Some properties and characterizations for generalized multivariate Pareto distributions

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Abstract

In this paper, several distributional properties and characterization theorems of the generalized multivariate Pareto distributions are studied. It is found that the multivariate Pareto distributions have many mixture properties. They are mixed either by geometric, Weibull, or exponential variables. The multivariate Pareto, \(\text{MP}^{(k)}(I)\), \(\text{MP}^{(k)}(II)\), and \(\text{MP}^{(k)}(IV)\) families have closure property under finite sample minima. The \(\text{MP}^{(k)}(III)\) family is closed under both geometric minima and geometric maxima. Through the geometric minima procedure, one characterization theorem for \(\text{MP}^{(k)}(III)\) distribution is developed. Moreover, the \(\text{MP}^{(k)}(III)\) distribution is proved as the limit multivariate distribution under repeated geometric minimization. Also, a characterization theorem for the homogeneous \(\text{MP}^{(k)}(IV)\) distribution via the weighted minima among the ordered coordinates is developed. Finally, the \(\text{MP}^{(k)}(II)\) family is shown to have the truncation invariant property.

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1. Introduction

Four generalized multivariate Pareto distributions, \(\text{MP}^{(k)}(I)\), \(\text{MP}^{(k)}(II)\), \(\text{MP}^{(k)}(III)\), \(\text{MP}^{(k)}(IV)\) are proposed by Arnold [1]. The first three of them are
identified as the special cases of the fourth in the following:

\[ \text{MP}^{(k)}(I)(\sigma, \alpha) = \text{MP}^{(k)}(IV)(0, \sigma, 1, \alpha), \]

\[ \text{MP}^{(k)}(II)(\mu, \sigma, \alpha) = \text{MP}^{(k)}(IV)(\mu, \sigma, 1, \alpha), \]

\[ \text{MP}^{(k)}(III)(\mu, \sigma, \gamma) = \text{MP}^{(k)}(IV)(\mu, \sigma, \gamma, 1). \]

According to Arnold’s definition, the joint survival function of the \( \text{MP}^{(k)}(IV) \) distribution is given as

**Definition 1.1.** Suppose \( X = (X_1, \ldots, X_k) \) is a \( k \)-dimensional random vector. If the joint survival function of \( X \) is

\[ F_X(x) = \left\{ 1 + \sum_{j=1}^{k} \left( \frac{x_j - \mu_j}{\sigma_j} \right)^{1/\gamma_j} \right\}^{-\alpha}, \]

for any \( x = (x_1, \ldots, x_k) \), \( x_i \geq \mu_i \), where \( \alpha > 0, \mu = (\mu_1, \ldots, \mu_k), \sigma = (\sigma_1, \ldots, \sigma_k), \gamma = (\gamma_1, \ldots, \gamma_k), \) each \( \mu_i \in \mathbb{R}, \sigma_i > 0, \gamma_i > 0 \) for \( \forall 1 \leq i \leq k \), then \( X \) is said to follow a \( \text{MP}^{(k)}(IV)(\mu, \sigma, \gamma, \alpha) \) distribution and is denoted by \( X \sim \text{MP}^{(k)}(IV)(\mu, \sigma, \gamma, \alpha) \).

It is easily discerned that these four \( k \)-dimensional \( \text{MP}^{(k)}(I), (II), (III), (IV) \) are qualified as multivariate Pareto distributions by virtue of having Pareto marginal variables. Also, it is easy to verify that the \( \text{MP}^{(k)}(I), (II) \) and \( (IV) \) families are closed with respect to conditional distributions. Yeh [6–8] studied some properties and inferences for these four multivariate Pareto distributions. Numerous papers dealing with bivariate and multivariate Pareto distributions have subsequently appeared in the literature after Arnold [1] (see [4, Chapter 52] and the references therein). In this paper, some other distributional properties of the \( \text{MP}^{(k)}(I), (II), (III), (IV) \) families are further studied. The characterization theorems for the \( \text{MP}^{(k)}(III) \) and \( \text{MP}^{(k)}(IV) \) are also proved. It is shown that the \( \text{MP}^{(k)}(II) \) has the truncation property.

These four generalized multivariate Pareto distributions are expected to fit the upper tails of some multivariate continuous income data and some other socio-economic multivariate variables. Within the hierarchy of the four generalized \( \text{MP}^{(k)}(I), (II), (III), (IV) \), the \( \text{MP}^{(k)}(II) \) family is most suited in reliability context owing to its truncation invariant property.

2. Mixture properties of the multivariate Pareto distributions

Let \( X = (X_1, X_2, \ldots, X_k) \) denote \( k \)-dimensional multivariate Pareto random vector, if some parameters of the \( \text{MP}^{(k)}(IV) \) distributions are themselves random variables, then some simple results hold, which are reported in this section.
Property 2.1. If \( \alpha \sim \text{geometric } (p) \) and given \( \alpha \), \( X|\alpha \sim \text{MP}^{(k)}(\text{IV})(\mu, \sigma, \gamma, \alpha) \), then \( X \sim \text{MP}^{(k)}(\text{III})(\mu, \sigma \gamma, \gamma) \), where the scale vector \( \sigma \gamma = (\sigma_1 \gamma_1, \ldots, \sigma_k \gamma_k) \).

In Property 2.1, if \( \alpha \) is assumed to have an exponential distribution with mean \( \lambda^{-1} \), then the survival function of \( X \) is given in the following property.

Property 2.2. If \( \alpha \sim \text{exponential } (\text{mean } = \lambda^{-1}) \), and given \( \alpha \), \( X|\alpha \sim \text{MP}^{(k)}(\text{IV})(\mu, \sigma, \gamma, \alpha) \), then the survival function of \( X \) is

\[
F_X(x) = \left\{ 1 + \frac{1}{\lambda} \ln \left( 1 + \sum_{i=1}^{k} \left( \frac{x_i - \mu_i}{\sigma_i} \right)^{1/\gamma_i} \right) \right\}^{-1}, \forall x \geq \mu.
\]

As mentioned in [7], there are two representations of the \( \text{MP}^{(k)}(\text{IV}) \) random vector given by Arnold [1], they are stated as the following two properties.

Property 2.3. Suppose that \( Z \sim \Gamma(a, 1) \) and that, given \( Z = z \), in \( X = (X_1, X_2, \ldots, X_k) \), each \( X_i|Z=z \) independent Weibull variable with conditional survival function \( P(X_i > x_i|Z=z) = e^{-(x_i - \mu_i)/\sigma_i}, \forall x_i > \mu_i \), then \( X \sim \text{MP}^{(k)}(\text{IV})(\mu, \sigma, \gamma, \alpha) \).

Property 2.4. Let \( \mathcal{X} = (X_1, \ldots, X_k) \sim \text{MP}^{(k)}(\text{IV})(\mu, \sigma, \gamma, \alpha) \), then for each \( i = 1, \ldots, k \), there exist \( k \) i.i.d. exponential \( (\lambda = 1) \) random variables \( W_i \) and a \( \Gamma(a, 1) \) variable \( Z \), \( W_i \)'s and \( Z \) are independent such that the \( X_i \)'s have the representation \( X_i = \mu_i + \sigma_i(W_i/Z)^{\gamma_i}, \ i = 1, 2, \ldots, k \).

Because \( \text{MP}^{(k)}(\text{II})(\mu, \sigma, \alpha) \) is a subclass of \( \text{MP}^{(k)}(\text{IV}) \) family by letting all \( \gamma_i = 1 \), as \( \gamma = (1, \ldots, 1) \), so there is an important representation of \( \text{MP}^{(k)}(\text{II}) \) distribution, it is stated as the following two corollaries.

Corollary 2.3.1. Suppose that \( Z \sim \Gamma(a, 1) \) and that, given \( Z = z \), in \( X \), each \( X_i|Z=z \) independent exponential with conditional survival function \( P(X_i > x_i|Z=z) = e^{-(x_i - \mu_i)/\sigma_i}, \forall x_i > \mu_i \). Then \( X \sim \text{MP}^{(k)}(\text{II})(\mu, \sigma, \alpha) \), where \( \mu, \sigma \) and \( \alpha \) are the same as in (1.1).

Corollary 2.4.1. Let \( \mathcal{X} = (X_1, \ldots, X_k) \sim \text{MP}^{(k)}(\text{II})(\mu, \sigma, \alpha) \), then for each \( i = 1, \ldots, k \), the \( X_i \)'s have the representation \( X_i = \mu_i + \sigma_i(W_i/Z)^{\gamma_i}, \text{ where } W_i \text{ i.i.d. } \exp(1) \text{ and } Z \sim \Gamma(a, 1) \), \( W_i \)'s and \( Z \) are independent.
This corollary can be regarded as the reverse statement of Corollary 2.3.1, hence for the MP\(^{(k)}(\text{II})\) family, suppose \(Z \sim \text{Gamma}(\alpha, 1)\), given \(Z = z\), the representation of each component \(\{(X_i - \mu_i)/\sigma_i\}\) independent \(\text{Exp}(\lambda)\) is an “if and only if” result.

3. Extreme order statistics of multivariate Pareto distributions

Analogous to Yeh’s multivariate Zipf results [9], the MP\(^{(k)}(\text{I})\), (II) and (IV) families are also closed under finite sample minima.

Let \(X^i = (X^i_1, X^i_2, \ldots, X^i_k)\) for each \(i = 1, 2, \ldots, n\) be a random sample from any one of the MP\(^{(k)}(\text{I})\), (II), (IV) populations, then the closure property of the sample minima is stated as follows.

**Property 3.1.** For independent \(X^1, X^2, \ldots, X^n\) with each

1. \(X^i \sim \text{MP}^{(k)}(\text{I})(\varrho, \varpi_i)\), then \(\min_{1 \leq i \leq n} X^i \sim \text{MP}^{(k)}(\text{I})(\varrho, \sum_{i=1}^n \varpi_i)\).
2. \(X^i \sim \text{MP}^{(k)}(\text{II})(\mu, \varrho, \varpi_i)\), then \(\min_{1 \leq i \leq n} X^i \sim \text{MP}^{(k)}(\text{II})(\mu, \varrho, \sum_{i=1}^n \varpi_i)\).
3. \(X^i \sim \text{MP}^{(k)}(\text{IV})(\mu, \varrho, \varpi_i, \varpi_i)\), then \(\min_{1 \leq i \leq n} X^i \sim \text{MP}^{(k)}(\text{IV})(\mu, \varrho, \varpi_i, \sum_{i=1}^n \varpi_i)\).

In Property 3.1, the sample minima \(\min_{1 \leq i \leq n} X^i\) is the \(k\)-dim random vector of the \(k\) coordinatewise sample minima of \(\{X^i\}_1^n\), i.e. \(\min_{1 \leq i \leq n} X^i \triangleq (X^i_1, X^i_2, \ldots, X^i_k)\), where \(X^i_j = \min\{X^i_1, X^i_2, \ldots, X^i_k\}\) for \(j = 1, 2, \ldots, k\).

It is easily verified that MP\(^{(k)}(\text{III})\) population has no closure property for sample minima, the reason is that MP\(^{(k)}(\text{III})\) is a special case of MP\(^{(k)}(\text{IV})\) by setting \(\varpi_i \equiv 1\). Then by Property 3.1(3), we have the following corollary:

**Corollary 3.1.1.** Suppose \(\{X^i\}_1^n\) is a random sample from the MP\(^{(k)}(\text{III})(\mu, \varrho, \varpi_i)\) population, then \(\min_{1 \leq i \leq n} X^i \sim \text{MP}^{(k)}(\text{IV})(\mu, \varrho, \varpi_i, n)\).

It will be shown in the following that the MP\(^{(k)}(\text{III})\) family is closed under geometric minima and geometric maxima.

**Property 3.2.** Suppose that \(N\) is a geometric variable with pmf \(P(N = n) = p(1 - p)^{n-1}, \ n = 1, 2, \ldots\), given \(N = n\), let \(\{X^i_j\}_1^n \overset{\text{i.i.d.}}{\sim} \text{MP}^{(k)}(\text{III})(\mu, \varrho, \varpi_i)\), and \(N\) is independent of \(\{X^i\}_1^n\). Let \(m = (X^1_1, X^2_1, \ldots, X^k_1)\) be the \(k\)-dim. geometric minima with \(X^i_{(j)} = \min\{X^i_1, X^i_2, \ldots, X^i_k\}\) for each \(j = 1, 2, \ldots, k\), then \(m\) is also MP\(^{(k)}(\text{III})\) distributed as \(m \sim \text{MP}^{(k)}(\text{III})(\mu, \varrho p^\mu, \varpi_i)\), where \(\varrho p^\mu = (\sigma_1 p^\mu, \ldots, \sigma_k p^\mu)\).
Note that if we compare the results of Properties 2.1 and 3.2, it is found that the geometric mixture of a MP\(^{(k)}\)(IV) random vector is merely a mixture formulation of the geometric minima of a random sample of the MP\(^{(k)}\)(III) population.

Similarly, the coordinatewise geometric maxima of \(\{X^j\}\) is denoted by \(X^{(j)} = \max\{X_1^j, X_2^j, \ldots, X_N^j\}\) for each \(j = 1, 2, \ldots, k\), and the \(k\)-dim. geometric maxima of \(\{X^j\}_1^N\) is \(M = (X^{(1)}, X^{(2)}, \ldots, X^{(k)})\), then the following weaker property is obtained.

**Property 3.3.** For each \(j = 1, 2, \ldots, k\), the \(k\) marginal distributions of \(M\), i.e., \(X^{(j)}\) are univariate Pareto(III) \((\mu_j, \sigma_j p^{-\gamma_j}, \gamma_j)\) distributed, but the \(k\)-dim. geometric maxima \(M\) itself is not a multivariate MP\(^{(k)}\)(III) vector.

Finally, it is noted that there is an interesting property among the coordinate extreme minima among the MP\(^{(k)}\)(IV) random vector which is stated in the following.

**Property 3.4.** Let \(X = (X_1, X_2, \ldots, X_k) \sim \text{MP}^{(k)}(\text{IV}) (\mu_1, \sigma_1, \gamma_1, \alpha)\), let \(X_1 = \min_{1 \leq i \leq k} X_i\), then \(X_1\) is univariate Pareto(IV) distributed as \(X_1 \sim P(\text{IV}) (\mu, (1/\sum_{i=1}^k (\frac{1}{\sigma_i})^{1/\gamma_i}), \gamma, \alpha)\), where, \(\mu \in \mathbb{R}\), \(\gamma > 0\) and \(\mu_1 = (\mu, \ldots, \mu)\), \(\gamma_1 = (\gamma, \ldots, \gamma)\) are \(k\)-dimensional.

### 4. Some limiting properties of the MP(III) distributions

#### 4.1. Characterization of the MP\(^{(k)}\)(III) distribution

Arnold and Laguna [2] and Arnold et al. [3] studied the characterization of the univariate Pareto(III) distribution via the scale transformation of the geometric minima.

It is observed from Property 3.2, that the MP\(^{(k)}\)(III) distribution is closed under geometric minima. This fact stimulates me to study the characterization of the MP\(^{(k)}\)(III) through the geometric minimization procedure.

For \(\sigma = (\sigma_1, \ldots, \sigma_k), \gamma = (\gamma_1, \ldots, \gamma_k)\), \(\sigma_i > 0, \gamma_i > 0, \forall i = 1, \ldots, k\). Let \(F_{\sigma, \gamma}\) denote the family of all distribution functions \(F(\cdot)\) with the property that the local behavior of \(F(\cdot)\) near its lower bound \(\chi \to 0\) is approximated by the asymptotic form

\[
\bar{F}(\chi) \sim 1 - \sum_{i=1}^k \left(\frac{x_i}{\sigma_i}\right)^{1/\gamma_i},
\]

where \(\bar{F}(\cdot)\) is the survival function of \(F(\cdot)\).

Then the following theorem leads to a characterization of the MP\(^{(k)}\)(III)(\(0, \sigma, \gamma\)) distribution.
Theorem 4.1. Let \( \{X^i\} \) be i.i.d. non-negative random vector with common cdf \( F_{X^i}(\cdot) \in F_{g,\gamma} \), i.e., for some \( \sigma > 0, \gamma > 0 \), \( F_{X^i}(\cdot) \) satisfying (4.1). For a fixed \( p \in (0, 1) \), suppose that \( N_p \sim \text{geometric}(p) \), \( N_p \) is independent of all the \( X^i \)'s and define \( n_p \) as the k-dim. geometric minima of \( \{X^i\} \). If \( p^{-\gamma}n_p \overset{d}{=} X^i \), then \( X^i \sim \text{MP}^{(k)}(\text{III})(0, \sigma, \gamma) \).

Proof. First, we have to claim that the common cdf of \( \{X^i\} \), \( F_{X^i}(\cdot) \) satisfies \( F_{X^i}(0) = P(X^i = 0) = 0 \).

Suppose that \( p^{-\gamma}n_p = (p^{-\gamma_1}m_1, \ldots, p^{-\gamma_k}m_k) \overset{d}{=} X^1 \) for some \( p \in (0, 1) \) and since \( \gamma_i > 0 \), so all \( p^{-\gamma_1} > 1 \) for \( i = 1, 2, \ldots, k \), and hence all the \( k \) coordinates \( m_i < X^1_i \), so \( m_p < X^1 \) and \( p^{-\gamma}n_p \overset{d}{=} X^1 \) imply that \( P(X^1 = 0) = 0 \). Also, since \( X^1 \overset{d}{=} X^2 \) and \( N_p \sim \text{geometric}(p) \) is independent of \( X^1, X^2 \), and from \( p^{-\gamma}n_p \overset{d}{=} X^1 \), so the two events \( \{X^1 = 0\} \) and \( \{n_p = 0\} \), are equivalent and hence

\[
P(X^1 = 0) = P(n_p = 0) = P \text{ (at least one of the } X^i = 0) \\
= P(X^1 = 0) + P(X^1 \neq 0 \cap X^2 = 0 \cap (N_p > 1)) \\
= P(X^1 = 0) \{1 + P(X^1 \neq 0) \cap (N_p > 1)\}
\]

so that \( P(X^1 = 0) = 0 \) or 1. Ignoring the trivial case of an \( X^1 \) degenerate at 0, we have \( P(X^1 = 0) = 0 \). Also, \( p^{-\gamma}n_p \overset{d}{=} X^1 \) implies that \( P(X^1 > X) > 0 \), for all \( X > 0 \).

The survival function of \( X^1 \), \( F_{X^1}(\cdot) \) is obtained by conditioning on \( n_p \), hence for any \( X > 0 \).

\[
F_{X^1}(X) = P(n_p \geq p^\gamma X) = \sum_{n=1}^{\infty} P(X^1 \geq p^\gamma X)^np(1-p)^{n-1} \\
= \frac{pF_{X^1}(p^\gamma X)}{1 - (1-p)F_{X^1}(p^\gamma X)}, \quad (4.2)
\]

where \( X = (x_1, \ldots, x_k), \ p^\gamma X = (p^\gamma x_1, \ldots, p^\gamma x_k) \).

Let \( \phi(X) = \frac{1-F_{X^1}(X)}{F_{X^1}(X)} \) substitute \( \phi(\cdot) \) in (4.2), we conclude that for all \( X > 0 \), \( \phi(X) = p^{-\gamma} \phi(p^\gamma X) \). It follows by iteration that

\[
\phi(X) = p^{-\ell} \phi((p^\gamma)^\ell X) \quad \text{for all } \ell \geq 1, \quad (4.3)
\]

where \( (p^\gamma)^\ell X = (p^{\gamma_1}x_1, p^{\gamma_2}x_2, \ldots, p^{\gamma_k}x_k) \), and hence

\[
\frac{1 - F_{X^1}(X)}{F_{X^1}(X)} = p^{-\ell} \frac{(1 - F_{X^1}((p^\gamma)^\ell X))}{F_{X^1}((p^\gamma)^\ell X)}, \quad (4.4)
\]
for all positive integers \( \ell \geq 1 \), since \( F_{X}(\cdot) \in F_{\bar{X}} \), and \((p')_{\bar{X}} \rightarrow 0 \) as \( \ell \rightarrow \infty \), (4.3) implies that

\[
\frac{1 - F_{X}(\bar{X})}{F_{X}(\bar{X})} = \lim_{\ell \rightarrow \infty} \frac{p^{-\ell}(1 - F_{X}(\langle p' \rangle_{\bar{X}}))}{F_{X}(\langle p' \rangle_{\bar{X}})}.
\]

(4.5)

then in the limit of (4.5), the asymptotic forms of the numerator and denominator are, respectively,

\[
p^{-\ell}(1 - F_{X}(\langle p' \rangle_{\bar{X}})) \sim (p^{-\ell}) \left( p^k \sum_{i=1}^{k} \left( \frac{x_i}{\sigma_i} \right)^{1/\gamma_i} \right) = \sum_{i=1}^{k} \left( \frac{x_i}{\sigma_i} \right)^{1/\gamma_i},
\]

so that

\[
\lim_{\ell \rightarrow \infty} \frac{1 - F_{X}(\langle p' \rangle_{\bar{X}}))}{F_{X}(\langle p' \rangle_{\bar{X}})} = \sum_{i=1}^{k} \left( \frac{x_i}{\sigma_i} \right)^{1/\gamma_i},
\]

and \( F_{X}(\langle p' \rangle_{\bar{X}}) \sim F_{X}(0) = 1 \), so that,

\[
1 - F_{X}(\bar{X}) = \sum_{i=1}^{k} \left( \frac{x_i}{\sigma_i} \right)^{1/\gamma_i},
\]

thus \( F_{X}(\bar{X}) \) is solved as \( F_{X}(\bar{X}) = \{1 + \sum_{i=1}^{k} (\frac{x_i}{\sigma_i})^{1/\gamma_i} \}^{-1} \) for any \( \bar{X} > 0 \), therefore, \( \bar{X}^{l} \sim MP^{(k)}(\text{III})(0, \sigma, \gamma) \) is followed.

In the above theorem, it will be noticed that \( p^{-\ell}m_{p}^{d} \bar{X}^{l} \) is required to hold for only one value of \( p \). The condition for \( F_{X}(\cdot) \), \( 1 - F_{X}(\bar{X}) \sim \sum_{i=1}^{k} (\frac{x_i}{\sigma_i})^{1/\gamma_i} \) as \( \bar{X} \rightarrow 0 \) can be dispensed with \( p^{-\ell}m_{p}^{d} \bar{X}^{l} \) for all \( p \in (0, 1) \).

4.2. Asymptotic results of repeated geometric minimization

A limit theorem lurking behind the characterization of \( MP^{(k)}(\text{III}) \) distribution is presented in Theorem 4.1. The \( MP^{(k)}(\text{III}) \) distribution can arise as the limit multivariate distribution under repeated geometric minimization.

Suppose we start with a sequence of i.i.d. random vectors with common joint survival function \( F_{1}(\cdot) \), i.e., assuming \( \bar{X}^{(1)}_{1}, \bar{X}^{(1)}_{2}, \ldots, \bar{X}^{(1)}_{n}, \ldots \sim i.i.d. F_{1}(\cdot) \), let \( N_{1} \sim \text{geometric}(p_{1}) \), define \( \bar{X}^{(1)}_{(N_{1})} = \min\{\bar{X}^{(1)}_{1}, \bar{X}^{(1)}_{2}, \ldots, \bar{X}^{(1)}_{N_{1}}\} \) as the \( k \)-dim geometric minima of \( \{\bar{X}^{(1)}_{i}\} \), assuming the survival function of \( \bar{X}^{(1)}_{(N_{1})} \) is \( F_{2}(\cdot) \).

Also, let \( \bar{X}^{(2)}_{1}, \bar{X}^{(2)}_{2}, \ldots, \bar{X}^{(2)}_{n}, \ldots \sim i.i.d. F_{2}(\cdot) \) and \( N_{2} \sim \text{geometric}(p_{2}) \), define \( \bar{X}^{(2)}_{(N_{2})} = \min\{\bar{X}^{(2)}_{1}, \bar{X}^{(2)}_{2}, \ldots, \bar{X}^{(2)}_{N_{2}}\} \), suppose \( \bar{X}^{(2)}_{(N_{2})} \sim \bar{F}_{3}(\cdot), \ldots \), in general, after \((\ell - 1)\) steps, let
Proof. In (4.6), let $X_1^{(\ell-1)}, X_2^{(\ell-1)}, \ldots, X_n^{(\ell-1)} \overset{\text{i.i.d.}}{\sim} \tilde{F}_{\ell-1}(\cdot)$, let $N_{\ell-1} \sim \text{geometric}(p_{\ell-1})$, define $X_{(N_{\ell-1})}^{(\ell-1)} = \min\{X_1^{(\ell-1)}, X_2^{(\ell-1)}, \ldots, X_{N_{\ell-1}}^{(\ell-1)}\}$, suppose $X_{(N_{\ell-1})}^{(\ell-1)} \sim \tilde{F}_{\ell}(\cdot)$.

For $\ell = 2, 3, \ldots$, the recursive relation among $\{\tilde{F}_\ell(\cdot)\}$ is derived by conditioning on $N_{\ell-1}$, hence for any $x > 0$, the survival function of $X_{(N_{\ell-1})}^{(\ell-1)}$ is

$$\tilde{F}_\ell(x) = \frac{p_{\ell-1} \tilde{F}_{\ell-1}(x)}{1 - (1 - p_{\ell-1}) \tilde{F}_{\ell-1}(x)},$$

(4.6) then under some suitable normalization on the repeated geometric minima will lead to a non-trivial multivariate limit law as a $\text{MP}^{(k)}(\text{III})$ distribution. It is the following theorem.

**Theorem 4.2.** Suppose that $F_1(\cdot) \in \mathcal{F}_{\sigma, \gamma}$ is a $k$-dim. distribution function satisfying (4.1) for some $\sigma > 0$ and $\gamma > 0$. For each $\ell = 2, 3, \ldots$, define $\tilde{F}_\ell(\cdot)$ sequentially in such a manner that $\tilde{F}_\ell(\cdot)$ is the survival function of a geometric $(p_{\ell-1})$ minima of a random sample of $\{X_1^{(\ell-1)}\}$ i.i.d. $\tilde{F}_{\ell-1}(\cdot)$, and let $\prod_{j=1}^{\ell-1} p_j \to 0$ as $\ell \to \infty$. Then

$$\lim_{\ell \to \infty} \tilde{F}_\ell \left( x \left( \prod_{j=1}^{\ell-1} p_j \right)^{\gamma} \right) = \left\{ 1 + \sum_{i=1}^{k} \left( \frac{x}{\sigma_i} \right)^{1/\gamma_i} \right\}^{-1},$$

(4.7)

for $x > 0$, and $x(\prod_{j=1}^{\ell-1} p_j)^{\gamma} = (x_1(\prod_{j=1}^{\ell-1} p_j)^{\gamma_1}, \ldots, x_k(\prod_{j=1}^{\ell-1} p_j)^{\gamma_k})$ or equivalently

$$\left( \prod_{j=1}^{\ell-1} p_j \right)^{-\gamma} X_{(N_{\ell-1})}^{(\ell-1)} \sim \text{MP}^{(k)}(\text{III})(0, \sigma, \gamma),$$

(4.8)

provided $\lim_{\ell \to \infty} (\prod_{j=1}^{\ell-1} p_j) = 0$.

**Proof.** In (4.6), let $\varphi_\ell(x) = \frac{1-\tilde{F}_\ell(x)}{\tilde{F}_\ell(x)}$, for each $\ell \geq 1$, we conclude that for all $x > 0$, $\varphi_\ell(x) = p_{\ell-1}^{-1} \varphi_{\ell-1}(x)$. It follows by iteration that

$$\varphi_\ell(x) = \left( \prod_{j=1}^{\ell-1} p_j \right)^{-1} \varphi_1(x)$$

(4.9)

for all positive integers $\ell \geq 1$, thus

$$\varphi_\ell \left( x \left( \prod_{j=1}^{\ell-1} p_j \right)^{\gamma} \right) = \left( \prod_{j=1}^{\ell-1} p_j \right)^{-1} \varphi_1 \left( x \left( \prod_{j=1}^{\ell-1} p_j \right)^{\gamma} \right) = \left( \prod_{j=1}^{\ell-1} p_j \right)^{-1} \frac{1 - \tilde{F}_1 \left( x \left( \prod_{j=1}^{\ell-1} p_j \right)^{\gamma} \right)}{\tilde{F}_1 \left( x \left( \prod_{j=1}^{\ell-1} p_j \right)^{\gamma} \right)},$$

(4.10)

where $x(\prod_{j=1}^{\ell-1} p_j)^{\gamma} = (x_1(\prod_{j=1}^{\ell-1} p_j)^{\gamma_1}, \ldots, x_k(\prod_{j=1}^{\ell-1} p_j)^{\gamma_k})$. 


Consider limits on both sides of (4.10), then

\[
\lim_{\ell \to \infty} \frac{(1 - \tilde{F}_\ell(x(\prod_{j=1}^{\ell-1} p_j)^{\frac{1}{\ell}}))}{\tilde{F}_\ell(x(\prod_{j=1}^{\ell-1} p_j)^{\frac{1}{\ell}})} = \lim_{\ell \to \infty} \left( \prod_{j=1}^{\ell-1} p_j \right)^{-1} \frac{(1 - \tilde{F}_1(x(\prod_{j=1}^{\ell-1} p_j)^{\frac{1}{\ell}}))}{\tilde{F}_1(x(\prod_{j=1}^{\ell-1} p_j)^{\frac{1}{\ell}})}.
\]

(4.11)

In the right limit of (4.11) since \(F_1(\cdot) \in \mathcal{F}_{\mathbb{G}},\) and \(\prod_{j=1}^{\ell-1} p_j\) diverges to 0 as \(\ell \to \infty,\) so the point \(x(\prod_{j=1}^{\ell-1} p_j)^{\frac{1}{\ell}} \to 0,\) and the asymptotic forms of the numerator and denominator are, respectively,

\[
\left( \prod_{j=1}^{\ell-1} p_j \right)^{-1} \left\{ 1 - \tilde{F}_1 \left( x \left( \prod_{j=1}^{\ell-1} p_j \right)^{\frac{1}{\ell}} \right) \right\} \\
\sim \left( \prod_{j=1}^{\ell-1} p_j \right)^{-1} \left( \prod_{j=1}^{\ell-1} p_j \right) \left( \sum_{i=1}^{k} \left( \frac{x_i}{\sigma_i} \right)^{1/\gamma_i} \right) = \sum_{i=1}^{k} \left( \frac{x_i}{\sigma_i} \right)^{1/\gamma_i},
\]

and \(\tilde{F}_1(x(\prod_{j=1}^{\ell-1} p_j)^{\frac{1}{\ell}}) \sim \tilde{F}_1(0) = 1,\) then the following limit is obtained:

\[
\lim_{\ell \to \infty} \tilde{F}_\ell \left( x \left( \prod_{j=1}^{\ell-1} p_j \right)^{\frac{1}{\ell}} \right) = \left\{ 1 + \sum_{i=1}^{k} \left( \frac{x_i}{\sigma_i} \right)^{1/\gamma_i} \right\}^{-1}.
\]

(4.12)

Because \(\tilde{F}_\ell(\cdot)\) is the survival function of the geometric \((p_{\ell-1})\) minima of a random sample from \(\tilde{F}_{\ell-1}(\cdot).\) Use the previous notation, let \(N_{\ell-1} \sim \) geometric \((p_{\ell-1})\), define the geometric minima as \(X^{(\ell-1)}_{(N_{\ell-1})}\), then (4.10) is equivalent to

\[
\left( \prod_{j=1}^{\ell-1} p_j \right)^{-\frac{1}{\gamma}} X^{(\ell-1)}_{(N_{\ell-1})} \sim \text{MP}^{(k)}(\text{III})((0, \sigma, \frac{1}{\gamma}),
\]

(4.13)

provided \(\lim_{\ell \to \infty} (\prod_{j=1}^{\ell-1} p_j) = 0. \)

Both of the functions \(\tilde{F}_\ell(\cdot)\) in Theorem 4.1 and \(\tilde{F}_1(\cdot)\) in Theorems 4.2 can be chosen to have support \(\mathcal{Y} > \mathcal{Y},\) rather than \(\mathcal{Y} > 0,\) one merely replaces \(\mathcal{Y}\) by \(\mathcal{Y} - \mathcal{Y}\) in all the equations, and throughout the proofs, then the limiting distribution will be changed to \(\text{MP}^{(k)}(\text{III})(0, \sigma, \frac{1}{\gamma}).\)

It is discerned from Theorems 4.1 and 4.2 that the form of the multivariate limiting distribution will depend only on the local behavior of \(\tilde{F}_\ell(\cdot)\) or \(\tilde{F}_1(\cdot)\) near its lower bound. This is remarkable because in the literature on univariate Pareto distribution, the great emphasis usually places on upper tails.
5. Characterizations of the homogeneous MP(\(k\))(IV) distribution

Yeh [7] studied the homogeneous MP(\(k\))(IV) distribution. It is defined by all the \(\gamma_i = \gamma\) for all \(i = 1, \ldots, k\) in the MP(\(k\))(IV)(\(0, \sigma, \gamma_1, \alpha\)) distribution, hence the survival function of the homogeneous MP(\(k\))(IV) distribution is \(F_X(x) = \left\{ 1 + \sum_{i=1}^k \left( \frac{x_i}{\sigma_i} \right)^{1/\gamma} \right\}^{-\alpha}\), for all \(x > 0\).

According to Yeh [7] and Section 3, Property 3.4 of this paper, it is found that the minima of the ordered coordinates of a homogeneous MP(\(k\))(IV) satisfying the homogeneous MP(\(k\))(IV) vector is a univariate Pareto (IV) variable. Actually, this property leads to a characterization of the homogeneous MP(\(k\))(IV) distribution [5]. Ma’s result is generalized to the more general homogenous MP(\(k\))(IV)(\(\mu, \sigma, \gamma_1, \alpha\)) family.

**Theorem 5.1.** Let \(X\) be a \(k\)-dim. random vector with some \(\mu = (\mu_1, \ldots, \mu_k)\), \(\mu_i \in \mathbb{R}, \gamma > 0, \sigma > 0\) such that for all \(x \geq \mu\), \(\sigma > 0\), the following two statements are equivalent:

1. For all \(\mathbf{a} = (a_1, \ldots, a_k) > 0\) such that \(\|\mathbf{a}\| \triangleq (\sum_{i=1}^k a_i^{1/\gamma})^\gamma = 1\), the weighted minima among the ordered coordinates in \(X\) is defined as \(m = \min_{1 \leq i \leq k} \left\{ \frac{X_i - \mu_i}{a_i \sigma_i} \right\}\) is a univariate Pareto(IV) variable, i.e., \(m \sim P(IV)(0, 1, \gamma, \alpha)\) for some \(\alpha > 0\).
2. The survival function of \(X\) is \(F_X(x) = \left\{ 1 + \sum_{i=1}^k \left( \frac{x_i - \mu_i}{\sigma_i} \right)^{1/\gamma} \right\}^{-\alpha}\) for all \(x \geq \mu\), i.e., \(X \sim MP(k)(IV)(\mu, \sigma, \gamma_1, \alpha)\).

**Proof.** (1) \(\Rightarrow\) (2): For arbitrary \(x \geq \mu\), let \(t = \left\| \frac{1}{\|\mathbf{a}\|} (X - \mu) \right\| \triangleq (\sum_{i=1}^k \left( \frac{X_i - \mu_i}{a_i \sigma_i} \right)^{1/\gamma})^\gamma\), then \(t^1/\gamma = \sum_{i=1}^k \left( \frac{x_i - \mu_i}{a_i \sigma_i} \right)^{1/\gamma}\), writing \(a_i = \frac{(x_i - \mu_i)}{\sigma_i}, i = 1, \ldots, k\), for this choice of \(a_i\) satisfying

\[
\|\mathbf{a}\| = \left( \sum_{i=1}^k a_i^{1/\gamma} \right)^\gamma = \left( \sum_{i=1}^k \left( \frac{(x_i - \mu_i)}{\sigma_i} \right)^{1/\gamma} \right)^\gamma = 1.
\]

The survival function of \(X\) is for any \(x \geq \mu\),

\[
F_X(x) = P\left( \frac{X_1 - \mu_1}{\sigma_1} > \frac{x_1 - \mu_1}{\sigma_1}, \ldots, \frac{X_k - \mu_k}{\sigma_k} > \frac{x_k - \mu_k}{\sigma_k} \right) = P\left( \min_{1 \leq i \leq k} \left\{ \frac{X_i - \mu_i}{a_i \sigma_i} \right\} > t \right) = (1 + t^{1/\gamma})^{-\alpha} = \left\{ 1 + \sum_{i=1}^k \left( \frac{x_i - \mu_i}{\sigma_i} \right)^{1/\gamma} \right\}^{-\alpha}.
\]

Thus, \(X \sim MP(k)(IV)(\mu, \sigma, \gamma_1, \alpha)\), and hence (1) \(\Rightarrow\) (2) is followed.
(2) ⇒ (1): Suppose \( X \sim MP^{(k)}(IV)(\mu, \sigma, \gamma_1, \alpha) \). Then the survival function of \( m \), is as defined in before,

\[
\bar{F}_m(t) = P \left( \frac{X_1 - \mu_1}{a_1 \sigma_1} > t, \ldots, \frac{X_k - \mu_k}{a_k \sigma_k} > t \right) \\
= P(X_1 > \mu_1 + a_1 \sigma_1 t, \ldots, X_k > \mu_k + a_k \sigma_k t) \\
= \left\{ 1 + \sum_{i=1}^{k} \left( \frac{\mu_i + a_i \sigma_i t - \mu_j}{\sigma_i} \right)^{1/\gamma} \right\}^{-\alpha} \\
= \left\{ 1 + \tau^{1/\gamma} \left( \sum_{i=1}^{k} a_i^{1/\gamma} \right) \right\}^{-\alpha} = (1 + \tau^{1/\gamma})^{-\alpha}.
\]

Hence the weighted minima \( m \) is a univariate Pareto variable, and \( m \sim P(IV)(0, 1, \gamma, \alpha) \), therefore, (2) ⇒ (1) is followed. □

6. Truncation property and residual life of the \( MP^{(k)}(II) \) distribution

Arnold [1] mentioned that a truncated univariate Pareto(II) distribution itself is a univariate Pareto(II) distribution. This truncation property holds parallel to the multivariate \( MP^{(k)}(II) \) distribution. It is the following property.

Property 6.1. If \( X \sim MP^{(k)}(II)(\mu, \sigma, \alpha) \), then the distribution of \( X \) truncated at the left at \( x_0 \) is still \( MP^{(k)}(II) \) distributed as

\[
X|X > x_0 \sim MP^{(k)}(II) \left( x_0, \left( 1 + \sum_{j=1}^{k} \left( \frac{x_j^0 - \mu_j}{\sigma_j} \right) \right) \sigma, \alpha \right),
\]

or equivalently, \( X - x_0|X > x_0 \sim MP^{(k)}(II) \left( 0, \left( 1 + \sum_{j=1}^{k} \left( \frac{x_j^0 - \mu_j}{\sigma_j} \right) \right) \sigma, \alpha \right) \) for any given \( x_0 > \mu \).

It is observed from Property 6.1 that the \( MP^{(k)}(II) \) family has the truncation property, hence within the hierarchy of the four generalized multivariate Pareto distributions, \( MP^{(k)}(I) \), (II), (III), (IV), the \( MP^{(k)}(II) \) family is most suited in both economic and reliability contexts.

In reliability contexts, the residual life topic is usually of interest to researcher. Thus we define:

Definition 6.1. For a general multivariate distribution function, \( F_X(\cdot) \), the residual life distribution of \( \tilde{F}_X(\cdot) \) at the truncation point \( \bar{x} \), denoted by \( F^{\bar{x}}(\cdot) \), is defined as

\[
F^{\bar{x}}(y) = P(\bar{x} < X \leq \bar{x} + y|Y > \bar{x}), \ \forall y \geq 0.
\]
The multivariate life distribution at truncation point \( \chi \) is only defined for \( \chi \)'s for which \( P(X > \chi) > 0 \). In most application, if \( \hat{F}_X(\cdot) \) is defined on all \( \chi \geq \mu \), then the assumption is modified to \( P(X > \chi) > 0 \) for all \( \chi \geq \mu \).

Among these four generalized MP\(^{(k)}\)(I), (II), (III), and (IV) distributions, only the MP\(^{(k)}\)(II), and hence MP\(^{(k)}\)(I), families have tractable multivariate residual life distributions. For any \( \chi \geq \mu \), the survival function of the residual life in MP\(^{(k)}\)(II)(\( \mu, \sigma, \alpha \)) family is

\[
\hat{F}_X(y) = P(Y > \chi + y | Y > \chi) = \left\{ \frac{1 + \sum_{i=1}^{k} \frac{y_i}{\sigma_i}}{1 + \sum_{i=1}^{k} \frac{x_i - \mu_i}{\sigma_i}} \right\}^{-\alpha}
\]

for all \( y \geq 0 \). The above resulting expression is also a MP\(^{(k)}\)(II)(0, 1 + \sum_{i=1}^{k} \frac{(\gamma_i - \mu_i)}{\sigma_i}) survival function for each \( \chi \geq \mu \). This observation is equivalent to the fact that the MP\(^{(k)}\)(II) family has the truncation property which is proved in Property 6.1.

**Appendix**

**Proof of Property 2.1.** In this case, the pmf of \( \alpha \) is \( P_\alpha(j) = P(\alpha = j) = p(1 - p)^{j-1}, \; j = 1, 2, \ldots \). Given \( X_i \sim MP^{(k)}(IV)(\mu, \sigma_i, \gamma_i, \alpha) \), the unconditional survival function of \( X \) is

\[
\hat{F}_X(\chi) = \sum_{j=1}^{\infty} P(X > \chi|X = j)P(\alpha = j) = \sum_{j=1}^{\infty} \left\{ 1 + \sum_{i=1}^{k} \frac{(x_i - \mu_i)}{\sigma_i} \right\}^{1/\gamma_i} - j \cdot p(1 - p)^{j-1}
\]

\[
= \left\{ 1 + \sum_{i=1}^{k} \frac{(x_i - \mu_i)}{p^{\gamma_i} \sigma_i} \right\}^{-1/\gamma_i} \quad \text{for all } \chi \geq \mu.
\]

Hence \( X \sim MP^{(k)}(III)(\mu, \sigma p^{\gamma}, \gamma) \), and Property 2.1 is followed. \( \Box \)

**Proof of Property 2.2.** The pdf of \( \alpha \) is \( f_\alpha(\alpha) = \begin{cases} \lambda e^{-\lambda \alpha}, & \alpha > 0, \\ 0, & \text{otherwise}, \end{cases} \) given \( \alpha > 0 \), \( X_i \sim MP^{(k)}(IV)(\mu, \sigma_i, \gamma_i, \alpha) \), then the resulting mixture is

\[
\hat{F}_X(\chi) = \int_{0}^{\infty} \left( 1 + \sum_{i=1}^{k} \frac{(x_i - \mu_i)}{\sigma_i} \right)^{-\alpha} \lambda e^{-\lambda x} = \lambda \int_{0}^{\infty} e^{-\lambda \sum_{i=1}^{k} \frac{(x_i - \mu_i)}{\sigma_i} - \alpha} e^{-\lambda x} dx
\]

\[
= \left\{ 1 + \frac{1}{\lambda} \ln \left( 1 + \sum_{i=1}^{k} \frac{(x_i - \mu_i)}{\sigma_i} \right) \right\}^{-1}, \quad \forall \chi \geq \mu. \quad \Box
\]
Proof of Property 3.4. For any $x \geq \mu$, the survival function of the coordinate extreme minimum is
\[
P(X_{(1)} \geq x) = P(X_1 \geq x, \ldots, X_k \geq x) = \left\{1 + \sum_{i=1}^{k} \left(\frac{x - \mu}{\sigma_i}\right)^{1/\gamma}\right\}^{-\gamma} = \left\{1 + \left(\frac{x - \mu}{1/\sum_{i=1}^{k} (1/\sigma_i)^{1/\gamma}}\right)^{1/\gamma}\right\}^{-\gamma}.
\]
Thus, $X_{(1)} \sim P(IV)(\mu, 1/\sum_{i=1}^{k} (1/\sigma_i)^{1/\gamma}, \gamma, \alpha)$ is followed. \qed

Proof of Property 6.1. Suppose $X \sim MP(k)(\mu, \sigma, \alpha)$, then for any $x \geq x_0$, the conditional survival function of $X$, truncated at the $x_0$ is
\[
P(X > x|X > x_0) = \frac{P(X > x)}{P(X > x_0)} = \left\{1 + \sum_{i=1}^{k} \left(\frac{x_i - \mu}{\sigma_i}\right)^{1/\gamma}\right\}^{-\gamma} \left\{1 + \left(\frac{x - \mu}{1/\sum_{i=1}^{k} (1/\sigma_i)^{1/\gamma}}\right)^{1/\gamma}\right\}^{-\gamma}
\]
\[
= \left\{1 + \sum_{i=1}^{k} \left(\frac{x_i - \mu}{\sigma_i}\right)\right\}^{-\gamma}
\]
\[
= \left\{1 + \sum_{i=1}^{k} \left(\frac{x_i - \mu}{\sigma_i}\right)\right\}^{-\gamma}
\]
\[
= \left\{1 + \sum_{i=1}^{k} \left(\frac{x_i - x_0}{\sigma_i}\right)\right\}^{-\gamma}
\]
\[
= \left\{1 + \sum_{i=1}^{k} \left(\frac{x_i - x_0}{\sigma_i}\right)\right\}^{-\gamma}
\]
\[
= \left\{1 + \sum_{i=1}^{k} \left(\frac{x_i - x_0}{\sigma_i}\right)\right\}^{-\gamma}
\]
\[
= \left\{1 + \sum_{i=1}^{k} \left(\frac{x_i - x_0}{\sigma_i}\right)\right\}^{-\gamma}
\]
\[
= \left\{1 + \sum_{i=1}^{k} \left(\frac{x_i - x_0}{\sigma_i}\right)\right\}^{-\gamma}
\]
\[
hence it is discerned that
\[
X|X > x_0 \sim MP(k)(\mu, \sigma, \alpha)
\]
or equivalently, $X - x_0|X > x_0 \sim MP(k)(\mu, \sigma, \alpha)$ is followed. \qed

References


