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Knaster's problem for almost $(Z_p)^k$-orbits

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ABSTRACT

In this paper some new cases of Knaster's problem on continuous maps from spheres are established. In particular, we consider an almost orbit of a p-torus X on the sphere, a continuous map f from the sphere to the real line or real plane, and show that X can be rotated so that f becomes constant on X.

1. Introduction

In [7] the following conjecture (Knaster's problem) was formulated.

Conjecture 1. *Let S^{d-1} be a unit sphere in $\mathbb{R}^d$. Suppose we are given $m = d - k + 1$ points $x_1, \dots, x_m \in S^{d-1}$ and a continuous map $f : S^{d-1} \to \mathbb{R}^k$. Then there exists a rotation $\rho \in SO(d)$ such that*

$$f\big(\rho(x_1)\big) = f\big(\rho(x_2)\big) = \dots = f\big(\rho(x_m)\big).$$

In papers [6,4] it was shown that for certain sets $\{x_1, \dots, x_m\} \subset S^{d-1}$ Knaster's conjecture fails, such counterexamples exist for every $k > 2$, for $k = 2$ and $d \geqslant 5$, for $k = 1$ and $d \geqslant 67$.

Still it is possible to prove Knaster's conjecture in some particular cases of sets. In [10] the set of points was some orbit of the action of a p-torus $G = (Z_p)^k$ on $\mathbb{R}[G]$ for $k = 1$ and on $\mathbb{R}[G] \oplus \mathbb{R}$ for $k = 2$. Here we prove some similar results, the set of points being a $(Z_p)^k$-orbit minus one point.

The group algebra $\mathbb{R}[G]$ is supposed to have left G-action, unless otherwise stated. Considered as a G-representation, $\mathbb{R}[G]$ may have a G-invariant inner product. In fact, the space of invariant inner products has the dimension equal to the

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number of distinct irreducible G-representations in $\mathbb{R}[G]$ (for a commutative G), for a p-torus $G = (Z_p)^k$ the dimension of this space is $\frac{p^k+1}{2}$ for odd p, and p^k for $p = 2$.

Definition 1. Denote $I[G] \subset \mathbb{R}[G]$ the G-invariant subspace in $\mathbb{R}[G]$ consisting of

$$\sum_{g\in G} \alpha_g g, \quad \text{with} \sum_{g\in G} \alpha_g = 0.$$

Note that its orthogonal complement (w.r.t. any G-invariant inner product) is the one-dimensional space with trivial G-action.

In the sequel we consider a p-torus $G = (Z_p)^k$ and denote $q = p^k$.

Theorem 1. *Let S^{q-2} be the unit sphere of $I[G]$ w.r.t. some G-invariant inner product, denoted by $(\cdot,\cdot)$. Then Conjecture* 1 *holds for $k = 1$, the rotations w.r.t. $(\cdot,\cdot)$, and the set $Gx \setminus \{x\}$, where $x \in S^{q-2}$ is any point.*

Theorem 2. *Let S^{q-1} be the unit sphere of $\mathbb{R}[G]$ w.r.t. some G-invariant inner product $(\cdot,\cdot)$. Then Conjecture* 1 *holds for $k = 2$, q odd, the rotations w.r.t. $(\cdot,\cdot)$, and the set $Gx \setminus \{x\}$, where $x \in S^{q-1}$ is any point.*

In fact, the last theorem may be formulated a little stronger. For example, Theorem 5 (see below) gives the following statement. Let $x \in S^{q-1}$ be as in the theorem, and let $f_1, f_2 : S^{q-1} \to \mathbb{R}$ be two continuous functions. Then for some rotation ρ and two constants c_1, c_2,

$$\forall g \in G, \quad f_1\big(\rho(gx)\big) = c_1,$$
$$\forall g \in G,\ g \neq e, \quad f_2\big(\rho(gx)\big) = c_2.$$

2. Equivariant cohomology of *G*-spaces

We consider topological spaces with continuous action of a finite group G and continuous maps between such spaces that commute with the action of G. We call them G-spaces and G-maps.

Let us consider the group $G = (Z_p)^k$ and list the results (mostly from [12]) that we need in this paper.

The cohomology is taken with coefficients in Z_p, in the notation we omit the coefficients.

Consider the algebra of G-equivariant (in the sense of Borel) cohomology of the point $A_G = H^*_G(\mathrm{pt}) = H^*(BG)$. For any G-space X the natural map $X \to \mathrm{pt}$ induces the natural map of cohomology $\pi^*_X : A_G \to H^*_G(X)$.

For a group $G = (Z_p)^k$ the algebra A_G (see [5]) has the following structure. For odd p, it has $2k$ multiplicative generators v_i, u_i with dimensions $\dim v_i = 1$ and $\dim u_i = 2$ and relations

$$v_i^2 = 0, \qquad \beta v_i = u_i.$$

Here we denote $\beta(x)$ the Bockstein homomorphism.

For a group $G = (Z_2)^k$ the algebra A_G is the algebra of polynomials of k one-dimensional generators v_i.

The powerful tool of studying G-spaces is the following spectral sequence (see [5,8]).

Theorem 3. *There exists a spectral sequence with E_2-term*

$$E_2^{x,y} = H^x\big(BG, \mathcal{H}^y(X, Z_p)\big),$$

that converges to the graded module, associated with the filtration of $H^*_G(X, Z_p)$.

The system of coefficients $\mathcal{H}^y(X, Z_p)$ is obtained from the cohomology $H^y(X, Z_p)$ by the action of $G = \pi_1(BG)$. The differentials of this spectral sequence are homomorphisms of $H^*(BG, Z_p)$-modules.

For every term $E_r(X)$ of this spectral sequence there is a natural map $\pi^*_r : A_G \to E_r(X)$.

Definition 2. Denote the kernel of the map π^*_r by $\mathrm{Ind}^r_G X$.

The ideal-valued index of a G-space was introduced in [3], the above filtered version was introduced in [11]. Recall the properties of $\mathrm{Ind}^r_G X$ that are obvious by the definition. We omit the subscript G when a single group is considered.

- If there is a G-map $f : X \to Y$, then $\mathrm{Ind}^r X \supseteq \mathrm{Ind}^r Y$.
- $\mathrm{Ind}^{r+1} X$ may differ from $\mathrm{Ind}^r X$ only in dimensions $\geqslant r$.
- $\bigcup_r \mathrm{Ind}^r X = \mathrm{Ind}\, X = \ker \pi^*_X : A_G \to H^*_G(X)$.

The first property in this list is very useful to prove nonexistence of G-maps. Following [12] we define a numeric invariant of this ideal filtering $\operatorname{Ind}_G^r X$.

Definition 3. Put

$$i_G(X) = \max\{r\colon \operatorname{Ind}_G^r X = 0\}.$$

It is easy to see that $i_G(X) \geqslant 1$ for any G-space X, $i_G(X) \geqslant 2$ for a connected G-space X, and $i_G(X)$ may be equal to $+\infty$. Moreover, for a G-space X without fixed points, G-homotopy equivalent to a finite G-CW-complex, $i_G(X) \leqslant \dim X + 1$.

From the definition of $\operatorname{Ind}_G^r X$ it follows that if there exists a G-map $f : X \to Y$, then $i_G(X) \leqslant i_G(Y)$ (the monotonicity property).

The definition of $i_G(X)$ can be further extended.

Definition 4. Define the index of a cohomology class $\alpha \in A_G$ on a G-space X by

$$i_G(\alpha, X) = \max\{r\colon \alpha \notin \operatorname{Ind}_G^r X\}.$$

It may equal $+\infty$ if $\alpha \notin \operatorname{Ind}_G X$.

It is clear from the definition that either $i_G(\alpha, X) = +\infty$, or $i_G(\alpha, X) \leqslant \dim \alpha$ and $i_G(\alpha, X) \leqslant \dim X + 1$ (for a finite G-CW-complex). Moreover, for any G-map $f : X \to Y$ we have the monotonicity property

$$i_G(\alpha, X) \leqslant i_G(\alpha, Y).$$

3. Reformulations

We reformulate Theorems 1 and 2 in a more general way.

Theorem 4. *Let S^{q-2} be the unit sphere of $I[G]$ w.r.t. some G-invariant inner product, and let $f : S^{q-2} \to \mathbb{R}$ be some continuous function. Consider $x \in S^{q-2}$, the vector $v = \sum_{g\in G} g \in \mathbb{R}[G]$ and some other vector $w \in \mathbb{R}[G]$, non-collinear to v.*

Then for some rotation $\rho \in SO(q-1)$ the vector $\sum_{g\in G} f(\rho(gx))g \in \mathbb{R}[G]$ is in the linear span of v and w.

Theorem 1 follows from this theorem in the following way. Put $w = e \in \mathbb{R}[G]$. Then by Theorem 4 there exists a rotation ρ such that for some $\alpha, \beta \in \mathbb{R}$,

$$\forall g \in G,\ g \neq e, \quad f\big(\rho(gx)\big) = \alpha, \qquad f\big(\rho(x)\big) = \alpha + \beta.$$

That is exactly the statement of Theorem 1.

Theorem 5. *Let S^{q-1} be the unit sphere of $\mathbb{R}[G]$ w.r.t. some G-invariant inner product, and let $f : S^{q-1} \to \mathbb{R}^2$ be some continuous map with coordinates f_1, f_2. Let q be odd. Consider $x \in S^{q-1}$, the vectors $v = \sum_{g\in G} g \oplus 0 \in \mathbb{R}[G] \oplus \mathbb{R}[G]$, $u = 0 \oplus \sum_{g\in G} g \in \mathbb{R}[G] \oplus \mathbb{R}[G]$ and some other vector $w \in \mathbb{R}[G] \oplus \mathbb{R}[G]$, non-coplanar to v, u.*

Then for some rotation $\rho \in SO(q)$ the vector

$$\sum_{g\in G} f_1\big(\rho(gx)\big)g \oplus \sum_{g\in G} f_2\big(\rho(gx)\big)g \in \mathbb{R}[G] \oplus \mathbb{R}[G]$$

is in the linear span of v, u, w.

Again, Theorem 2 (and its stronger version in the remark after Theorem 2) follows from this theorem by taking a vector $w = e \oplus 0$, similar to the previous remark.

4. Proof of Theorem 4 in the case of odd *q*

In this section $q = p^k$, p is an odd prime, $G = (Z_p)^k$. Define for any $\rho \in SO(q-1)$,

$$\phi(\rho) = \sum_{g\in G} f\big(\rho\big(g(x)\big)\big)g \in \mathbb{R}[G].$$

For any $h \in G$ we have

$$\phi\big(\rho \circ h^{-1}\big) = \sum_{g \in G} f\big(\rho\big(h^{-1}g(x)\big)\big)g = \sum_{g \in G} f\big(\rho\big(h^{-1}g(x)\big)\big)hh^{-1}g = \sum_{g \in G} f\big(\rho\big(g(x)\big)\big)hg.$$

Thus the map $\phi : SO(q-1) \to \mathbb{R}[G]$ is a G-map for the left action of G on $SO(q-1)$ by right multiplications by $g^{-1} \in G$, and for the standard left action of G on $\mathbb{R}[G]$.

Denote for any $g \in G$ by $L_g = (v, gw) \subset \mathbb{R}[G]$ the 2-dimensional subspaces. Assume the contrary: that is the image of ϕ does not intersect $\bigcup_{g \in G} L_g$. So ϕ maps $SO(q-1)$ to the space $Y = \mathbb{R}[G] \setminus \bigcup_{g \in G} L_g$. The natural projection $\pi : Y \to \mathbb{R}[G]/(v) = V$ gives a homotopy equivalence between Y and $V \setminus \bigcup_{g \in G} \mathbb{R}\pi(gw)$, the latter space is homotopically a $(q-2)$-dimensional sphere without several points, hence it is a wedge of $(q-3)$-dimensional spheres. G acts on Y without fixed points, so $i_G(Y) \leqslant q-2$.

In [10] it was shown that $i_G(SO(q-1)) = q-1$ w.r.t. the considered G-action. Here we give a short explanation. In the spectral sequence of Theorem 3 all multiplicative generators of $H^*(SO(q-1), Z_p)$ are transgressive, because they are pullbacks of the transgressive generators of $H^*(SO(q-1), Z_p)$ in the spectral sequence of the fiber bundle $\pi_{SO(q-1)} : ESO(q-1) \to BSO(q-1)$. So the first nonzero $\operatorname{Ind}_G^r SO(q-1)$ corresponds to the first nonzero characteristic class of the G-representation $I[G]$ in the cohomology ring A_G. It was shown in [10] that this is the Euler class of $I[G]$ of dimension $q-1$.

So we have a contradiction with the monotonicity of $i_G(X)$.

5. Proof of Theorem 5

Similar to the previous proof, we consider the G-map $\phi : SO(q) \to \mathbb{R}[G] \oplus \mathbb{R}[G]$, given by the formula

$$\phi(\rho) = \sum_{g \in G} f_1\big(\rho\big(g(x)\big)\big)g \oplus \sum_{g \in G} f_2\big(\rho\big(g(x)\big)\big)g \in \mathbb{R}[G] \oplus \mathbb{R}[G].$$

Take the composition $\psi = \pi \cdot \phi$ with the projection $\pi : \mathbb{R}[G] \oplus \mathbb{R}[G] \to I[G] \oplus I[G] = V$. Assume the contrary: that is the map ϕ does not intersect the linear span of u and v in $\mathbb{R}[G] \oplus \mathbb{R}[G]$ and ψ does not intersect the linear span of gw for any $g \in G$ in V, which means that the image of ψ is in the space $Y = V \setminus \bigcup_{g \in G} \mathbb{R}\pi(gw)$.

Let $e \in A_G$ be the Euler class of V. From the spectral sequence of Theorem 3 it is obvious that $i_G(e, V \setminus \{0\}) = 2q-2$, because the spectral sequence for $V \setminus \{0\}$ has the only nontrivial differential that kills the Euler class e. Since $Y \subset V \setminus \{0\}$, then $i_G(e, Y) < +\infty$. Similar to the previous proof, the space Y is homotopically a wedge of $(2q-4)$-dimensional spheres, so $i_G(e, Y) \leqslant \dim Y + 1 = 2q-3$.

In [10] it was shown that $i_G(e, SO(q)) = 2q-2$, because e is in the image of the transgression in the spectral sequence and e is not contained in the ideal of A_G, generated by the characteristic classes of $SO(q)$ of lesser dimension. So we again have a contradiction with the monotonicity of $i_G(e, X)$.

6. Proof of Theorem 4 in the case of even q

In this section $q = 2^k$, $G = (Z_2)^k$. We use the notation from the odd case in Section 4. Note that the case $q = 2$ is trivial, and if $q \geqslant 4$ then G acts on $I[G]$ by transforms with positive determinant, so the group $SO(q-1)$ can be considered as the configuration space.

Assume the contrary: the image $\phi(SO(q-1))$ is in $Y = \mathbb{R}[G] \setminus \bigcup_{g \in G} L_g$.

Denote the Stiefel–Whitney classes of $I[G]$ in A_G by w_k. We need the following lemma, stated in [10], based on results from [2,9].

Lemma 1. *The only nonzero Stiefel–Whitney classes of $I[G]$ are $w_{q-2^l} \in A_G$ $(l = 0, \dots, k)$, the classes w_{q-2^l} $(l = 0, \dots, k-1)$ are algebraically independent and form a regular sequence, hence w_{q-1} is nonzero and not contained in the ideal of A_G, generated by w_k with $k < q-1$.*

Similar to the proof of Theorem 5 in Section 5, we find that $i_G(w_{q-1}, Y) \leqslant \dim Y + 1 = q-2$.

Now we apply the spectral sequence of Theorem 3 to the G-space $SO(q-1)$. The action of G on $SO(q-1)$ is the restriction of action of $SO(q-1)$ on itself, the latter group being connected, hence G acts trivially on $H^*(SO(q-1), Z_2)$.

The results of [1] imply that the differentials in this spectral sequence are generated by transgressions that send the primitive (in terms of [1]) elements of $H^*(SO(q-1), Z_2)$ to the Stiefel–Whitney classes w_k (see Proposition 23.1 in [1]). Thus Lemma 1 implies that $i_G(w_{q-1}, SO(q-1)) = q-1$, and the existence of the G-map ϕ contradicts the monotonicity of $i_G(w_{q-1}, X)$.

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