1))}\frac{\partial Z_1}{\partial y_k},$$

where

$$Z_1 = -\frac{a_{12}(w_1(y_k) + \alpha)}{b_1}\exp\left(\frac{L_2 - a_{12}(w_1(y_k) + \alpha)}{b_1}\right).$$

Further, by (2.6) and (4.6) the derivative of the Z_1 with respect to y_k, we have

$$\frac{\partial Z_1}{\partial y_k} = -\frac{a_{12}w_1(y_k)\,[b_1 - a_{12}(w_1(y_k) + \alpha)]\,(b_1 - (1-q)a_{12}y_k)}{b_1^2 y_k(b_1 - a_{12}w_1(y_k))}\exp\left(\frac{L_2 - a_{12}(w_1(y_k) + \alpha)}{b_1}\right). \tag{4.8}$$

Note that $(\phi(t), \psi(t))$ be a positive order-1 periodic solution of system (1.1) which starts from the point (h_2, η_0), so η_0 is a fixed point of $P_2(q, \alpha, y_k)$. Therefore from (4.6)–(4.8), we have

$$\frac{\partial P_2(q, \alpha, \eta_0)}{\partial y_k} = \frac{w_1(\eta_0)[b_1 - a_{12}(w_1(\eta_0) + \alpha)](b_1 - (1-q)a_{12}w_1(\eta_0))}{(b_1 - a_{12}w_1(\eta_0))(w_1(\eta_0) + \alpha)(b_1 - a_{12}\eta_0)},$$

where

$$w_1(\eta_0) = -\frac{b_1}{a_{12}} W_0 \left(-\frac{(1-q)a_{12}\eta_0}{b_1} \exp\left(\frac{L_1 - (1-q)a_{12}\eta_0}{b_1} \right) \right). \tag{4.9}$$

If (4.5) holds, then $|\partial P_2(q, \alpha, \eta_0)/\partial y_k| < 1$. Thus, the solution $(\varphi(t), \psi(t))$ of system (1.1) is orbitally asymptotically stable and has asymptotic phase property. This completes the proof. □

Remark 2. From Theorem 3, we note that the solution $(\varphi(t), \psi(t))$ of system (1.1) is unstable if $\mu > 1$, and is critical case when $\mu = 1$.

4.2. The case of $h_1 \le (b_2/a_{21}) < h_2$

Theorem 4. *Suppose that $p, q \in (0, 1)$ and $h_1 \le (b_2/a_{21}) < h_2$. If the following condition*

$$0 < \alpha < \rho_1 + \frac{b_1}{a_{12}} W_0 \left(-(1-q) \exp\left(\frac{L_4 - (1-q)b_1}{b_1} \right) \right) := \alpha^*$$

holds, where

$$\rho_1 = \begin{cases} -\dfrac{b_1}{a_{12}} W_0 \left(-\exp\left(\dfrac{-L_3 - b_1}{b_1} \right) \right), & \text{The trajectory } O^+\left(\left(h_2, \dfrac{b_1}{a_{12}} \right), 0 \right) \text{ of system (2.1) intersects with } \sum^{h_1}; \\ \dfrac{b_1}{a_{12}}, & \text{Otherwise}, \end{cases}$$

$L_3 = a_{21}(h_2 - h_1) - b_2 \ln(h_2/h_1)$ and $L_4 = a_{21}(h_1 - (1-p)h_2) - b_2 \ln(h_1/(1-p)h_2)$. Then system (1.1) has a positive order- 1 periodic solution.

Proof. In view of the geometrical construction of the phase space of the system (2.1), if the trajectory Γ of system (2.1) which starts from $Q_2(h_2, b_1/a_{12})$ on $\sum^{h_2}$ intersects with $\sum^{h_1}$ at the point $Q_1(h_1, \rho_1)$, where $\rho_1 \le b_1/a_{12}$. Then, by (4.1), we can determine ρ_1 from the following relation

$$b_1 \ln\left(\frac{b_1}{a_{12}\rho_1} \right) - a_{12}\left(\frac{b_1}{a_{12}} - \rho_1 \right) = a_{21}(h_2 - h_1) - b_2 \ln\left(\frac{h_2}{h_1} \right) = L_3,$$

$$-\frac{a_{12}\rho_1}{b_1} \exp\left(-\frac{a_{12}\rho_1}{b_1} \right) = -\exp\left(\frac{-L_3 - b_1}{b_1} \right).$$

From Lemma 1, it follows that $L_3 < 0$. Hence, we only need to consider the branch $W_0(z)$ of the Lambert W function. Therefore, it follows that

$$\rho_1 = -\frac{b_1}{a_{12}} W_0 \left(-\exp\left(\frac{-L_3 - b_1}{b_1} \right) \right).$$

Otherwise, that is the trajectory Γ does not intersect with $\sum^{h_1}$, then let $\rho_1 = b_1/a_{12}$.

Suppose that the trajectory $O^+(Q_2, t_0)$ starts from the point $Q_2(h_2, b_1/a_{12})$ which intersects $\sum^{p}$ and $\sum^{h_1}$ at the points $Q_3((1-p)h_2, (1-q)b_1/a_{12})$ and $Q_4(h_1, \rho_2)$ due to impulsive effects. Similarly, we can obtain ρ_2 from the following relation

$$\rho_2 = -\frac{b_1}{a_{12}} W_0 \left(-(1-q) \exp\left(\frac{L_4 - (1-q)b_1}{b_1} \right) \right).$$

Thus, for any $0 < \alpha < \alpha^* = \rho_1 - \rho_2$, the trajectory of system (1.1) which starts from the point $G_1(h_2, y_1)$ $(y_1 < b_1/a_{12})$ will intersects with $\sum^{h_1}$ and $\sum^{h_2}$ infinite times due to the impulsive effects.

Similar to the proof of Theorem 2, we obtain that system (1.1) has a positive order-1 periodic solution and this completes the proof. □

Remark 3. From the geometrical construction of the phase space of the system (2.1), we note that the trajectory of system (1.1) may intersects with $\sum^{h_2}$ finite times if $\alpha > \alpha^*$. So, $0 < \alpha < \alpha^*$ is a sufficient condition for system (1.1) has a positive order-1 periodic solution.

Finally, on the stability of positive order-1 periodic solution of system (1.1), we have the following result.

Theorem 5. *Suppose that conditions of Theorem 4 hold. Let $(\varphi(t), \psi(t))$ be a positive order-1 periodic solution of system (1.1) which starts from the point (h_2, θ_0). If the condition*

$$\nu = \left|\frac{w_1(\theta_0)[b_1 - a_{12}(w_1(\theta_0)+\alpha)](b_1-(1-q)a_{12}w_1(\theta_0))}{(b_1-a_{12}w_1(\theta_0))(w_1(\theta_0)+\alpha)(b_1-a_{12}\theta_0)}\right| < 1$$

holds, where $w_1(\theta_0)$ is given in (4.9). Then $(\varphi(t), \psi(t))$ is locally orbitally asymptotically stable and which has asymptotic phase property.

The proof of Theorem 5 is similar to that of Theorem 3, we therefore omit it here.

5. Example, numerical simulation and discussing

In this paper, we investigate a class of a Lotka–Volterra predator–prey model with state dependent impulsive effects. By using Poincaré map and Lambert W function, we give the criteria for the existence and stability of semi-trivial solution and positive periodic solution of system (1.1).

In order to check the validity of our results, we consider the following two species Lotka–Volterra predator–prey systems with state dependent impulsive effects

$$\begin{cases} \left.\begin{aligned} \frac{dx(t)}{dt} &= x(t)\,[0.3 - a_{11}x(t) - 0.2y(t)] \\ \frac{dy(t)}{dt} &= y(t)\,[-0.18 + 0.2x(t)] \end{aligned}\right\} & x \neq h_1, h_2, \\ \left.\begin{aligned} \Delta x(t) &= 0 \\ \Delta y(t) &= y(t^+) - y(t) = \alpha \end{aligned}\right\} & x = h_1, \\ \left.\begin{aligned} \Delta x(t) &= x(t^+) - x(t) = -px(t) \\ \Delta y(t) &= y(t^+) - y(t) = -qy(t) \end{aligned}\right\} & x = h_2, \end{cases} \tag{5.1}$$

where $a_{11} \geq 0, \alpha \geq 0, p, q \in (0,\ 1), h_1 > 0, h_2 > 0$ and $(1-p)h_2 < h_1$. Now, we consider the impulsive effects influences on the dynamics of system (5.1).

Example 1. Existence and stability of semi-trivial periodic solution with $\alpha = 0$.

In system (5.1), let $a_{11} = 0.1, p = 0.4, q = 0.3, \alpha = 0$ and $h_2 = 1.1$. It is easy to compute that system (5.1) has the following semi-trivial periodic solution for $(k-1)T < t \leq kT (k = 1, 2, \ldots)$,

$$\begin{cases} \varphi(t) = \dfrac{33\exp(0.3(t-(k-1)T))}{39 + 11\exp(0.3(t-(k-1)T))}, \\ \psi(t) = 0, \end{cases} \tag{5.2}$$

where $T = 0.3^{-1}\ln(39/19)$. From Theorem 1, we can easily compute

$$\begin{aligned} \lambda &= (1-q)\,(1-p)^{\frac{b_2}{b_1}}\left[\frac{b_1-(1-p)a_{11}h_2}{b_1-a_{11}h_2}\right]^{\frac{a_{21}}{a_{11}}-\frac{b_2}{b_1}} \\ &= 0.7 \times 0.6^{0.6} \times \left(\frac{0.234}{0.19}\right)^{1.4} \approx 0.68966 < 1. \end{aligned}$$

So, condition (3.2) holds. Therefore, from Theorem 1, system (5.1) has a typical stable of semi-trivial periodic solution (5.2), which shown in Fig. 1(a). However, if we choose $p = 0.4, q = 0.1, \alpha = 0$ and $h_2 = 1.6$ in system (5.1), we can compute $\lambda \approx 1.12211 > 1$. In this case, system (5.1) has a unstable of semi-trivial periodic solution. Numerical simulation of the result can be seen in Fig. 1(b).

Example 2. Existence and stability of positive periodic solutions of system (5.1) with $h_2 \leq (b_2/a_{21})$.

In system (5.1), let $a_{11} = 0, p = 0.4, q = 0.3, \alpha = 0.15, h_1 = 0.6$ and $h_2 = 0.85$. We easily verify that

$$\begin{aligned} \alpha < \alpha^0 &= \frac{b_1}{a_{12}}W_0\left(-\exp\left(\frac{L^* - b_1}{b_1}\right)\right) + \frac{b_1}{a_{12}} \\ &\approx 1.5 + W_0(-0.3143) \\ &= 1.5 - 0.8356 = 0.9614. \end{aligned}$$

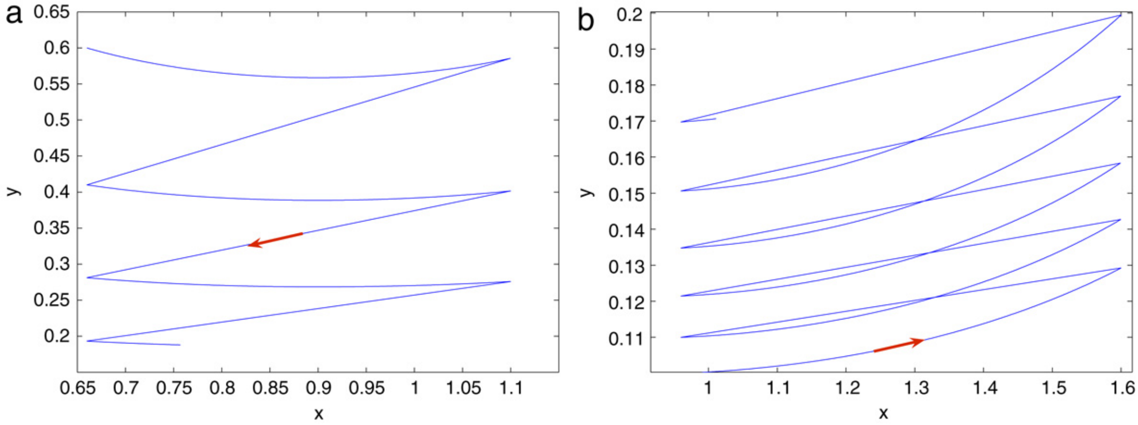

Fig. 1. The trajectory of system (5.1) with $a_{11} = 0.1, p = 0.4$ and $\alpha = 0$: (a) $q = 0.3$ and $h_1 = 1.1$; (b) $q = 0.1$ and $h_2 = 1.6$.

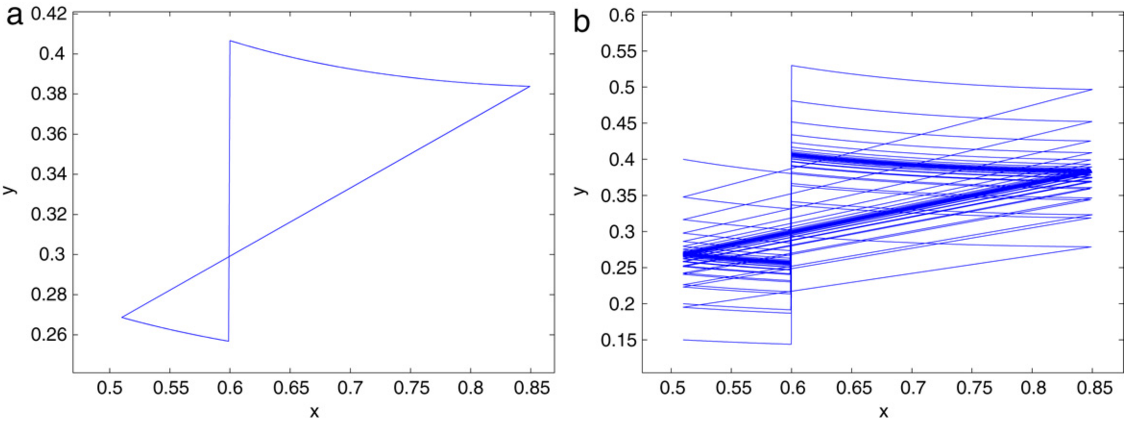

Fig. 2. The trajectory of system (5.1) with $a_{11} = 0, p = 0.4, q = 0.3, h_1 = 0.6, h_2 = 0.85$ and $\alpha = 0.15$.

Therefore, the conditions of Theorem 2 hold. So, system (5.1) has a positive order-1 periodic solution. From numerical simulations, we note that the solution $(\phi(t), \psi(t))$ starts from (0.85, 0.3839) is a order-1 periodic solution, which is shown in Fig. 2(a). Further, it is easily compute

$$\begin{aligned}\mu &= \left|\frac{w_1(\eta_0)[b_1 - a_{12}(w_1(\eta_0)+\alpha)](b_1 - (1-q)a_{12}w_1(\eta_0))}{(b_1 - a_{12}w_1(\eta_0))(w_1(\eta_0)+\alpha)(b_1 - a_{12}\eta_0)}\right| \\ &\approx \left|\frac{-1.5W_0(-0.1280)[0.3 - 0.2(-1.5W_0(-0.1280)+0.15)]}{(0.3 + 0.2 \times 1.5W_0(-0.1280))(-1.5W_0(-0.1280)+0.15)} \times \frac{0.3 + 0.7 \times 0.2 \times 1.5W_0(-0.1280)}{0.3 - 0.2 \times 0.3839}\right| \\ &= 0.6352 < 1.\end{aligned}$$

So, the conditions of Theorem 3 hold, and then the positive order-1 periodic solution $(\phi(t), \psi(t))$ is locally orbitally asymptotically stable and has asymptotic phase property, which is shown in Fig. 2(b).

Example 3. Existence and stability of positive periodic solutions of system (5.1)with $h_1 \le (b_2/a_{21}) < h_2$.

In system (5.1), let $a_{11} = 0, p = 0.4, q = 0.3, h_1 = 0.75$ and $h_2 = 1.1$. It is easily compute $\alpha^* = 1.5(W_0(-0.3417) - W_0(-0.3665)) \approx 0.3470$. So, system (5.1) has a positive order-1 periodic solution when $\alpha < \alpha^*$, and may have no positive order-1 periodic solution when $\alpha > \alpha^*$. Which is shown in Fig. 3(a). Further, it is easily compute $\nu \approx 0.72988 < 1$ when $\alpha = 0.11$. So, the conditions of Theorem 5 hold, and then the positive order-1 periodic solution $(\phi(t), \psi(t))$ is locally orbitally asymptotically stable and has asymptotic phase property, which is shown in Fig. 3(b).

In system (5.1), let $a_{11} = 0.1, p = 0.4, q = 0.3, h_1 = 0.6, h_2 = 0.85$ and $\alpha = 0.15$. From numerical simulations, we note that the solution $(\phi(t), \psi(t))$ of system (5.1) starts from (0.85, 0.3546) is a order-1 periodic solution, and is locally orbitally asymptotically stable and has asymptotic phase property. These are shown in Fig. 4. Further, let $a_{11} = 0.1, p = 0.4, q = 0.3, h_1 = 0.75$ and $h_2 = 1.1$ in system (5.1). By numerical simulation, we obtain that system (5.1) has no positive order-1 periodic solution when $\alpha = 0.75$, and has a positive order-1 periodic solution and which is locally orbitally asymptotically

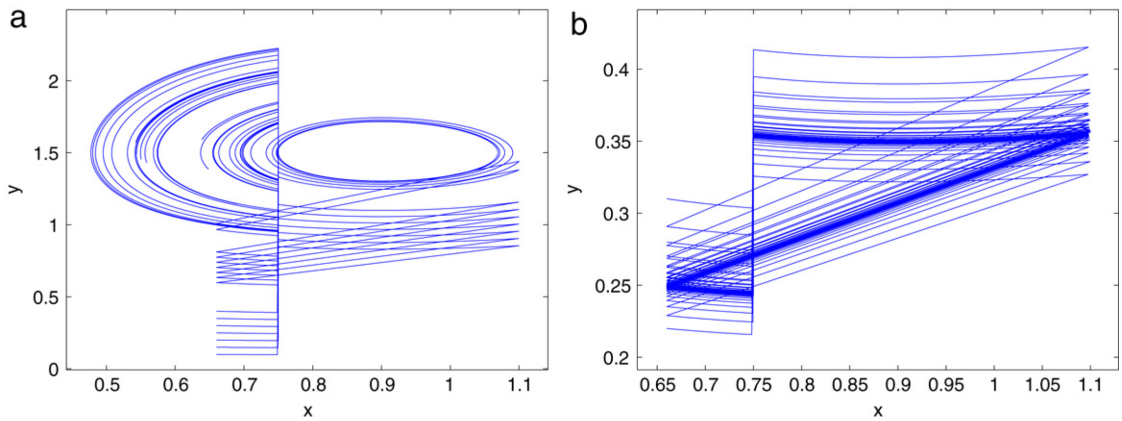

Fig. 3. The trajectory of system (5.1) with $a_{11} = 0$, $p = 0.4$, $q = 0.3$, $h_1 = 0.75$ and $h_2 = 1.1$: (a) $\alpha = 0.75$; (b) $\alpha = 0.11$.

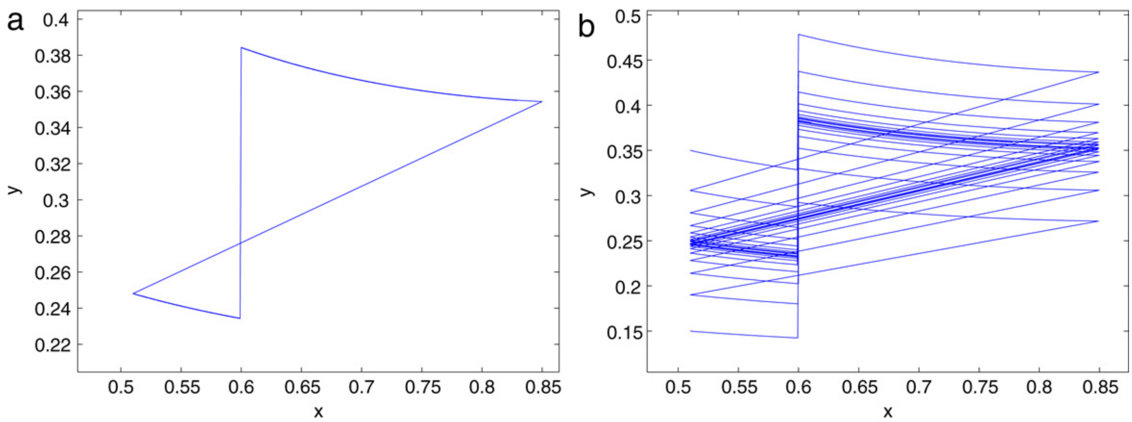

Fig. 4. The trajectory of system (5.1) with $a_{11} = 0.1$, $p = 0.4$, $q = 0.3$, $h_1 = 0.6$, $h_2 = 0.85$ and $\alpha = 0.15$.

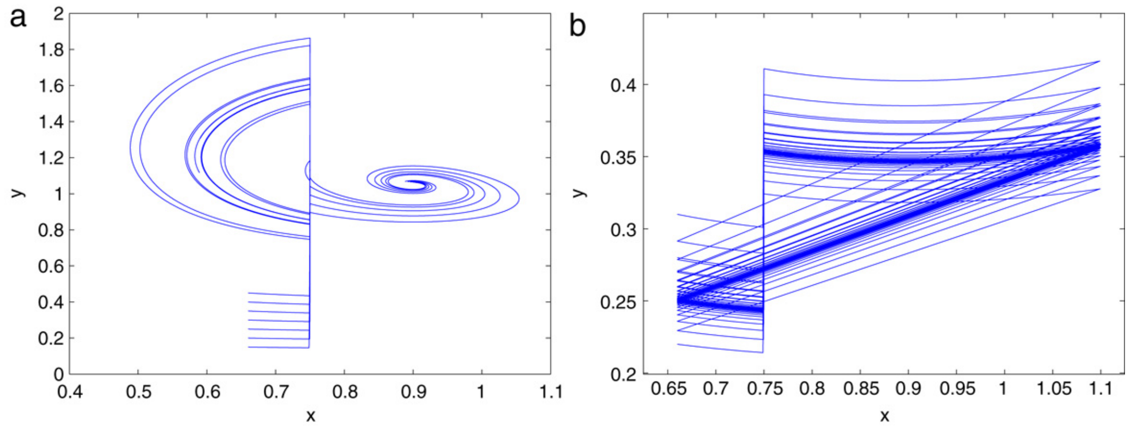

Fig. 5. The trajectory of system (5.1) with $a_{11} = 0$, $p = 0.4$, $q = 0.3$, $h_1 = 0.75$ and $h_2 = 1.1$: (a) $\alpha = 0.75$; (b) $\alpha = 0.11$.

stable and has asymptotic phase property when $\alpha = 0.11$. These are shown in Fig. 5. Thus, we have an interesting open problem: for any $p, q \in (0, 1)$, there is a constant $\alpha^* > 0$, such that for any $\alpha \in (0, \alpha^*)$, system (1.1) has a positive order-1 periodic solution and which has asymptotic phase property under some conditions.

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