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Absolute reflexive retracts and absolute bipartite retracts*

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Abstract

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It is a well-known phenomenon in the study of graph retractions that most results about absolute retracts in the class of bipartite (irreflexive) graphs have analogues about absolute retracts in the class of reflexive graphs, and vice versa. In this paper we make some observations that make the connection explicit. We develop four natural transformations between reflexive graphs and bipartite graphs which preserve the property of being an absolute retract, and allow us to derive results about absolute reflexive retracts from similar results about absolute bipartite retracts and conversely. Then we introduce generic notions that specialize to the appropriate concepts in both cases. This paves the way to a unified view of both theories, leading to absolute retracts of general (i.e., partially reflexive) graphs.

Introduction and definitions

It is our intention here to attempt to explain the striking resemblance between the theory of absolute retracts of reflexive graphs, and the theory of absolute retracts

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of bipartite graphs. In the next section we introduce four transformations between reflexive graphs and bipartite graphs, which map absolute reflexive retracts to absolute bipartite retracts and conversely; these allow us to derive one set of results from the other corresponding set of results. Finally, we also suggest a general theory of absolute retracts that generalizes both these cases.

A *graph* G is a pair of finite sets (V,E), where E (the set of *edges* of G) consists of some unordered pairs vv' of (not necessarily distinct) elements of V (the set of *vertices* of G). An edge vv is called a *loop* of G. If G contains all loops vv (v in V), G is called *reflexive*; if G contains no loop it is called *irreflexive*. (To emphasize that neither restriction applies we sometimes call a general G *partially reflexive*.) The *neighbourhood* of x in G, denoted as $N_G(x)$ (or just $N(x)$ if G is understood), consists of all the vertices y such that $xy \in E$; note that $x \in N(x)$ if and only if x is a loop of G. A graph G is *bipartite* if its vertex set V can be partitioned as $X \cup Y$ in such a way that every edge of G is of the form xy with $x \in X$ and $y \in Y$; if this is the case we say that $X \cup Y$ is a *bipartition* of G. Note that a bipartite graph is by definition irreflexive. A *walk* in G is a sequence $v_0, e_1, v_1, e_2, \dots, e_k, v_k$ of vertices and edges of G, such that $e_i = v_{i-1}v_i$ (for all $i = 1, 2, \dots, k$); the integer k is called the *length* of the walk. We shall always assume in this paper that our graphs are *connected*, i.e., that any two vertices are joined by a walk. The *distance* in G of vertices x and y, denoted by $d_G(x,y)$ (or just $d(x,y)$ if G is clear from the context), is the minimum length of any walk that starts in x and ends in y. If $G=(V,E)$ and $G'=(V',E')$ are two graphs such that V is a subset of V' and E a subset of E', we say that G is a *subgraph* of G'. A *clique* of a graph is a maximal complete subgraph. A *biclique* of a bipartite graph is a maximal complete bipartite subgraph. The subgraph of $G=(V,E)$ *induced* by a subset $\tilde{V}$ of V is $\tilde{G}=(\tilde{V},\tilde{E})$ where $\tilde{E}$ consists of precisely all those $xy \in E$ for which both $x \in \tilde{V}$ and $y \in \tilde{V}$. We say that G is an *isometric* subgraph of G', if $d_G(x,y) = d_{G'}(x,y)$ for each pair of vertices x, y of G.

A subgraph G of G' is a *retract of* G' if there is a mapping $f: V' \to V$ such that (a) f is edge-preserving, i.e., $f(x)f(y) \in E$ for each $xy \in E'$, and (b) f is fixed on each vertex of G, i.e., $f(x)=x$ for all $x \in V$; in such a case the mapping f is called a *retraction* of G' onto G. It is easy to see that if G is a retract of G' then it is an isometric subgraph of G'. If $N_G(x)$ is a subset of $N_G(y)$ for some $y \neq x$ we say that x is *covered* (by y) in G. We say that $v_1, v_2, \dots, v_n$ is an *elimination ordering* of $G=(V,E)$, if $V=\{v_1, v_2, \dots, v_n\}$, $v_{n-1}v_n$ is an edge of G, and, for each $i=1,2,\dots,n-2$, v_i is covered by some v_j in the subgraph of G induced by $\{v_i, v_{i+1}, \dots, v_n\}$. A graph which admits an elimination ordering is called *dismantlable*. If G_i, $i \in I$, is a family of graphs, then the *product* of G_i has vertices $(x_i)_{i\in I}$ with each x_i a vertex of G_i, and $(x_i)_{i\in I}(y_i)_{i\in I}$ is an edge in the product just if each $x_i y_i$ is an edge of G_i. An *irreflexive path* is a graph $G=(V,E)$ with $V=\{1,2,\dots,i\}$ and $E=\{12,\dots,(i-1)i\}$; a *reflexive path* is a graph $G=(V,E)$ with $V=\{1,2,\dots,i\}$ and $E=\{12,\dots,(i-1)i\} \cup \{jj\colon j=1,2,\dots,i\}$.

An *absolute retract in the class of reflexive graphs*, or an *absolute reflexive retract*, is a reflexive graph G such that whenever G is an isometric subgraph of a reflexive graph G', then G is a retract of G'. An *absolute retract in the class of bipartite graphs*, or an *absolute bipartite retract*, is a bipartite graph G such that whenever G is an isometric subgraph of a bipartite graph G', then G is a retract of G'. Absolute reflexive and bipartite retracts have been extensively studied, [1, 2, 4, 5, 10–16, 18, 19], and have several known characterizations, cf. Theorems A and B below. One tool that was useful in many such characterizations concerns the Helly properties of discs (all vertices within a certain distance from a fixed vertex form a *disc*), or equivalently the absence of "holes". The formal definitions of these concepts differ in the two cases (reflexive versus bipartite graphs), and can be found in [4, 5, 14], and [2, 10–13]. We give here instead general definitions that apply in both cases (although these definitions are technically slightly different from the ones given in [14]). In fact, in the last section we take advantage of this new formulation to suggest a general theory of absolute retracts.

Let G be an arbitrary graph and $k \geq 3$ an integer; a *k-hole* in G is a set of vertices $v_1, v_2, \ldots, v_k$ of G and a set of nonnegative integers $d_1, d_2, \ldots, d_k$, such that for each $i \neq j$ there is in G a walk from v_i to v_j of length precisely $d_i + d_j$, and there is in G no vertex x which admits, for each $i = 1, 2, \ldots, k$, a walk from x to v_i of length precisely d_i. A *hole* is any *k-hole*, $k \geq 3$; a *unit hole* is a *k-hole* with $d_1 = d_2 = \cdots = d_k = 1$, and an *almost-unit hole* is a *k-hole* with $d_2 = d_3 = \cdots = d_k = 1$. The *pth iterated neighbourhood* of vertex x in graph G, denoted by $N_G^p(x)$ (or just $N^p(x)$), is defined recursively by $N^1(x) = N(x)$ and $N^{p+1}(x) = N(N^p(x))$, where $N(S) =$

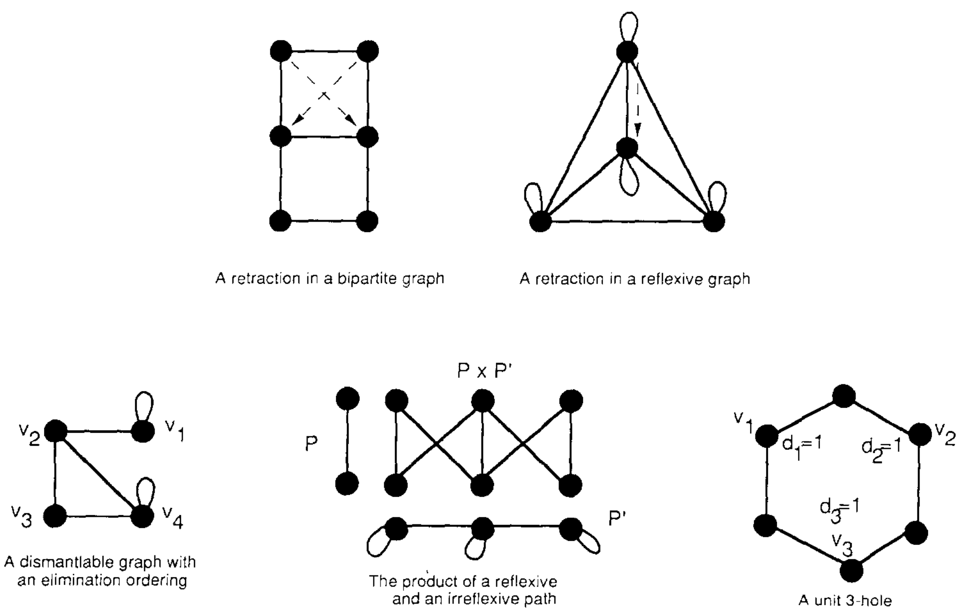

Fig. 1. Some examples illustrating the definitions.

$\bigcup_{x \in S} N(x)$ for a set S of vertices. Equivalently, $N_G^p(x)$ consists of all vertices of G which admit a walk of length p from x. A family of sets S_i, $i \in I$, has the *Helly property*, if any subfamily S_j, $j \in J$, in which any two members have a nonempty intersection, has itself a nonempty intersection. It is a simple exercise to verify that the family of all iterated neighbourhoods in G has the Helly property if and only if G has no holes. We shall say that G is *clique-Helly* if the family of cliques of G has the Helly property, and that G is *neighbourhood-Helly* if the family of neighbourhoods of G has the Helly property, or equivalently, if G has no unit holes. (See Fig. 1.)

There is clearly a strong connection between the study of retracts and the metric properties of graphs. The reader interested in the latter subject is directed to consult the survey [20].

The following theorems summarize some results in [4, 5, 14–16, 18, 19] and [2, 16, 18], and manifest the similarity between the reflexive and the bipartite cases.

Theorem A. *Let G be a reflexive graph. Then the following conditions are equivalent*:

(1) *G is an absolute reflexive retract.*
(2) *G has no holes.*
(3) *G has no 3-holes and no unit holes.*
(4) *G has no almost-unit holes.*
(5) *G is dismantlable and clique-Helly.*
(6) *G is the retract of a product of reflexive paths.*

Theorem B. *Let H be a bipartite graph. Then the following conditions are equivalent*:

(1′) *H is an absolute bipartite retract.*
(2′) *H has no holes.*
(3′) *H has no 3-holes and no unit holes.*
(4′) *H has no almost-unit holes.*
(5′) *H is dismantlable and neighbourhood-Helly.*
(6′) *H is the retract of a product of irreflexive paths.*

Transformations between reflexive and bipartite graphs

In this section we shall consistently denote reflexive graphs by G, G', etc., while H, H', etc., shall stand for bipartite graphs.

The first transformation associates with a reflexive graph G the bipartite graph $B(G)$, called the *bigraph* of G. If V is the vertex set of G then the vertex set of $B(G)$ consists of two disjoint copies V' and V'' of V, with $v' \in V'$ and $w'' \in V''$ adjacent in $B(G)$ just if the corresponding vertices v and w are adjacent in G. (Note that $v'v''$ is always an edge of $B(G)$, because vv is always an edge of G.) This is a standard operation for general graphs, and instances of its use include [8,9]. For future reference we examine how the distances in $B(G)$ relate to the distances in G: Let x, y be two vertices of G. Then we have

$$d_{B(G)}(x',y') = d_{B(G)}(x'',y'') = 2\lceil d_G(x,y)/2\rceil,$$
$$d_{B(G)}(x',y'') = d_{B(G)}(x'',y') = 2\lceil (d_G(x,y)-1)/2\rceil + 1.$$

The second transformation associates with a bipartite graph H, and a bipartition $X \cup Y$ of H, two reflexive graphs $S_X(H)$ and $S_Y(H)$, called the *sesqui-powers* of H. Both sesqui-powers are defined on the vertex set of H: in $S_X(H)$ two vertices are adjacent just if they are identical, or are adjacent in H, or both belong to X and have a common neighbour in H; $S_Y(H)$ is defined analogously. (Two vertices are adjacent in the second power of H if there is a walk of length two joining them in H; thus $S_X(H)$ and $S_Y(H)$ have some claim to being one-and-a-half-th powers of H, whence their name.) We again calculate how the distances in $S = S_X(H)$ relate to the distances in H (a similar calculation holds for $S_Y(H)$): Let u, v be two vertices of H. Then

- for $u, v \in X$ we have $d_H(u,v)$ even and $d_S(u,v) = d_H(u,v)/2$,
- for $u \in X$ and $v \in Y$ we have $d_H(u,v)$ odd and $d_S(u,v) = [d_H(u,v)+1]/2$,
- for $u, v \in Y$ we have $d_H(u,v)$ even and $d_S(u,v) = [d_H(u,v)+2]/2$.

The third transformation assigns to a reflexive graph G the bipartite graph $I(G)$, known as the *vertex-clique incidence graph*. The vertices of $I(G)$ are the vertices of G together with the cliques, i.e., maximal complete subgraphs, of G. The edges of $I(G)$ are precisely the unordered pairs vC where v is a vertex of G and C is a clique of G containing v. This is also a well-known transformation for general graphs.

The last operation assigns to a bipartite graph H the reflexive graph $E(H)$, which we call the *edge graph* of H. The vertices of $E(H)$ are the edges of H, and two such vertices are adjacent just if they (as edges of H) intersect, or lie on a four-cycle of H. While the edge graph contains the well-known line graph, we believe this construct to be new.

We remark in passing that B and E are actually functors between the category of reflexive graphs and homomorphisms and the category of bipartite graphs and homomorphisms. Similarly, the pair S_X, S_Y can be viewed as a functor from the category of bipartite graphs to the paired category of reflexive graphs.

We now proceed to prove that each of the four transformations preserves important properties related to retracts. This will permit us to transfer results about one kind of absolute retracts to similar results about the other.

Theorem 1. *Let G be a reflexive graph and $B(G)$ its bigraph. Then*

(a) *G is an absolute reflexive retract if and only if $B(G)$ is a bipartite absolute retract.*

(b) *G has no holes if and only if $B(G)$ has no holes.*

(c) *G has no unit holes if and only if $B(G)$ has no unit holes.*

(d) *G has no* 3*-holes if and only if* $B(G)$ *has no* 3*-holes.*
(e) *G has no almost-unit holes if and only if* $B(G)$ *has no almost-unit holes.*
(f) *G is dismantlable if and only if* $B(G)$ *is dismantlable.*
(g) *G is clique-Helly if and only if* $B(G)$ *is neighbourhood-Helly.*

Proof. (a) Suppose that G is an absolute reflexive retract, and that $B(G)$ is an isometric subgraph of a bipartite graph H with a bipartition $X \cup Y$; suppose further that X contains V' and Y contains V'' (from the definition of $B(G)$). We first construct from H a reflexive graph G' containing G as an isometric subgraph: this is done by identifying each $v' \in V'$ with the corresponding $v'' \in V''$ in $B(G)$, and adding all loops xx. If G were not isometric in G' then for some vertices $x = v_0$, $y = v_k$ of G, a walk $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ in G' has $k < d_G(x, y)$. We may assume that k is the least integer for which such a walk exists; then v_i is not a vertex of G, for $1 \le i \le k-1$. If v_1 is in X then define $v_0^* = v_0''$ and $e_1^* = v_0'' v_1$, otherwise define $v_0^* = v_0'$ and $e_1^* = v_0' v_1$; define v_k^* and e_k^* in a similar fashion with respect to v_{k-1}. Then it easily follows that the walk $v_0^*, e_1^*, v_1, e_2, \ldots, e_k^*, v_k^*$ in H has $k < d_{B(G)}(v_0^*, v_k^*)$ contrary to the assumption that $B(G)$ is isometric in H. Since G is an absolute reflexive retract, there is a retraction $f: G' \to G$. We then define $r: H \to B(G)$ as follows: if x is a vertex of $B(G)$ then $r(x) = x$; otherwise if $f(x) = v$ and $x \in X$ then $r(x) = v'$, and if $f(x) = v$ and $x \in Y$ then $r(x) = v''$. It is straightforward to verify that r is a retraction. Thus $B(G)$ is an absolute bipartite retract.

To prove the converse, let $G = (V, E)$ be a reflexive graph, and suppose that $B(G)$ is an absolute bipartite retract. We will show that whenever G is an isometric subgraph of a reflexive graph G' then G is a retract of G'. We do this by induction on n', the number of vertices of G'. The base case $n' = |V|$ is trivial, so suppose $n' > |V|$. It is easy to see that $B(G)$ is an isometric subgraph of $B(G')$. Thus there is a retraction $f: B(G') \to B(G)$. Let v be a vertex of G' that does not belong to G; note that the image $f(v')$ under the retraction must be some u'. We claim that the graph G'', obtained from G' by identifying u and v, contains G as an isometric subgraph; this will prove that G is a retract of G'', and hence also of G' (to define a retraction $G' \to G$ map v to u and all other vertices just as in G''). Suppose that G'' contains a walk of length k joining two vertices x, y of G with $k < d_G(x, y)$. Then there are two walks of length k in $B(G'')$, each joining a pair of vertices of $B(G)$ whose distance in $B(G)$ is greater than k. Now observe that $B(G'')$ is obtained from $B(G')$ by identifying the two pairs of vertices u', v' and u'', v''. Let B^* be obtained from $B(G')$ by just identifying u' and v'. It is easy to see that one of the above two walks of length k in $B(G'')$ is also a walk in B^*, joining two vertices of $B(G)$ of distance greater than k. Note that, since the retraction f maps v' to u', $B(G)$ is a retract of B^* (map all vertices according to f). This contradicts the fact that $B(G)$ is not isometric in B^*, and completes the proof of (a).

Statements (b), (c), (d), and (e) all follow from the observation that

$$N_{B(G)}^k(u') = \{v'' : v \in N_G^k(u)\} \quad \text{if } k \text{ is odd},$$

$$N^k_{B(G)}(u') = \{v' \colon v \in N^k_G(u)\} \quad \text{if } k \text{ is even,}$$

and similarly for $N^k_{B(G)}(u'')$.

As for (f), any elimination ordering $z_1, z_2, \ldots, z_n$ of G can be turned into the elimination ordering $z'_1, z''_1, z'_2, z''_2, \ldots, z'_n, z''_n$ of $B(G)$. Conversely, it is easy to see that every elimination ordering of $B(G)$ may be modified to produce another elimination ordering in which each x'' immediately follows x'. (This depends on the fact that if a graph F is dismantlable and x is a vertex covered in F, then $F-x$ is also dismantlable.) In such an elimination ordering of $B(G)$ we may substitute x for the pair x', x'' (for all x), to obtain an elimination ordering of G.

Finally, (g) follows from the easy observation that the bicliques in $B(G)$ are all of the form $B(C)$ where C is a clique in G. □

Theorem 2. *Let H be a bipartite graph, with a bipartition $X \cup Y$, and let $S_X(H)$ and $S_Y(H)$ be the sesqui-powers of H. Then*

(a) *H is an absolute bipartite retract if and only if both $S_X(H)$ and $S_Y(H)$ are absolute reflexive retracts.*

(b) *H has no holes if and only if both $S_X(H)$ and $S_Y(H)$ have no holes.*

(c) *H is dismantlable and neighbourhood-Helly if and only if both $S_X(H)$ and $S_Y(H)$ are dismantlable and clique-Helly.*

Proof. (a) Let H be an absolute bipartite retract, and suppose $S_X(H)$ is an isometric subgraph of G. Let $\tilde{H}$ be the bipartite graph obtained from G by (i) the deletion of all loops, (ii) the deletion of all edges joining two vertices of X, and (iii) the subdivision of each edge joining a vertex not in $X \cup Y$ with a vertex not in Y. We claim that H is an isometric subgraph of $\tilde{H}$. Otherwise, for some vertices $u = v_0$, $v = v_n$ of H, a walk $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ in $\tilde{H}$ has length $k < d_H(u, v)$. We may assume that k is the least integer for which such a walk exists; then v_i is not a vertex of H, for $1 \le i \le k-1$. It follows from the definition of $\tilde{H}$ that $d_{\tilde{H}}(u, v) = 2d_G(u, v)$ if $u, v \in X$, $d_{\tilde{H}}(u, v) = 2d_G(u, v) - 1$ if $u \in X$, $v \in Y$, and $d_{\tilde{H}}(u, v) = 2d_G(u, v) - 2$ if $u, v \in Y$. Say, $u, v \in X$ (the other cases are similar): Since $S = S_X(H)$ is isometric in G, $d_{\tilde{H}}(u, v) = 2d_S(u, v) = d_H(u, v)$, contrary to the assumption that $k < d_H(u, v)$.

For the converse, let H be a bipartite graph, with bipartition $X \cup Y$, which is not an absolute bipartite retract. Then H is not a retract of some bipartite graph H' containing H as an isometric subgraph. Let $X' \cup Y'$ be a bipartition of H' with X' containing X and Y' containing Y. Suppose $S_X(H)$ is an absolute reflexive retract. Then $S_X(H)$ is a retract of $S_{X'}(H')$, since $S_X(H)$ is an isometric subgraph of $S_{X'}(H')$. Among all possible retractions of $S_{X'}(H')$ onto $S_X(H)$, let r minimize the number a of vertices of X' which are mapped to Y, and, subject to this, the number b of vertices of Y' which are mapped to X. Observe that $a + b \ne 0$, since H is not a retract of H'. Observe, too, that the neighbourhood in $S_{X'}(H')$ of any vertex of Y' is a complete graph. It now follows from the minimality of a that $a = 0$: Indeed, any $v \in X'$ with $r(v) \in Y$ has $r(w) \in X$ for all neighbours w of v in $S_{X'}(H')$, and hence could be mapped instead to any one of these $r(w)$. Thus, $b \neq 0$. Choose $v \in Y'$ such that $r(v) \in X$, and let C be the set of all $r(w)$ where w is a neighbour of v in $S_{X'}(H')$. By the above observation, C is a complete subgraph of $S_X(H)$, and, since $a=0$, C lies entirely in X. By the minimality of b, the vertices in C do not have a common neighbour in Y, for otherwise v could have been mapped to it. As a subset of H, C consists of vertices that are pairwise at distance two, but which do not have a common neighbour. Now form the reflexive graph R obtained from $S_Y(H)$ by adding one new vertex adjacent just to (itself and) the vertices of C. Clearly, $S_Y(H)$ is isometric in R and is not a retract of R. Therefore $S_Y(H)$ is not an absolute reflexive retract.

(b) Assume that $v_1, v_2, \dots, v_k$ with $d_1, d_2, \dots, d_k$ is a hole in $S_X(H)$ and that $v_1, v_2, \dots, v_t$ are vertices from X, and $v_{t+1}, v_{t+2}, \dots, v_k$ vertices from Y. Let $d_i' = 2d_i$ when $i = 1, 2, \dots, t$ and $d_i' = 2d_i - 1$ when $i = t+1, t+2, \dots, k$. We claim that $v_1, v_2, \dots, v_k$ with $d_1', d_2', \dots, d_k'$ is a hole in H. To verify this we first need to check that any pair v_i, v_j of vertices has a walk of length $d_i' + d_j'$ in H. Let $v_{i,j}$ be a vertex in $S_X(H)$ which has a walk of length d_i to v_i and a walk of length d_j to v_j in $S_X(H)$. Such a vertex must exist, for each $i \neq j$, because $v_1, v_2, \dots, v_k$ with $d_1, d_2, \dots, d_k$ is a hole in $S_X(H)$. It is now a routine exercise to verify that $v_{i,j}$ has a walk of length d_i' to v_i and a walk of length d_j' to v_j in H. Finally, we need to verify that there is in H no vertex v with walks of lengths d_i' to v_i for all $i = 1, 2, \dots, k$. An argument similar to the above exercise shows that such a vertex v would have walks in $S_X(H)$ of lengths d_i to v_i for all $i = 1, 2, \dots, k$, contrary to the fact that $v_1, v_2, \dots, v_k$ with $d_1, d_2, \dots, d_k$ is a hole.

For the converse, assume that $v_1, v_2, \dots, v_k$ with $d_1, d_2, \dots, d_k$ is a hole in H and that $v_1, v_2, \dots, v_t$ are vertices from X, and $v_{t+1}, v_{t+2}, \dots, v_k$ vertices from Y. It is easy to see (since H is bipartite) that the parity of all $d_1, d_2, \dots, d_t$ is the same, and is the opposite of the parity of all $d_{t+1}, d_{t+2}, \dots, d_k$. Suppose all of $d_1, d_2, \dots, d_t$ are even, and all of $d_{t+1}, d_{t+2}, \dots, d_k$ odd. Let $\tilde{d}_i = d_i/2$ for $i = 1, 2, \dots, t$, and $\tilde{d}_i = (d_i + 1)/2$ for $i = t+1, t+2, \dots, k$. Then $v_1, v_2, \dots, v_k$ with $\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_k$ is a hole in $S_X(H)$.

(c) Let H be dismantlable, and $z_1, z_2, \dots, z_n$ an elimination ordering of H. Suppose the vertices of X appear there in the order $x_1, x_2, \dots, x_k$, and let $y_1, y_2, \dots, y_m$ be any ordering of the vertices of Y. Then we claim that $y_1, y_2, \dots, y_m, x_1, x_2, \dots, x_k$ is an elimination ordering of $S_X(H)$. Indeed, each y_i is covered by any x_j adjacent to it in H, and each x_i is covered by the same x_s as in the ordering $z_1, z_2, \dots, z_n$. Thus $S_X(H)$ is dismantlable, and a similar argument proves that $S_Y(H)$ is also dismantlable. If H is neighbourhood-Helly then each clique of $S_X(H)$ consists of some y in Y together with all its neighbours in H. Moreover, two such cliques cannot share the vertex in Y, by maximality of each clique. Therefore, the fact that $S_X(H)$ is clique-Helly follows directly from the fact that H is neighbourhood-Helly, and a similar argument proves that $S_Y(H)$ is also clique-Helly.

Conversely, if both $S_X(H)$ and $S_Y(H)$ are clique-Helly, and if $N_H(x_1), N_H(x_2), \dots, N_H(x_p)$ (each x_i in X) are pairwise intersecting neighbourhoods in H, then each $N_H(x_i) \cup \{x_i\}$ is contained in some clique C_i of $S_Y(H)$, and the cliques $C_1, C_2, \dots, C_p$

are also pairwise intersecting. Since $S_Y(H)$ is clique-Helly, the neighbourhoods $N_H(x_1), N_H(x_2), \dots, N_H(x_p)$ have a common point. Since a similar argument applies to $N_H(y_1), N_H(y_2), \dots, N_H(y_p)$ (each y_i in Y), H is neighbourhood-Helly. Finally, we come to the dismantlability of H. Note that here it is not sufficient to assume that both $S_X(H)$ and $S_Y(H)$ are dismantlable; indeed, this does not imply that H is dismantlable as can be seen by taking for H the six-cycle. However, we are assuming that $S_X(H)$ and $S_Y(H)$ are clique-Helly, i.e. (cf. above), that H is neighbourhood-Helly. Let $y_1, y_2, \dots, y_r, x_1, x_2, \dots, x_p$ be an elimination ordering of $S = S_X(H)$; such an ordering must exist because for each i, the vertex y_i is covered in $S_X(H) - y_1 - \cdots - y_{i-1}$ by any x_i adjacent to it in H, and $S_X(H) - y_1 - \cdots - y_i$ is also dismantlable. We shall now construct an elimination ordering of H in which the vertices of X appear in the order $x_1, x_2, \dots, x_p$. Assume that x_1 is covered by some x_j in $S_X(H) - y_1 - \cdots - y_r$. If each $y \in N_H(x_1)$ is adjacent to x_j, then x_1 is also covered in H. Otherwise suppose that $z_1, z_2, \dots, z_q$ are all the neighbours of x_1 (in H) not adjacent to x_j (in H). For each z_i we have $N_H(z_i)$ included in $N_S(x_1)$, and $N_S(x_1)$ included in $N_S(x_j)$; since H is neighbourhood-Helly, there must exist a vertex y in Y adjacent to x_j and all of $N_H(z_i)$. This allows us to dismantle the $z_1, z_2, \dots, z_q$ and then x_1. (In other words, every z_i is covered by y and x_1 is covered by x_j in $H - z_1 - z_2 - \cdots - z_q$.) In either case, we obtain a subgraph H' of H which is also neighbourhood-Helly, contains $x_2, x_3, \dots, x_p$, and such that $S_X(H')$ is dismantlable. Hence we can continue this way to obtain an elimination ordering of H. □

Theorem 3. *Let G be a reflexive graph and $I(G)$ its vertex-clique incidence graph. Then*

(a) *G is dismantlable if and only if $I(G)$ is dismantlable.*

(b) *G is clique-Helly if and only if $I(G)$ is neighbourhood-Helly.*

(c) *G is an absolute reflexive retract if and only if $I(G)$ is an absolute bipartite retract.*

Proof. (a) Let $v_1, v_2, \dots, v_n$ be an elimination ordering of G. We define an elimination ordering of $I(G)$ by inserting each clique C as soon as possible into $v_1, v_2, \dots, v_n$. (C can certainly be inserted by the time all but one of its vertices have been removed.) Indeed, if v_i is covered by v_j in the subgraph of G induced by $\{v_i, v_{i+1}, \dots, v_n\}$, then any clique of that subgraph which contains v_i also contains v_j. Conversely, from an elimination ordering of $I(G)$ one obtains an elimination ordering of G simply by the deletion of all cliques. To see this, observe that two vertices are adjacent in G if and only if there is a clique containing both.

(b) Observe that every clique C of G equals the neighbourhood of C in $I(G)$. Hence if $I(G)$ is neighbourhood-Helly then G is clique-Helly. For the converse, it only remains to consider a family of pairwise intersecting neighbourhoods centered at vertices, $v_1, v_2, \dots, v_k$. The neighbourhood of a vertex v_i in $I(G)$ consists of all the cliques of G which contain v_i. Thus any two v_i lie in a common clique, i.e., are adjacent. Therefore $v_1, v_2, \dots, v_k$ induce a complete subgraph, and hence belong to one clique. Thus $I(G)$ is neighbourhood-Helly.

(c) now follows from (a), (b), and Theorems A and B. □

Theorem 4. *Let H be a bipartite graph, and let $E(H)$ be its edge graph. Then*

(a) *H is dismantlable if and only if $E(H)$ is dismantlable.*

(b) *H is neighbourhood-Helly if and only if $E(H)$ is clique-Helly.*

(c) *H is an absolute bipartite retract if and only if $E(H)$ is a reflexive absolute retract.*

Proof. (a) Assume that H is dismantlable and $z_1, z_2, \dots, z_n$ is an elimination ordering. Define an ordering $e_1, e_2, \dots, e_m$ of $E(H)$ as follows: $e_1, e_2, \dots, e_k$, for some k, are all the edges of H incident with z_1 (in any order), $e_{k+1}, e_{k+2}, \dots, e_{k+t}$, for some t, all the edges incident with z_2, and so on. We claim that $e_1, e_2, \dots, e_m$ is an elimination ordering of $E(H)$. Specifically, we note that if $e_i = z_r z_s$, with $r < s$, and if z_r is covered in H by some z_p with $p > r$, then $z_p z_s$ is some e_j with $j > i$. It is now easy to verify that e_j covers e_i in $E(H)$. Conversely, assume that $E(H)$ is dismantlable. Let $X \cup Y$ be a bipartition of H and suppose that $x_1 y_1$, with $x_1 \in X$, $y_1 \in Y$, is the first element in an elimination ordering of $E(H)$, covered by some $x_i y_i$, $x_i \in X$, $y_i \in Y$; without loss of generality we may assume that $x_1 \neq x_i$. Then x_1 is covered by x_i in H: Indeed, any edge $x_1 y$ is adjacent in $E(H)$ to $x_1 y_1$ and hence to $x_i y_i$. Therefore either $y = y_i$ or $x_1 y x_i y_i$ is a four-cycle of H; in either case $x_i y$ is also an edge of H. Thus we can begin an elimination ordering of H with x_1. By the first part of this proof we can modify the elimination ordering of $E(H)$ by first removing all edges of H incident with x_1. Then $E(H - x_1)$ is still dismantlable, and we can continue to obtain an elimination ordering of H.

(b) If edges $x_i y_i$ $(i = 1, 2, \dots, k)$ form a clique in $E(H)$, then the vertices x_i, y_i $(i = 1, 2, \dots, k)$ form a biclique in H, and conversely, the edges of a biclique in H form a clique in $E(H)$. Two such cliques have a vertex in common if and only if the bicliques have an edge in common. Now suppose H is neighbourhood-Helly, and we have a set of cliques in $E(H)$ which have pairwise nonempty intersection. Let X_i, Y_i be the vertex sets of the corresponding bicliques in H (we are assuming again that $X \cup Y$ is a fixed bipartition of H and that each X_i is a subset of X, each Y_i a subset of Y). Since any two bicliques have a common edge, the neighbourhoods $N_H(x)$ for all x in the union of all X_i are pairwise intersecting. Hence there is a vertex y_0 common to all $N_H(x)$. By the maximality of the bicliques, y_0 belongs to each Y_i. Similarly, we find a vertex x_0 common to all X_i, so that $x_0 y_0$ is an edge common to all bicliques of H and hence to all the cliques of $E(H)$.

Conversely, suppose that $E(H)$ is clique-Helly, and $N_H(x_i)$ $(i = 1, 2, \dots, m)$ are pairwise intersecting. Extend each $N_H(x_i)$ to a biclique. Then, by our hypothesis and the above observations, there is an edge xy common to all these bicliques. In particular, y belongs to each $N_H(x_i)$ $(i = 1, 2, \dots, m)$; hence H is neighbourhood-Helly.

(c) again follows from (a), (b), and Theorems A and B. $\square$

Conclusions

The reader has probably noticed the redundancy in our treatment of Theorems 1 and 2: Indeed, having proved statement (a) in Theorem 1 or 2 implies all the other statements of that theorem by appealing to Theorems A and B. However, we chose to prove Theorems 1 and 2 without appealing to Theorems A and B, because *we are now able to derive equivalences in Theorem* A *from those in Theorem* B *and vice versa*: For instance, the equivalence of (1) and (2) follows from the equivalence of (1′) and (2′) by applying Theorem 1(a) and (b):

$$\begin{tikzcd}
G \text{ is an absolute reflexive retract} \arrow[r, leftrightarrow, "\text{Theorem 1(a)}"'] \arrow[d, leftrightarrow, "\text{Theorem A (1) and (2)}"'] & B(G) \text{ is an absolute bipartite retract} \arrow[d, leftrightarrow, "\text{Theorem B (1′) and (2′)}"] \\
G \text{ has no holes} \arrow[r, leftrightarrow, "\text{Theorem 1(b)}"] & B(G) \text{ has no holes}
\end{tikzcd}$$

(In fact, any of the pictured equivalences follows from the other three.) Similarly, the equivalence of (1) and (3) follows from the equivalence of (1′) and (3′) and Theorem 1(a), (c) and (d); the equivalence of (1) and (4) from the equivalence of (1′) and (4′) and Theorem 1(a) and (e); the equivalence of (1) and (5) from the equivalence of (1′) and (5′) and Theorem 1(a), (f), and (g). In the same spirit, the equivalence of (1′) and (2′) follows from the equivalence of (1) and (2) and Theorem 2(a) and (b); the equivalence of (1′) and (5′) from that of (1) and (5) and Theorem 2(a) and (c).

Finally, we would like to give a definition of absolute retract that applies to general (partially reflexive) graphs, and specializes to the appropriate concepts when the graph is reflexive or bipartite. For this purpose we need to strengthen the definition of an isometric subgraph. We shall say that the subgraph G of a graph G' is a *strongly isometric* subgraph of G' if for each pair of vertices x, y of G and each walk from x to y in G' there exists in G a walk from x to y of the same length. Note that for reflexive graphs a subgraph is strongly isometric if and only if it is isometric, and that a bipartite graph G is a strongly isometric subgraph of G' if and only if G' is also bipartite and contains G as an isometric subgraph. Hence we now define an *absolute retract* to be a graph G such that whenever G is a strongly isometric subgraph of a graph G', then G is a retract of G'. It is clear that an absolute retract which is reflexive is an absolute reflexive retract, and an absolute retract which is bipartite is an absolute bipartite retract. In a future paper we hope to give characterizations of these absolute retracts that give a common generalization of Theorems A and B, thus shedding further light on their similarity.

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