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Limit Theorems for Randomly Weighted Sums of Random Elements in Normed Linear Spaces

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Let $\{V_n; n \geqslant 1\}$ be a sequence of random elements in a separable normed linear space E, uniformly dominated by a random variable V. Let $\{A_{nk};\ k=1, 2, ..., n;\ n \geqslant 1\}$ be a triangular array of random variables. In this paper, conditions for the convergence of $\sum_{k=1}^{n} A_{nk} V_k$ to zero (in probability and completely) are obtained. No geometric condition on E is imposed.

1. INTRODUCTION

If E denotes a separable normed linear space with norm $\|\cdot\|$, and (Ω, A, P) is a probability space, a random element V in E is a function from Ω into E which is A-measurable with respect to the Borel subsets of E. A random variable is a random element in R, he reads.

A sequence of random elements $\{V_n\}$ in E is said to be uniformly dominated by a random variable X (or by a random element V) if, for each $x > 0$,

$$P[\|V_n\| \geqslant x\| \leqslant P[|X| \geqslant x] \qquad \text{for every } n \geqslant 1$$

or

$$P[\|V_n\| \geqslant x] \leqslant P[\|V\| \geqslant x] \qquad \text{for every } n \geqslant 1, \text{ respectively.}$$

This condition was introduced by Rohatgi [3] in order to weaken the classic requirement of identical distribution of a sequence of random variables.

In this paper, two results of convergence (in probability and completely) for randomly weighted sums of random elements in a normed linear space without geometric conditions are established. The limit theorems obtained are always refered to the convergence in the norm topology.

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These theorems extend the original results of Rohatgi for weighted sums of random variables, in the way followed by Bozorgnia, Bhaskara Rao, and Taylor.

The basic techniques that we use in the proofs are close to Taylor's, but we do not need conditions of convexity on the space where the random elements take values.

2. Convergence in Probability

Rohatgi [3] stated a well-known result of convergence in probability for sums, weighted by constants, of independent random variables uniformly dominated by a random variable.

This result was extended by Bozorgnia and Rao [1] to sums, weighted by constants, of random elements taking values in a separable Banach space:

Theorem 2.1 [1]. *Let $\{V_n; n \geqslant 1\}$ be a sequence of pairwise independent random elements taking values in a separable Banach space and uniformly dominated by a random element V.*

Assume that $E\|V\|^r < \infty$ for some $0 < r < 1$. Let $\{a_{nk}; n, k \geqslant 1\}$ be a double sequence of real numbers satisfying the following conditions:

(a) $\lim_{n \to \infty} \max_{k \geqslant 1} |a_{nk}| = 0$,

(b) $\sum_{k \geqslant 1} |a_{nk}|^r \leqslant S$ *for every $n \geqslant 1$, where S is a positive constant.*

Then, $S_n = \sum_{k \geqslant 1} a_{nk} V_k$ converges to zero in probability.

Taylor, Raina, Calhoun, and Daffer [5] obtain a result in this way for sums of random elements in a Banach space of type p, $1 < p \leqslant 2$, weighted by random variables:

Theorem 2.2 [5]. *Let $\{X_k\}$ be random elements in B, where B is a Banach space of type p, $1 < p \leqslant 2$, and let $EX_n = 0$ for each n, and $\{X_k\}$ uniformly dominated by a positive random variable X, where $xP[X^r > x] \to 0$ as $x \to \infty$, for some r, $1 < r < p$. Let $\{a_{nk}\}$ be random variables such that:*

(1) *$\{a_{nk} X_k\}$ are independent for each n,*

(2) *for each n, a_{nk} is independent of X_k for each k,*

(3) $\max_k |a_{nk}| \to 0$ *a.s.,*

(4) $\sum_{k=1}^{n} |a_{nk}|^r \leqslant M$ *a.s. for each n, where M is a constant.*

Then, $S_n = \sum_{k=1}^{n} a_{nk} X_k \to 0$ in probability.

In this paper, an extension of the original result of Rohatgi for randomly weighted sums of random elements in a normed linear space (not necessarily a Banach space) without geometric conditions is obtained.

We have not imposed conditions a.s. on the random weights as in Theorem 2.2, but conditions on some of their moments; moreover, only a condition of uncorrelation between the random weights and the truncated random elements is required.

Here is the result:

THEOREM 2.3. *Let* $\{V_n\}$ *be a sequence of random elements in a separable normed linear space, uniformly dominated by a random variable* V, *with* $E|V|^r<\infty$ *for some* r, $0<r<1$.

Let $\{A_{nk}; k=1, 2, ..., n; n\geqslant 1\}$ *be an array of random variables such that*:

(1) $\lim_{n\to\infty}(\max_k E|A_{nk}|^r)=0$,

(2) $\sum_{k=1}^n E^r|A_{nk}|^r \leqslant C$ *for each* n, *where* C *is a constant*,

(3) $|A_{nk}|^r$ *and* $W_{nk}^r=\|V_k\|^r I_{[\|V_k\|<E^{-1}|A_{nk}|^r]}$ *are uncorrelated random variables, for each* k *and every* n.

Then, $S_n=\sum_{k=1}^n A_{nk}V_k\to 0$ *in probability.*

Proof. Since $E|V|^r<\infty$, we have

$$xP[|V|\geqslant x^{1/r}]\to 0 \qquad \text{when} \quad x\to\infty. \tag{2.1}$$

Define

$$V_{nk}=\begin{cases}A_{nk}V_k & \text{if } \|V_k\|<E^{-1}|A_{nk}|^r\\ 0 & \text{if } \|V_k\|\geqslant E^{-1}|A_{nk}|^r\end{cases}$$

for each $k=1, 2, ..., n$ and every $n=1, 2, ...$.

Denote $S_{nn}=\sum_{k=1}^n V_{nk}$.

Later, we shall use the notation $\|V_{nk}\|=|A_{nk}|\,W_{nk}$, where

$$W_{nk}=\begin{cases}\|V_k\| & \text{if } \|V_k\|<E^{-1}|A_{nk}|^r\\ 0 & \text{if } \|V_k\|\geqslant E^{-1}|A_{nk}|^r.\end{cases}$$

It follows that

$$\begin{aligned}P[S_{nn}\neq S_n]&\leqslant\sum_{k=1}^n P[\|V_k\|\geqslant E^{-1}|A_{nk}|^r]\\ &\leqslant\sum_{k=1}^n P[|V|\geqslant E^{-1}|A_{nk}|^r]\\ &=\sum_{k=1}^n E^r|A_{nk}|^r E^{-r}|A_{nk}|^r P[|V|\geqslant E^{-1}|A_{nk}|^r].\end{aligned}$$

Using (2.1) and the conditions (1)–(2), given $\varepsilon > 0$ we choose n_0 such that $P[S_{nn} \neq S_n] \leqslant C\varepsilon$ for all $n > n_0$.

Therefore, it suffices to prove that $S_{nn} \to 0$ in probability.

For this $\varepsilon > 0$, choose M such that

$$x^r P[|V| \geqslant x] \leqslant \varepsilon \qquad \text{for } x \geqslant M.$$

Then, for any $N > M$,

$$\begin{aligned}
\int_0^N x^{r-1} P[|V| \geqslant x]\, dx &= \int_0^M x^{r-1} P[|V| \geqslant x]\, dx + \int_M^N x^{r-1} P[|V| \geqslant x]\, dx \\
&\leqslant \int_0^M x^{r-1}\, dx + \int_M^N \varepsilon x^{-1}\, dx \\
&\leqslant H + \varepsilon \ln N, \qquad \text{where } H = H(M, \varepsilon).
\end{aligned}$$

Note that the condition $1 - r < 1$ assures the convergence of the integral. Therefore, for any $s > 0$,

$$N^{-s} \int_0^N x^{r-1} P[|V| \geqslant x]\, dx \leqslant N^{-s} H + \varepsilon N^{-s} \ln N,$$

and so

$$N^{-s} \int_0^N x^{r-1} P[|V| \geqslant x]\, dx \to 0 \qquad \text{when } N \to \infty. \tag{2.2}$$

Integration by parts yields

$$\begin{aligned}
\int_0^N x^r\, dP[\|V_k\| < x] &= x^r P[\|V_k\| < x] \,|_0^N - r \int_0^N x^{r-1} P[\|V_k\| < x]\, dx \\
&= -N^r P[\|V_k\| \geqslant N] + r \int_0^N x^{r-1} P[\|V_k\| \geqslant x]\, dx \\
&\leqslant r \int_0^N x^{r-1} P[|V| \geqslant x]\, dx. \qquad (2.3)
\end{aligned}$$

Moreover,

$$\begin{aligned}
P[\|S_{nn}\| > \varepsilon] &\leqslant P\left[\sum_{k=1}^n \|V_{nk}\| > \varepsilon\right] \leqslant \varepsilon^{-r} E\left(\sum_{k=1}^n |A_{nk}|\, W_{nk}\right)^r \\
&\leqslant \varepsilon^{-r} \sum_{k=1}^n E(|A_{nk}|^r\, W_{nk}^r) \qquad \text{since } 0 < r < 1.
\end{aligned}$$

By the hypothesis of uncorrelation (3) and by (2.3),

$$\begin{aligned}\sum_{k=1}^{n} E(|A_{nk}|^r W_{nk}^r) &= \sum_{k=1}^{n} E|A_{nk}|^r EW_{nk}^r \\ &= \sum_{k=1}^{n} E|A_{nk}|^r \int_0^{E^{-1}|A_{nk}|^r} x^r \, dP[\|V_k\| < x] \\ &\leqslant r \sum_{k=1}^{n} E|A_{nk}|^r \int_0^{E^{-1}|A_{nk}|^r} x^{r-1} P[|V| \geqslant x] \, dx \\ &= r \sum_{k=1}^{n} E^r |A_{nk}|^r E^{1-r} |A_{nk}|^r \int_0^{E^{-1}|A_{nk}|^r} x^{r-1} P[|V| \geqslant x] \, dx.\end{aligned}$$

Using (1) and (2.2), for any arbitrary and fixed $\delta > 0$,

$$E^{1-r} |A_{nk}|^r \int_0^{E^{-1}|A_{nk}|^r} x^{r-1} P[|V| \geqslant x] \, dx \leqslant \delta$$

for n sufficiently large.

Therefore, for n sufficiently large,

$$E\left(\sum_{k=1}^{n} |A_{nk}| W_{nk} \right)^r \leqslant rC\delta$$

and hence $P[\|S_{nn}\| > \varepsilon] \to 0$ when $n \to \infty$.

Then, $S_{nn} \to 0$ in probability, which implies the convergence in probability to zero of S_n.

3. Complete Convergence

Rohatgi [3] also obtains a result of convergence a.s. enlarging the imposed conditions to the constant weights and to the moments of the random variables. This result was extended by Bozorgnia and Rao [1] for random elements in a separable Banach space and constant weights. Later, Taylor, Raina, Calhoun, and Daffer [5] considered the case of random weights, obtaining the next theorem:

Theorem 3.1. [5]. *Let $\{V_n\}$ be independent random elements in a separable Banach space of type p, $1 < p \leqslant 2$, with $EV_n = 0$ for every n, and uniformly dominated by a positive random variable X with $EX^{1+1/\alpha} < \infty$, $\alpha > 0$.*

Let $\{a_{nk}\}$ be a Toeplitz array of row-wise independent random variables *such that* $\max_k |a_{nk}| \leqslant Bn^{-\alpha}$ *a.s. For each n, assume that* $\{a_{nk}\}$ *and* $\{V_k\}$ *are independent.*

Then, $\|\sum_{k=1}^{\infty} a_{nk} V_k\| \to 0$ *completely, and hence a.s.*

We obtain a result in this way for sums, weighted by a triangular array of random variables, of random elements in a normed linear space without geometric conditions, and without conditions of independence between random weights and random elements or between the random elements.

Here is the result:

THEOREM 3.2. *Let* $\{V_n\}$ *be a sequence of random elements in a separable normed linear space, uniformly dominated by a random variable* V, *with* $E|V|^r \leqslant C < \infty$, *for some* $1 \leqslant r < \infty$.

Let $\{A_{nk}; k=1, 2, ..., n; n \geqslant 1\}$ *be an array of random variables such that* $\max_k |A_{nk}| \leqslant Mn^{-\alpha}$ *a.s.* ($M>0$, *constant*) *for some* $\alpha > 1 + 1/r$.

Then, $S_n = \sum_{k=1}^{n} A_{nk} V_k \to 0$ *completely, an hence a.s.*

Proof. By the condition of uniform domination,

$$E\|V_n\|^r \leqslant E|V|^r \leqslant C < \infty \qquad \text{for every } n. \tag{3.1}$$

Let $\varepsilon > 0$ given. Then

$$\begin{aligned} P[\|S_n\| > \varepsilon] &\leqslant P\left[\sum_{k=1}^{n} |A_{nk}|\, \|V_k\| > \varepsilon\right] \\ &\leqslant P\left[\sum_{k=1}^{n} (\max_k |A_{nk}|)\, \|V_k\| > \varepsilon\right] \leqslant P\left[\sum_{k=1}^{n} \|V_k\| > (\varepsilon/M)\, n^{\alpha}\right]. \end{aligned}$$

Therefore

$$P[\|S_n\| > \varepsilon] \leqslant An^{-\alpha r} E\left(\sum_{k=1}^{n} \|V_k\|\right)^r,$$

where $A = A(\varepsilon, M, r)$.

Using Minkowski's inequality and (3.1),

$$\begin{aligned} P[\|S_n\| > \varepsilon] &\leqslant An^{-\alpha r}\left[\sum_{k=1}^{n} E^{1/r}\, \|V_k\|^r\right]^r \\ &\leqslant An^{-\alpha r} Cn^r = ACn^{-r(\alpha-1)}. \end{aligned}$$

The condition $\alpha > 1 + 1/r$ implies $-r(\alpha - 1) < -1$.

Therefore $\sum_{n=1}^{\infty} P[\|S_n\| > \varepsilon] < \infty$, i.e., $\{S_n\}$ converges completely to zero when $n \to \infty$.

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