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Continuity of spectrum and approximate point spectrum on operator matrices

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ABSTRACT

Let X and Y be given Banach spaces. For $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$, let M_C be the operator defined on $X \oplus Y$ by $M_C = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$. In this paper we give conditions for continuity of τ at M_C through continuity of τ at A and B, where τ can be equal to the spectrum or approximate point spectrum.

1. Preliminaries and notations

Let X and Y be Banach spaces and let $\mathcal{B}(X, Y)$ denote the space of all bounded linear operators from X to Y; abbreviate $\mathcal{B}(X, X)$ to $\mathcal{B}(X)$. For $T \in \mathcal{B}(X)$, let $\sigma(T)$, $\sigma_l(T)$, $\sigma_r(T)$, $\sigma_{ap}(T)$ and $\sigma_{su}(T)$ denote respectively the spectrum, the left spectrum, the right spectrum, the approximate point spectrum and the surjective spectrum of T.

If $T \in \mathcal{B}(X)$ we write $N(T)$ and $R(T)$ for the null space and range of T. Also, let $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim X/R(T)$, when theses spaces are finite dimensional. We set $\alpha(T) = \infty$ and $\beta(T) = \infty$, when $N(T)$ and $X/R(T)$ are not finite dimensional. An operator $T \in \mathcal{B}(X)$ is called upper semi-Fredholm, respectively lower semi-Fredholm, if it has closed range and $\alpha(T) < \infty$, respectively $\beta(T) < \infty$. The set of all upper (resp. lower) semi-Fredholm operators in $\mathcal{B}(X)$ is denoted by $\Phi_+(X)$ (resp. $\Phi_-(X)$). We say that $T \in \mathcal{B}(X)$ is a semi-Fredholm operator if $T \in \Phi_-(X) \cup \Phi_+(X) = \Phi_\pm(X)$, and T is a Fredholm operator if $T \in \Phi_-(X) \cap \Phi_+(X) = \Phi(X)$. The index of a semi-Fredholm operator T is defined as $i(T) = \alpha(T) - \beta(T)$. For an operator $T \in \mathcal{B}(X)$, the ascent $\operatorname{asc}(T)$ and the descent $\operatorname{des}(T)$ are given by $\operatorname{asc}(T) = \inf\{n \in \mathbb{N} \mid N(T^n) = N(T^{n+1})\}$ and $\operatorname{des}(T) = \inf\{n \in \mathbb{N} \mid R(T^n) = R(T^{n+1})\}$, respectively; the infimum over the empty set is taken to be ∞.

Let $K(X)$ denote the set of all compact linear operators in $\mathcal{B}(X)$. If $\pi : \mathcal{B}(X) \to \mathcal{B}(X)/K(X)$ is the canonical map, then the essential spectrum of an operator $T \in \mathcal{B}(X)$, $\sigma_e(T)$, is the spectrum of $\pi(T)$ in the Calkin algebra $\mathcal{B}(X)/K(X)$. Also, the left essential spectrum $\sigma_{le}(T)$ (the right essential spectrum $\sigma_{re}(T)$) is the left spectrum (right spectrum) of $\pi(T)$. We set $\sigma_{lre}(T) = \sigma_{le}(T) \cap \sigma_{re}(T)$. Now, let $\sigma_{s-F}(T)$ denote the set of all $\lambda \in \mathbb{C}$ such that $\lambda - T$ is not semi-Fredholm. It is clear that $\sigma_{s-F}(T) \subseteq \sigma_{lre}(T)$, but the opposite inclusion is not always satisfied in general Banach spaces. These classes of operators coincide in the case of Hilbert spaces.

The Weyl spectrum, the Browder spectrum and the set of Riesz points of $T \in \mathcal{B}(X)$ are defined respectively by $\sigma_w(T) = \{\lambda \in \mathbb{C} \mid \lambda - T$ is not a Fredholm operator of index $0\}$, $\sigma_b(T) = \{\lambda \in \mathbb{C} \mid \lambda - T$ is not a Fredholm operator with finite ascent and descent$\}$ and $\pi_0(T) = \{\lambda \in \mathbb{C} \mid \lambda$ is an isolated eigenvalues of T of finite algebraic multiplicity$\}$. Following [1], we say that $T \in \mathcal{B}(X)$ satisfies Browder's theorem, if $\sigma(T) \setminus \sigma_w(T) = \pi_0(T)$.

The next concepts are part of classical point set topology:

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Letting $\{K_n\}$ be a sequence of non-empty compact subsets of $\mathbb{C}$, define

- $\liminf K_n = \{\lambda \in \mathbb{C} \mid \text{for every } \epsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that } B(\lambda,\epsilon) \cap K_n \neq \emptyset \text{ for all } n \geqslant N\}$.
- $\limsup K_n = \{\lambda \in \mathbb{C} \mid \text{for every } \epsilon > 0, \text{ there exists } J \subseteq \mathbb{N} \text{ infinite such that } B(\lambda,\epsilon) \cap K_n \neq \emptyset \text{ for all } n \in J\}$.

It is well known that

(i) $\lambda \in \liminf K_n$ if and only if there exists $\{\lambda_n\} \subseteq \mathbb{C}$ such that $\lim_{n\to\infty} \lambda_n = \lambda$ and $\lambda_n \in K_n$ for all $n \in \mathbb{N}$.
(ii) $\lambda \in \limsup K_n$ if and only if there exists an increasing sequence of natural numbers $n_1 < n_2 < n_3 < \cdots$ and points $\lambda_{n_k} \in K_{n_k}$ such that $\lim_{k\to\infty} \lambda_{n_k} = \lambda$.

Let $T_n, T \in \mathcal{B}(X)$. We say that T_n converge in norm to T, and is denoted by $T_n \to T$, if $\lim_{n\to\infty} \|T_n - T\| = 0$. A function τ, defined on $\mathcal{B}(X)$, whose values are non-empty compact subsets of $\mathbb{C}$ is said to be upper (lower) semi-continuous at T, when if $T_n \to T$ then $\limsup \tau(T_n) \subseteq \tau(T)$ ($\tau(T) \subseteq \liminf \tau(T_n)$). It is known that if τ is bounded on convergent sequences, then τ is continuous in "the Hausdorff metric" if and only if τ is both upper and lower semi-continuous at T.

2. Spectral continuity on Banach spaces

Throughout this paper, X and Y denote Banach spaces. Let $T, T_n \in \mathcal{B}(X)$ be such that T_n converge in norm to T. By [2, Lemma 3] it is easy to see that for every $\lambda \in \operatorname{iso}\sigma(T)$ we have that $\lambda \in \liminf \sigma(T_n)$. Even we have more, if $\lambda \in \pi_0(T)$, then there exists a sequence of complex numbers $\{\lambda_n\}$ such that $\lambda_n \in \pi_0(T_n)$, for every positive integer n, and $\lambda_n \to \lambda$ (see, for example, [3, Corollary 2.13]). Hence, we have the next lemma:

Lemma 1. *π_0 is lower semi-continuous.*

A bounded linear operator $T \in \mathcal{B}(X)$ is said to have the single-valued extension property (SVEP, for short) at $\lambda \in \mathbb{C}$, if for every open neighborhood U_λ of λ, the only analytic function $f : U_\lambda \to X$ which satisfies the equation $(T - \mu)f(\mu) = 0$ for all $\mu \in U_\lambda$ is the function $f \equiv 0$.

We use $\mathcal{S}(T)$ to denote the open set where T fails to have SVEP and we say that T has SVEP if $\mathcal{S}(T)$ is the empty set.

Taking $T \in \mathcal{B}(X)$, define

$$\phi_+(T) = \bigl\{\lambda \in \mathbb{C} \bigm| \lambda - T \in \Phi_\pm(X),\ N(\lambda - T) \text{ is complemented and } i(\lambda - T) > 0\bigr\},$$
$$\phi_-(T) = \bigl\{\lambda \in \mathbb{C} \bigm| \lambda - T \in \Phi_\pm(X),\ R(\lambda - T) \text{ is complemented and } i(\lambda - T) < 0\bigr\}.$$

Let $\phi_{+\infty}(T)$ (and $\phi_{-\infty}(T)$) denote respectively the set of $\lambda \in \phi_+(T)$ ($\lambda \in \phi_-(T)$) such that $i(\lambda - T) = \infty$ ($i(\lambda - T) = -\infty$). We set $\phi_{\pm\infty}(T) = \phi_{+\infty}(T) \cup \phi_{-\infty}(T)$. It is not difficult to prove that all these sets are open, and with these sets, [4, Lemma 3.1] can be extended to general Banach spaces. In fact:

Lemma 2. *If $T_n \to T$ in $\mathcal{B}(X)$ and $\lambda \notin \overline{\phi_{\pm\infty}(T)}$ is such that, for every $\epsilon > 0$, the ball $B(\lambda,\epsilon)$ contains a component of $\sigma_{lre}(T)$, then $\lambda \in \liminf \sigma_{s-F}(T_n)$.*

Observe that if $i(\lambda - T) \geqslant 0$ for every $\lambda \notin \sigma_{lre}(T)$, then the set $\phi_{\pm\infty}(T)$ in Lemma 2 can be replaced by $\phi_{+\infty}(T)$. The next theorem gives a new sufficient condition for the continuity of the approximate point spectrum.

Theorem 3. *Let $T \in \mathcal{B}(X)$ such that T^* has SVEP at every $\beta \notin \sigma_{lre}(T)$. If for each $\lambda \in \sigma_{lre}(T) \setminus \overline{\phi_+(T)}$ and $\epsilon > 0$, the ball $B(\lambda,\epsilon)$ contains a component of $\sigma_{lre}(T)$, then σ and σ_{ap} are continuous at T.*

Proof. First, we are going to show that σ_{ap} is continuous at T. Using [4, Lemma 1.8] it is easy to see that σ_{ap} is always upper semi-continuous in the algebra $\mathcal{B}(X)$. Let $\{T_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(X)$ be a sequence such that $T_n \to T$, and let $\lambda \in \sigma_{ap}(T)$.

Case I: $\lambda \notin \sigma_{lre}(T)$.

In this case $\lambda - T$ is a semi-Fredholm operator and T^* has SVEP at λ, so by [5, Corollary 3.19], $i(\lambda - T) \geqslant 0$. Suppose that $i(\lambda - T) = 0$. Since T^* has SVEP at every $\beta \notin \sigma_w(T)$, it follows that T^* satisfies Browder's theorem, and consequently, T satisfies too. Thus $\lambda \in \sigma(T) \setminus \sigma_w(T) = \pi_0(T)$. Consequently by Lemma 1, $\lambda \in \liminf \pi_0(T_n) \subseteq \liminf \sigma_{ap}(T_n)$.

Now, let $i(\lambda - T) > 0$. If $\lambda \notin \liminf \sigma_{ap}(T_n)$, then there exist $\epsilon_1 > 0$ and an increasing sequence of natural numbers $n_1 < n_2 < n_3 \cdots$ such that $B(\lambda,\epsilon_1) \cap \sigma_{ap}(T_{n_k}) = \emptyset$ for all $k \in \mathbb{N}$. As $\lambda \notin \sigma_{ap}(T_{n_k})$, $\lambda - T_{n_k}$ is an injective operator with closed range. This implies that $\lambda - T_{n_k} \in \Phi_+(X)$ and $i(\lambda - T_{n_k}) \leqslant 0$. On the other hand, $\lambda - T_{n_k} \to \lambda - T$, so by the continuity of index, it follows that $i(\lambda - T) \leqslant 0$ which is a contradiction. That proves $\lambda \in \liminf \sigma_{ap}(T_n)$.

Case II: $\lambda \in \sigma_{lre}(T)$.

By the proof of case I, we have that $\phi_+(T) \subseteq \liminf \sigma_{ap}(T_n)$. Thus $\overline{\phi_+(T)} \subseteq \liminf \sigma_{ap}(T_n)$ because $\liminf \sigma_{ap}(T_n)$ is a closed set. Therefore if $\lambda \in \overline{\phi_+(T)}$, then $\lambda \in \liminf \sigma_{ap}(T_n)$.

Let $\lambda \notin \overline{\phi_+(T)}$. By the hypothesis and Lemma 2, we get that $\lambda \in \liminf \sigma_{s-F}(T_n)$. But since $\sigma_{s-F}(T_n) \subseteq \sigma_{ap}(T_n)$ for all $n \in \mathbb{N}$, it follows that $\lambda \in \liminf \sigma_{ap}(T_n)$. Therefore σ_{ap} is continuous at T.

By [2, Theorem 1] σ is upper semi-continuous. Let $\lambda \in \sigma(T)$. If $\lambda \in \sigma_{ap}(T)$, then by continuity of σ_{ap} at T, $\lambda \in \liminf \sigma_{ap}(T_n) \subseteq \liminf \sigma(T_n)$. Now if $\lambda \in \sigma(T) \setminus \sigma_{ap}(T)$, then $\lambda - T$ is a semi-Fredholm operator such that $i(\lambda - T) < 0$. If there exists an increasing sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ such that $\lambda - T_{n_k}$ is invertible for all $k \in \mathbb{N}$, then by continuity of index, $i(\lambda - T) = 0$ which is a contradiction. Thus there is $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$, $\lambda \in \sigma(T_n)$. Therefore $\lambda \in \liminf \sigma(T_n)$. □

Observe that in a previous theorem the assumed hypotheses are not necessary for the continuity of σ_{ap} at T as the following example illustrates. Let $\alpha_{nk} = (1 + \frac{1}{n})\exp(2\pi i \frac{k}{n})$ for all $n \in \mathbb{N}$ and $1 \leqslant k \leqslant n$, and let $M : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the diagonal operator defined by

$$M = \begin{pmatrix} \alpha_{11} & & & \\ & \alpha_{21} & & \\ & & \alpha_{22} & \\ & & & \ddots \end{pmatrix}.$$

Then $\sigma_p(M) = \{\alpha_{nk} \mid n \in \mathbb{N},\ 1 \leqslant k \leqslant n\}$ where each eigenvalue has geometry multiplicity one. Also $\sigma_{ap}(M) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\} \cup \{\alpha_{nk} \mid n \in \mathbb{N},\ 1 \leqslant k \leqslant n\}$. Moreover

$$\phi_\pm(M) = \emptyset, \qquad \pi_0(M) = \{\alpha_{nk}\} \quad \text{and} \quad \sigma_{lre}(M) = \bigl\{\lambda \in \mathbb{C} \bigm| |\lambda| = 1\bigr\}.$$

Observe that $\sigma_{ap}(M) = \overline{\pi_0(M)}$, and by Lemma 1, $\overline{\pi_0(M)} \subseteq \overline{\liminf \pi_0(A_n)} \subseteq \liminf \sigma_{ap}(A_n)$ for all sequence $\{A_n\}$ in $\mathcal{B}(\ell^2(\mathbb{N}))$ such that $A_n \to M$. So σ_{ap} is continuous at M. However, for each $\lambda \in \sigma_{lre}(T) \setminus \overline{\phi_+(T)} = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, there exists $\epsilon > 0$ such that $B(\lambda, \epsilon)$ does not contain a component of $\sigma_{lre}(T)$.

In Theorem 3 the set of points λ for which the ball $B(\lambda, \epsilon)$ contains a component of $\sigma_{lre}(T)$ for all $\epsilon > 0$ may be reduced. Indeed:

Theorem 4. *Let $T \in \mathcal{B}(X)$ such that T^* has SVEP at every $\beta \notin \sigma_{lre}(T)$. If for each $\lambda \in \sigma_{lre}(T) \setminus \overline{\phi_+(T) \cup \pi_0(T)}$ and $\epsilon > 0$, the ball $B(\lambda, \epsilon)$ contains a component of $\sigma_{lre}(T)$, then σ and σ_{ap} are continuous at T.*

We know that $\sigma_{lre}(T) = \sigma_{lre}(T^*)$ and $\phi_-(T) = \phi_+(T^*)$ for any bounded operator T. Thus by duality and Theorem 4 we have the following corollary.

Corollary 5. *Let $T \in \mathcal{B}(X)$ such that T has SVEP at every $\beta \notin \sigma_{lre}(T)$. If for each $\lambda \in \sigma_{lre}(T) \setminus \overline{\phi_-(T) \cup \pi_0(T)}$ and $\epsilon > 0$, the ball $B(\lambda, \epsilon)$ contains a component of $\sigma_{lre}(T)$, then σ and σ_{su} are continuous at T.*

An operator $T \in \mathcal{B}(X)$ is called a shift if $\alpha(T) = 0$, $\beta(T) = 1$ and $\bigcap_{n\in\mathbb{N}} T^n(X) = \{0\}$. If T is a shift and an isometry, then T is called a shift isometry.

Example 1. Let X be a Banach space. If $T \in \mathcal{B}(X)$ is a shift isometry, then

(a) σ and σ_{su} are continuous at T;
(b) σ and σ_{ap} are continuous at T^*.

It is clear that T is a Fredholm operator with $i(T) = -1$. Moreover $\sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leqslant 1\}$ (see [6, Theorem 6(a)]). Let $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$, then from [6, Theorem 6(b)] and [6, Proposition 2(a)], it follows that $\lambda - T$ is a shift operator. Therefore $i(\lambda - T) < 0$ and $R(\lambda - T)$ is complemented (because $\beta(\lambda - T) < \infty$). Thus

$$\sigma(T) = \bigl\{\lambda \in \mathbb{C} \bigm| |\lambda| \leqslant 1\bigr\} \subseteq \overline{\phi_-(T)}.$$

On the other hand, shift operators have SVEP (see [6, Proposition 6(a)]). Consequently by Corollary 5, σ and σ_{su} are continuous at T. By duality, σ and σ_{ap} are continuous at T^*.

3. Continuity of spectra on operator matrices

In this section we give sufficient conditions for the continuity of spectrum on the set of all upper-triangular operator matrices. For $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$ define the operator M_C on $X \oplus Y$ as

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}.$$

It is well known that for all $\Sigma \in \{\sigma, \sigma_e, \sigma_w, \sigma_b\}$ we have $\Sigma(M_C) \subseteq \Sigma(A) \cup \Sigma(B)$, and $\Sigma(A) \cap \Sigma(B) = \emptyset$ implies $\Sigma(M_C) = \Sigma(A) \cup \sigma(B)$ (see, for example, [7–9]). For the approximate point spectrum of operator matrices the situation is more complicated.

Making an examination of the proof of Theorem 5.2 (ii $\Rightarrow$ iii) of [7], one can prove the following lemma, that is a version of the second part of Corollary 5.3 of [7] for the case of Banach spaces X and Y.

Lemma 6. *Let X and Y be Banach spaces. If $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$, then*

$$\bigcap_{C \in \mathcal{B}(Y,X)} \sigma_{ap}(M_C) \supseteq \sigma_{ap}(A) \cup \{\lambda \in \mathbb{C} \mid R(\lambda - B) \text{ is not closed}\} \cup \{\lambda \in \mathbb{C} \mid \beta(\lambda - A) = 0 \text{ and } \alpha(\lambda - B) > 0\}$$

and

$$\bigcap_{C \in \mathcal{B}(Y,X)} \sigma_{su}(M_C) \supseteq \sigma_{su}(B) \cup \{\lambda \in \mathbb{C} \mid R(\lambda - A) \text{ is not closed}\} \cup \{\lambda \in \mathbb{C} \mid \beta(\lambda - A) > 0 \text{ and } \alpha(\lambda - B) = 0\}.$$

The next theorem is a generalization of Theorem 2.1 of [8]. The proof of this theorem can be extended for the case when $\Sigma = \sigma_e$ or σ_b.

Theorem 7. *Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $\Sigma \in \{\sigma, \sigma_e, \sigma_w, \sigma_b\}$ such that $\Sigma(A) \cap \Sigma(B) = \emptyset$. Then Σ is continuous at A and B if and only if Σ is continuous at M_C for every $C \in \mathcal{B}(Y, X)$.*

In the previous theorem, M_C satisfies that $\sigma(M_C) = \sigma(A) \cup \sigma(B)$. This condition and the continuity of σ at A and B are not enough for σ to be continuous at M_C. In fact, let U be the unilateral shift on $\ell^2(\mathbb{N})$ and let M be the operator defined on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ as

$$M = \begin{bmatrix} U & 0 \\ 0 & U^* \end{bmatrix}.$$

Then $\sigma(M) = \sigma(U) \cup \sigma(U^*)$, and by Example 1, U and U^* are continuity points of σ. However in [8] it is proved that M is not a continuity point of σ. To see this, consider

$$M_n = \begin{bmatrix} U & \frac{1}{n}(I - UU^*) \\ 0 & U^* \end{bmatrix}.$$

Then $M_n \to M$, M_1 is a unitary operator, and each M_n is similar to M_1. So for every n, $\sigma(M_n) = \sigma(M_1) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, and $\sigma(M) = \{\lambda \in \mathbb{C} \mid |\lambda| \leqslant 1\}$. Therefore $\sigma(M_n) \nrightarrow \sigma(M)$.

Of course, if $T_n = \left[\begin{smallmatrix} A_n & C_n \\ 0 & B_n \end{smallmatrix}\right]$ is a sequence of upper-triangular operator matrices such that $T_n \to M_C$ and $\sigma(T_n) = \sigma(A_n) \cup \sigma(B_n)$ for each n large, then $\sigma(A_n) \to \sigma(A)$ and $\sigma(B_n) \to \sigma(B)$ imply $\sigma(T_n) \to \sigma(M_C)$.

S.V. Djordjević and Y.M. Han have shown (see [8, Theorem 2.4]) that on Hilbert spaces σ is continuous at M_C when it satisfies Browder's theorem and σ_{ap} and σ_{su} are continuous at A and B respectively.

Theorem 8. *Let $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ such that*

(i) $\sigma(A) = \sigma_{ap}(A)$;
(ii) *σ_{ap} is continuous at A;*
(iii) *σ is continuous at B.*

Then σ is continuous at M_C for every $C \in \mathcal{B}(Y, X)$.

Proof. Since the spectrum is upper semi-continuous at every operator T [2, Theorem 1], it is sufficient to show that σ is lower semi-continuous at M_C (in the set of all upper-triangular operator matrices).

Let $\{M_n\}_{n\in\mathbb{N}}$ be a sequence of upper-triangular operator matrices, $M_n = \left[\begin{smallmatrix} A_n & C_n \\ 0 & B_n \end{smallmatrix}\right]$, such that $M_n \to M_C$. Taking an arbitrary $\lambda \in \sigma(M_C)$, since $\sigma(M_C) \subseteq \sigma(A) \cup \sigma(B)$, if follows that $\lambda \in \sigma(A)$ or $\lambda \in \sigma(B)$.

Case I: $\lambda \in \sigma(A)$.

By a hypothesis, $\lambda \in \sigma_{ap}(A)$, and from continuity of σ_{ap} at A, it follows that $\lambda \in \liminf \sigma_{ap}(A_n)$. According to Lemma 6, $\sigma_{ap}(A_n) \subseteq \sigma_{ap}(M_n) \subseteq \sigma(M_n)$ for all $n \in \mathbb{N}$. Consequently $\lambda \in \liminf \sigma(M_n)$.

Case II: $\lambda \in \sigma(B) \setminus \sigma(A)$.

Since σ is continuous at B, we can take $\lambda_n \in \sigma(B_n)$ for all $n \in \mathbb{N}$, such that $\lambda_n \to \lambda$. Suppose that there exists an increasing sequence of natural numbers $n_1 < n_2 < n_3 \cdots$ such that $\lambda_{n_k} \notin \sigma(M_{n_k})$ for all $k \in \mathbb{N}$. Observe that $\lambda_{n_k} \in (\sigma(A_{n_k}) \cup \sigma(B_{n_k})) \setminus \sigma(M_{n_k})$, thus $\lambda_{n_k} \in \sigma(A_{n_k}) \cap \sigma(B_{n_k})$. Therefore, for every $k \in \mathbb{N}$, $\lambda_{n_k} \in \sigma(A_{n_k})$. This implies that $\lambda \in \limsup \sigma(A_n)$. Thus from upper semi-continuity of σ, it follows that $\lambda \in \sigma(A)$ – a contradiction. Consequently there is $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$, $\lambda_n \in \sigma(M_n)$. Therefore $\lambda \in \liminf \sigma(M_n)$. $\square$

Let M be the matrix given in the example above, then

$$M^* = \begin{bmatrix} U^* & 0 \\ 0 & U \end{bmatrix}.$$

Observe that by Example 1, σ_{ap} is continuous at U^* and σ is continuous at U. In [10, Example 2], it is shown that $\mathcal{S}(M^*) = \mathcal{S}(U^*)$. The operator U^* does not have SVEP at 0 (see the proof of [6, Theorem 7(c)]), thus $0 \in \mathcal{S}(M^*)$. Moreover, since $i(M^*) = i(U^*) + i(U) = 0$ (see [11, Corollary 5]), it follows that $0 \in \mathcal{S}(M^*) \setminus \sigma_w(M^*)$. Therefore M^* does not satisfy Browder's theorem. However $\sigma(U^*) = \sigma_{ap}(U^*)$, thus by Theorem 8, σ is continuous at M^*. With this example we see that Theorem 8 is a generalization of [8, Theorem 2.4].

Remark 1. In Theorem 8 the hypothesis (i) can be replaced by the condition A^* has SVEP at every $\lambda \notin \sigma_{s-F}(T)$.

Let H be a Hilbert space, and let $A \in \mathcal{B}(H)$ be a continuity point of σ. We know [12, Theorem 3.1] that for each $\lambda \in \sigma(A) \setminus \overline{\phi_\pm(A)}$ and $\epsilon > 0$, the ball $B(\lambda, \epsilon)$ contains a component of $\pi_0(A) \cup \sigma_{lre}(A)$. Thus, if $\lambda \in \sigma_{lre}(A) \setminus \overline{\phi_\pm(A) \cup \pi_0(T)}$, then for every $\epsilon > 0$, the ball $B(\lambda, \epsilon)$ contains a component of $\sigma_{lre}(A)$. With this and Theorem 4, we have the following corollary.

Corollary 9. *Let H and K be Hilbert spaces. If $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(K)$ are such that*

(i) *A^* has SVEP at every $\lambda \notin \sigma_{lre}(A)$;*
(ii) *σ is continuous at A and B,*

then σ is continuous at M_C for every $C \in \mathcal{B}(K, H)$.

Theorem 10. *Let X and Y be Banach spaces. If $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ are such that*

(i) *$\sigma(A) = \sigma_{ap}(A)$;*
(ii) *σ_{ap} is continuous at A and B,*

then σ_{ap} is continuous at M_C for every $C \in \mathcal{B}(Y, X)$.

Proof. Let $\{M_n\}_{n\in\mathbb{N}}$ be a sequence of upper-triangular operator matrices, $M_n = \left[\begin{smallmatrix} A_n & C_n \\ 0 & B_n \end{smallmatrix}\right]$, such that $M_n \to M_C$.

Let $\lambda \in \sigma_{ap}(M_C) (\subset \sigma_{ap}(A) \cup \sigma_{ap}(B))$.

Case I: $\lambda \in \sigma_{ap}(A)$.

Then, similar to the proof of Theorem 8, we have by continuity of σ_{ap} at A that

$$\lambda \in \sigma_{ap}(A) \subseteq \liminf \sigma_{ap}(A_n) \subseteq \liminf \sigma_{ap}(M_n).$$

Case II: $\lambda \in \sigma_{ap}(B) \setminus \sigma_{ap}(A)$.

The point λ is not in $\sigma(A)$, because $\sigma(A) = \sigma_{ap}(A)$. By continuity of σ_{ap} at B there exists $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ such that $\lambda_n \to \lambda$ and $\lambda_n \in \sigma_{ap}(B_n)$ for all $n \in \mathbb{N}$. If there exists an increasing sequence of natural numbers $n_1 < n_2 < \cdots$ such that $\lambda_{n_k} \in \sigma(A_{n_k})$, then $\lambda \in \limsup \sigma(A_n)$. So by upper semi-continuity of σ, $\lambda \in \sigma(A)$ which is a contradiction. Thus, there exists a positive integer n_0 such that $A_n - \lambda_n$ is invertible, for every $n \geqslant n_0$. Consider $n \geqslant n_0$. Since $\lambda_n \in \sigma_{ap}(B_n)$ it follows that $R(\lambda_n - B_n)$ is not closed or $\lambda_n - B_n$ is not injective. If $R(\lambda_n - B_n)$ is not closed, then by Lemma 6, $\lambda_n \in \sigma_{ap}(M_n)$. Now if $\lambda_n - B_n$ is not injective, then $\alpha(\lambda_n - B_n) > 0$, but $\beta(\lambda_n - A_n) = 0$ (because $\lambda_n - A_n$ is invertible). Again, by Lemma 6, we have $\lambda_n \in \sigma_{ap}(M_n)$. Therefore for every $n \geqslant n_0$, $\lambda_n \in \sigma_{ap}(M_n)$, that implies $\lambda \in \liminf \sigma_{ap}(M_n)$. $\square$

By duality, we can prove the following statement.

Theorem 11. *Let X and Y be Banach spaces. If $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ are given such that*

(i) $\sigma(B) = \sigma_{su}(B)$;
(ii) *σ_{su} is continuous at A and B,*

then σ_{su} is continuous at M_C for every $C \in \mathcal{B}(Y, X)$.

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