JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **164**, 104–115 (1992)

A Note on the Convergence of Certain Families of Multiple Hypergeometric Series

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Submitted by J. L. Brenner

Received July 23, 1990

In this sequel to an earlier work on the subject, the authors present a further analysis of the convergence problem of Kampé de Fériet's double hypergeometric series (and also of Lauricella's multiple hypergeometric series) when the arguments take on values on the boundaries of the known open regions of convergence of these series. © 1992 Academic Press, Inc.

1. INTRODUCTION AND MAIN RESULTS

In 1921 Kampé de Fériet [3] initiated the study of the case

$$B = B' \qquad \text{and} \qquad D = D'$$

of the double hypergeometric series:

$$F^{A:B;B'}_{C:D;D'}\left[\begin{matrix}(a):(b);(b'); \\ (c):(d);(d');\end{matrix}\; x, y\right] = \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_{m+n}\prod_{j=1}^{B}(b_j)_m\prod_{j=1}^{B'}(b'_j)_n}{\prod_{j=1}^{C}(c_j)_{m+n}\prod_{j=1}^{D}(d_j)_m\prod_{j=1}^{D'}(d'_j)_n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.1}$$

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0022-247X/92 $3.00

where $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n=0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n \in \mathbb{N} = \{1, 2, 3, ...\}. \end{cases} \tag{1.2}$$

A systematic (and detailed) study of the double series (1.1) is an important problem, since the definition (1.1) incorporates many simpler (and useful) double hypergeometric series, including (for example) the four Appell series $F_1, ..., F_4$, and their seven confluent forms:

$$\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2,$$

which were introduced by Pierre Humbert (1891–1953). Furthermore, (1.1) reduces when $y = 0$ to the generalized hypergeometric series:

$${}_{A+B}F_{C+D}\left[\begin{matrix} a_1, ..., a_A, b_1, ..., b_B; \\ c_1, ..., c_C, d_1, ..., d_D; \end{matrix} \; x\right].$$

(For definitions and various important properties of these simpler functions, see Appell and Kampé de Fériet [1]; see also Srivastava and Karlsson [5].)

For the double hypergeometric series (1.1), Srivastava and Daoust [4] deduced from much more general results (proved by them) that

(i) if $A+B > C+D+1$ and $A+B' > C+D'+1$, then the series (1.1) diverges whenever $x \neq 0$ and $y \neq 0$;

(ii) if $A+B = C+D+1$ and $A+B' = C+D'+1$, then the series (1.1) converges absolutely, provided that

$$\begin{aligned} \max\{|x|, |y|\} < 1 \quad & \text{when} \quad A \leqq C, \\ |x|^{1/(A-C)} + |y|^{1/(A-C)} < 1 \quad & \text{when} \quad A > C; \end{aligned} \tag{1.3}$$

(iii) if $A+B < C+D+1$ and $A+B' < C+D'+1$, then the series (1.1) converges absolutely for all $x, y \in \mathbb{C}$;

it is understood (in each situation) that no zeros appear in the denominator of (1.1).

In Case (ii) above, it is not yet known as to what additional constraints (if any) would guarantee the convergence of the double series (1.1) when x and y lie on the boundaries of the regions described by (1.3). The main object of the present paper is to solve this problem completely. We also consider the corresponding problem involving the convergence of the multiple hypergeometric series:

$$F_{C:D';\dots;D^{(n)}}^{A:B';\dots;B^{(n)}}\left[\begin{matrix}(a):(b');\dots;(b^{(n)});\\(c):(d');\dots;(d^{(n)});\end{matrix}\; x_1,\dots,x_n\right]$$
$$=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m_1+\cdots+m_n}\prod_{j=1}^{B'}(b_j')_{m_1}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n}}{\prod_{j=1}^{C}(c_j)_{m_1+\cdots+m_n}\prod_{j=1}^{D'}(d_j')_{m_1}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n}}$$
$$\cdot\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!},\tag{1.4}$$

which unifies and extends the four Lauricella series $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$, and $F_D^{(n)}$ in n variables. In fact, as already observed in the literature [5, pp. 37–38], the multiple hypergeometric series (1.4) is a special case of the generalized Lauricella series in several variables, which was introduced by Srivastava and Daoust in 1969.

Our solutions of the convergence problem for the double hypergeometric series (1.1) are contained in Theorems 1, 2, and 3 below.

THEOREM 1. *Let* $A+B=C+D+1$, $A+B'=C+D'+1$, *and* $A=C$. *Then the series*

$$F_{A:B\;\;:B'}^{A:B+1:B'+1}\left[\begin{matrix}(a):(b);(b');\\(c):(d);(d');\end{matrix}\; x,y\right]\tag{1.5}$$

(i) *converges absolutely when* $|x|=1$ *and* $|y|=1$, *if and only if*

$$\lambda=\operatorname{Re}\left(\sum_{j=1}^{A}a_j+\sum_{j=1}^{B+1}b_j-\sum_{j=1}^{A}c_j-\sum_{j=1}^{B}d_j\right)<0,$$
$$\delta=\operatorname{Re}\left(\sum_{j=1}^{A}a_j+\sum_{j=1}^{B'+1}b_j'-\sum_{j=1}^{A}c_j-\sum_{j=1}^{B'}d_j'\right)<0,$$

and

$$\varepsilon=\operatorname{Re}\left(\sum_{j=1}^{A}a_j+\sum_{j=1}^{B+1}b_j+\sum_{j=1}^{B'+1}b_j'-\sum_{j=1}^{A}c_j-\sum_{j=1}^{B}d_j-\sum_{j=1}^{B'}d_j'\right)<0;$$

(ii) *converges conditionally when* $|x|=1$ *and* $|y|=1$ $(x\neq 1;\ y\neq 1)$, *if*

$$\lambda<1,\qquad \delta<1,\qquad \textit{and}\qquad \varepsilon<2;$$

(iii) *diverges when* $|x|=1$ *and* $|y|=1$, *if at least one of the following three conditions does not hold true*:

$$\lambda\leqq 1,\qquad \delta\leqq 1,\qquad \textit{and}\qquad \varepsilon<2.$$

THEOREM 2. *Let $A + B = C + D + 1$, $A + B' = C + D' + 1$, and $C - A = \kappa > 0$. Then the series*

$$F^{A\ :B+\kappa+1;B'+\kappa+1}_{A+\kappa:B\ \ ;B'}\left[\begin{matrix}(a):(b);(b');\\(c):(d);(d');\end{matrix}\ x,y\right] \tag{1.6}$$

(i) *converges absolutely when $|x| = 1$ and $|y| = 1$, if and only if*

$$\lambda = \mathrm{Re}\left(\sum_{j=1}^{A} a_j + \sum_{j=1}^{B+\kappa+1} b_j - \sum_{j=1}^{A+\kappa} c_j - \sum_{j=1}^{B} d_j\right) < 0$$

and

$$\delta = \mathrm{Re}\left(\sum_{j=1}^{A} a_j + \sum_{j=1}^{B'+\kappa+1} b'_j - \sum_{j=1}^{A+\kappa} c_j - \sum_{j=1}^{B'} d'_j\right) < 0;$$

(ii) *converges conditionally when $|x| = 1$ and $|y| = 1$ $(x \neq 1;\ y \neq 1)$, if*

$$\lambda < 1 \qquad \text{and} \qquad \delta < 1.$$

THEOREM 3. *Let $A + B = C + D + 1$, $A + B' = C + D' + 1$, and $A - C = \kappa > 0$. Then the series*

$$F^{A+\kappa:B+1;B'+1}_{A\ \ :B+\kappa;B'+\kappa}\left[\begin{matrix}(a):(b);(b');\\(c):(d);(d');\end{matrix}\ x,y\right] \tag{1.7}$$

converges absolutely when

$$|x|^{1/\kappa} + |y|^{1/\kappa} = 1 \qquad (x \neq 0;\ y \neq 0), \tag{1.8}$$

if

$$\varepsilon = \mathrm{Re}\left(\sum_{j=1}^{A+\kappa} a_j + \sum_{j=1}^{B+1} b_j + \sum_{j=1}^{B'+1} b'_j - \sum_{j=1}^{A} c_j - \sum_{j=1}^{B+\kappa} d_j - \sum_{j=1}^{B'+\kappa} d'_j\right) + \kappa < 1. \tag{1.9}$$

2. DEMONSTRATIONS OF THEOREMS 1, 2, AND 3

2.1. *Proof of Theorem* 1

Case (i). Denoting the general term of the series (1.5) by $A_{mn}x^m y^n$, and making use of the familiar asymptotic estimate:

$$\frac{\Gamma(a+n)}{\Gamma(b+n)} \sim n^{\mathrm{Re}(a-b)} \qquad (n \to \infty), \tag{2.1}$$

we have

$$\begin{aligned}|A_{mn}x^m y^n| &= \left|\frac{\prod_{j=1}^{A}(a_j)_{m+n}\prod_{j=1}^{B+1}(b_j)_m\prod_{j=1}^{B'+1}(b'_j)_n}{\prod_{j=1}^{A}(c_j)_{m+n}\prod_{j=1}^{B}(d_j)_m\prod_{j=1}^{B'}(d'_j)_n}\frac{x^m}{m!}\frac{y^n}{n!}\right| \\
&= \left|\Gamma\begin{bmatrix}(c_j), (d_j), (d'_j)\\ (a_j), (b_j), (b'_j)\end{bmatrix}\right| |x|^m\,|y|^n \\
&\quad\cdot\left|\Gamma\begin{bmatrix}a_1+m+n, \ldots, a_A+m+n\\ c_1+m+n, \ldots, c_A+m+n\end{bmatrix}\right| \\
&\quad\cdot\left|\Gamma\begin{bmatrix}b_1+m, \ldots, b_B+m, b_{B+1}+m\\ d_1+m, \ldots, d_B+m, 1+m\end{bmatrix}\right| \\
&\quad\cdot\left|\Gamma\begin{bmatrix}b'_1+n, \ldots, b'_{B'}+n, b'_{B'+1}+n\\ d'_1+n, \ldots, d'_{B'}+n, 1+n\end{bmatrix}\right| \\
&\sim H_1(m+n)^{\alpha}m^{\beta-1}n^{\gamma-1} \qquad (m\to\infty; n\to\infty), \end{aligned} \tag{2.2}$$

where H_1 is a constant, $|x|=|y|=1$,

$$\alpha=\operatorname{Re}\left(\sum_{j=1}^{A}a_j-\sum_{j=1}^{C}c_j\right), \qquad \beta=\operatorname{Re}\left(\sum_{j=1}^{B}b_j-\sum_{j=1}^{D}d_j\right),$$
$$\gamma=\operatorname{Re}\left(\sum_{j=1}^{B'}b'_j-\sum_{j=1}^{D'}d'_j\right), \tag{2.3}$$

C, D, and D' (and A, B, and B') being always specified in the context, and

$$\Gamma\begin{bmatrix}\alpha_1, \ldots, \alpha_p\\ \beta_1, \ldots, \beta_q\end{bmatrix}=\frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}{\Gamma(\beta_1)\cdots\Gamma(\beta_q)}.$$

To prove Case (i) of Theorem 1, it is sufficient to apply the following result.

LEMMA 1. *Let* α, β, $\gamma\in\mathbb{R}$. *Then the series*

$$\sum_{m,n=1}^{\infty}(m+n)^{\alpha}m^{\beta-1}n^{\gamma-1} \tag{2.4}$$

converges if and only if

$$\alpha+\beta<0, \qquad \alpha+\gamma<0, \qquad \textit{and} \qquad \alpha+\beta+\gamma<0.$$

Proof. Let us fix $m=1$ $(n=1)$ and obtain the condition $\alpha+\gamma<0$ $(\alpha+\beta<0)$. Hence $2\alpha+\beta+\gamma<0$; and if $\alpha\geqq 0$, then $\alpha+\beta+\gamma<0$.

In the case when $\alpha<0$, we have

$$\sum_{m,n=1}^{\infty}(m+n)^{\alpha}m^{\beta-1}n^{\gamma-1}=\sum_{m\geqq n\geqq 1}^{\infty}(m+n)^{\alpha}m^{\beta-1}n^{\gamma-1}+\sum_{1\leqq m<n}^{\infty}(m+n)^{\alpha}m^{\beta-1}n^{\gamma-1}, \qquad (2.5)$$

and

$$(m+n)^{\alpha}m^{\beta-1}n^{\gamma-1}=m^{\alpha+\beta-1}n^{\gamma-1}\left(1+\frac{n}{m}\right)^{\alpha}.$$

When $m\geqq n$, the hypothesis $\alpha<0$ implies that

$$2^{\alpha}\leqq\left(1+\frac{n}{m}\right)^{\alpha}<1.$$

Now if we denote the simultaneous convergence or divergence of two series by the equivalence symbol $\simeq$, then the following relations are easily verified:

$$\begin{aligned}\sum_{m\geqq n\geqq 1}^{\infty}(m+n)^{\alpha}m^{\beta-1}n^{\gamma-1} &\simeq \sum_{m\geqq n\geqq 1}^{\infty}(m+n)^{\alpha+\beta-1}n^{\gamma-1}\\ &=\sum_{m=1}^{\infty}\left(m^{\alpha+\beta-1}\sum_{n=1}^{m}n^{\gamma-1}\right)\\ &\simeq\begin{cases}\dfrac{1}{\gamma}\displaystyle\sum_{m=1}^{\infty}m^{\alpha+\beta+\gamma-1} & (\gamma>0),\\ \displaystyle\sum_{m=1}^{\infty}m^{\alpha+\beta-1}\ln m & (\gamma=0),\\ \displaystyle\sum_{m=1}^{\infty}m^{\alpha+\beta-1} & (\gamma<0).\end{cases}\end{aligned} \qquad (2.6)$$

Hence the condition $\alpha+\beta+\gamma<0$ follows, and the proof of the necessity part of Lemma 1 is completed.

Next we observe that, if $\alpha\geqq 0$, then

$$(m+n)^{\alpha}m^{\beta-1}n^{\gamma-1}\leqq m^{\alpha+\beta-1}n^{\gamma+\alpha-1}.$$

Thus the convergence of the series (2.4) follows from the conditions $\alpha+\beta<0$ and $\alpha+\gamma<0$. On the other hand, if $\alpha<0$, then the convergence of the first series on the right-hand side of (2.5) follows from the relationship (2.6). The convergence of the second series on the right-hand side of (2.5) can be proved similarly.

The proof of Lemma 1 is thus completed.

Case (ii). We need the following result in this case.

LEMMA 2. *Let* $\alpha, \beta, \gamma \in \mathbb{R}$, *and*

$$\varepsilon_{mn} = (m+n)^{\alpha} m^{\beta-1} n^{\gamma-1}.$$

Then, for the equalities:

$$\lim_{m,n\to\infty} \varepsilon_{mn} = \lim_{m\to\infty} \lim_{n\to\infty} \varepsilon_{mn} = \lim_{n\to\infty} \lim_{m\to\infty} \varepsilon_{mn} = 0$$

to hold true, it is necessary and sufficient that

$$\alpha+\beta+\gamma<2, \qquad \alpha+\beta \leqq 1, \qquad \textit{and} \qquad \alpha+\gamma \leqq 1.$$

The proof of Lemma 2 is easy, and we omit the details involved. Setting

$$A_{mn} x^m y^n = u_{mn} \cdot v_{mn},$$

where

$$v_{mn} = x^m y^n$$

and

$$u_{mn} = \frac{\prod_{j=1}^{A} (a_j)_{m+n} \prod_{j=1}^{B+1} (b_j)_m \prod_{j=1}^{B'+1} (b'_j)_n}{\prod_{j=1}^{A} (c_j)_{m+n} \prod_{j=1}^{B} (d_j)_m \prod_{j=1}^{B'} (d'_j)_n} \frac{1}{m!\, n!}, \tag{2.7}$$

let us use the conditions $|x| = |y| = 1$ $(x \neq 1;\ y \neq 1)$,

$$\alpha+\beta<1, \qquad \alpha+\gamma<1, \qquad \textit{and} \qquad \alpha+\beta+\gamma<2.$$

Then it is readily seen that

1. The partial sums $S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} v_{ij}$ are bounded;
2. $\{u_{mn}\}$ converges uniformly to zero when $m \to \infty$ and $n \to \infty$;
3. the series

$$\sum_{m=0}^{\infty} |u_{m,0} - u_{m+1,0}|, \qquad \sum_{n=0}^{\infty} |u_{0,n} - u_{0,n+1}|,$$

and

$$\sum_{m,n=0}^{\infty} |u_{mn} - u_{m+1,n} - u_{m,n+1} + u_{m+1,n+1}|$$

are convergent.

By appealing now to Theorem 1.1.3 of Yanushauskas [6, p. 17], we obtain the conditional convergence of the series (1.5) under the specified conditions.

Case (iii). It is obvious that the condition

$$\lim_{m,n\to\infty} u_{mn} = 0$$

is necessary for the convergence of the double sequence $\{u_{mn}\}$. Therefore, the validity of the assertion of Theorem 1 in Case (iii) follows from the relationship (2.2) and Lemma 2.

2.2. *Proof of Theorem* 2

Case (i). With the help of the asymptotic estimate (2.1), we find for the general term of the series (1.6) that

$$\begin{aligned} |B_{mn}x^m y^n| &= \left| \frac{\prod_{j=1}^{A}(a_j)_{m+n} \prod_{j=1}^{B+\kappa+1}(b_j)_m \prod_{j=1}^{B'+\kappa+1}(b'_j)_n}{\prod_{j=1}^{A+\kappa}(c_j)_{m+n} \prod_{j=1}^{B}(d_j)_m \prod_{j=1}^{B'}(d'_j)_n} \frac{x^m}{m!}\frac{y^n}{n!} \right| \\ &\sim H_2 (m+n)^{\alpha+\kappa} m^{\beta-\kappa-1} n^{\gamma-\kappa-1} |x|^m |y|^n \left[\frac{m!\,n!}{(m+n)!}\right]^{\kappa} \\ &\qquad (m\to\infty;\, n\to\infty), \end{aligned} \tag{2.8}$$

where H_2 is a constant, and α, β, γ are defined by (2.3) with the numbers of parameters appropriately specified as in (2.8).

By Stirling's formula:

$$l! \sim \sqrt{2\pi}\, l^{l+1/2} e^{-l} \qquad (l\to\infty),$$

we have

$$\frac{m!\,n!}{(m+n)!} \sim \sqrt{2\pi}\, \frac{m^{m+1/2}\cdot n^{n+1/2}}{(m+n)^{m+n+1/2}} \qquad (m\to\infty;\, n\to\infty).$$

Then, for $|x| = 1$ and $|y| = 1$, we find from (2.8) that

$$\begin{aligned} |B_{mn}x^m y^n| &\sim (2\pi)^{\kappa} H_2 (m+n)^{\alpha+(1/2)\kappa} m^{\beta-(1/2)\kappa-1} n^{\gamma-(1/2)\kappa-1} \left[\frac{m^m n^n}{(m+n)^{m+n}}\right]^{\kappa} \\ &= (2\pi)^{\kappa} H_2 B'_{mn}. \end{aligned}$$

Furthermore, for sufficiently large m and n,

$$\frac{m^m n^n}{(m+n)^{m+n}} = \frac{1}{(1+n/m)^m} \cdot \frac{1}{(1+m/n)^n} < \frac{1}{n^p} \cdot \frac{1}{m^p} \qquad (p \in \mathbb{R}^+).$$

Hence we obtain

$$B'_{mn} < (m+n)^{\alpha+(1/2)\kappa}\, m^{\beta-(1/2)\kappa-1-p\kappa}\, n^{\gamma-(1/2)\kappa-1-p\kappa}. \tag{2.9}$$

In accordance with Lemma 1, the double series with a general term of the type (2.9) converges for sufficiently large values of p. So, by taking into account the conditions $\lambda = \alpha + \beta < 0$ and $\delta = \alpha + \gamma < 0$, we conclude that the series

$$\sum_{m,n=1}^{\infty} B'_{mn}$$

converges. The sufficiency part of the conditions in Case (i) of Theorem 2 is proved.

The necessity part becomes evident if we fix $m = 0$ $(n = 0)$ in the series (1.6).

Case (ii) of Theorem 2 can be proved by the same technique as in Theorem 1.

2.3. *Proof of Theorem* 3

Let us consider the general term of the series (1.7):

$$|C_{mn} x^m y^n| \sim H_3 (m+n)^{\alpha+\kappa}\, m^{\beta+\kappa-1}\, n^{\gamma+\kappa-1} |x|^m |y|^n \left[\frac{(m+n)!}{m!\,n!}\right]^{\kappa}$$
$$= H_3 C'_{mn} |x|^m |y|^n, \tag{2.10}$$

where H_3 is a constant.

By the principle of mathematical induction, it is not difficult to obtain the inequality:

$$\left[\frac{(m+n)!}{m!\,n!}\right]^{\kappa} \leq \frac{[\kappa(m+n)]!}{(\kappa m)!\,(\kappa n)!}. \tag{2.11}$$

Now choose $M \in \mathbb{N}$ so that

$$\kappa M + \beta + \kappa - 1 > 0 \qquad \text{and} \qquad \kappa M + \gamma + \kappa - 1 > 0.$$

Then we obtain the inequality:

$$\frac{m^{\kappa M+\beta+\kappa-1}\, n^{\kappa M+\gamma+\kappa-1}}{(m+n)^{2\kappa M+2\kappa+\beta+\gamma-2}} < 1.$$

Hence

$$\sum_{m,n=1}^{\infty} C'_{mn}|x|^m|y|^n$$

$$= \sum_{m,n=1}^{\infty} (m+n)^{\alpha+\beta+\gamma+\kappa-2} \frac{m^{\kappa M+\beta+\kappa-1} n^{\kappa M+\gamma+\kappa-1}}{(m+n)^{2\kappa M+2\kappa+\beta+\gamma-2}}$$

$$\cdot\left[\frac{(m+n)^{2M}\cdot(m+n)!}{m^M n^M (m!\,n!)}\right]^{\kappa} |x|^m|y|^n$$

$$< \sum_{m,n=1}^{\infty} (m+n)^{\alpha+\beta+\gamma+\kappa-2}\left[\frac{(m+n)^{2M}\cdot(m+n)!}{m^M n^M (m!\,n!)}\right]^{\kappa} |x|^m|y|^n$$

$$\simeq \sum_{m,n=1}^{\infty} (m+n)^{\alpha+\beta+\gamma+\kappa-2}\left[\frac{(m+n+2M)!}{(m+M)!\,(n+M)!}\right]^{\kappa} |x|^m|y|^n$$

$$\leqq |xy|^{-M} \sum_{m,n=1}^{\infty} (m+n)^{\alpha+\beta+\gamma+\kappa-2} \frac{[\kappa(m+n+2M)]!}{[\kappa(m+M)]!\,[\kappa(n+M)]!}$$

$$\cdot(\sqrt[\kappa]{|x|})^{\kappa(M+m)}\,(\sqrt[\kappa]{|y|})^{\kappa(M+n)}$$

$$\leqq |xy|^{-M} \sum_{l=2}^{\infty} l^{\alpha+\beta+\gamma+\kappa-2}(\sqrt[\kappa]{|x|}+\sqrt[\kappa]{|y|})^{\kappa(l+2M)}$$

$$= |xy|^{-M} \sum_{l=2}^{\infty} l^{\alpha+\beta+\gamma+\kappa-2} \qquad (\text{since } \sqrt[\kappa]{|x|}+\sqrt[\kappa]{|y|}=1). \qquad (2.12)$$

The last series converges if $\alpha+\beta+\gamma+\kappa<1$. This evidently completes the proof of Theorem 3.

3. Convergence of the Multiple Hypergeometric Series (1.4)

Our demonstrations of the preceding section can be appropriately extended to obtain the corresponding convergence conditions for the multiple hypergeometric series (1.4) when

$$\Delta_k \equiv 1+C+D^{(k)}-A-B^{(k)}=0 \qquad (k=1, ..., n). \qquad (3.1)$$

For the convenience of the interested reader, we summarize our results as follows.

Theorem 4. *Let* $\Delta_k=0$ $(k=1, ..., n)$, $A=C$, *and*

$$|x_1|=\cdots=|x_n|=1.$$

Then the series (1.4)

(i) *converges absolutely if and only if*

$$\rho_k = \mathrm{Re}\left(\sum_{j=1}^{A} a_j + \sum_{j=1}^{B^{(k)}} b_j^{(k)} - \sum_{j=1}^{C} c_j - \sum_{j=1}^{D^{(k)}} d_j^{(k)}\right) < 0 \qquad (k=1, ..., n) \tag{3.2}$$

and

$$\sigma = \mathrm{Re}\left(\sum_{j=1}^{A} a_j + \sum_{k=1}^{n} \sum_{j=1}^{B^{(k)}} b_j^{(k)} - \sum_{j=1}^{C} c_j - \sum_{k=1}^{n} \sum_{j=1}^{D^{(k)}} d_j^{(k)}\right) < 0; \tag{3.3}$$

(ii) *converges conditionally when* $x_k \neq 1$ $(k=1, ..., n)$, *if*

$$\rho_k < 1 \ (k=1, ..., n) \qquad \textit{and} \qquad \sigma < n;$$

(iii) *diverges if at least one of the following* $n+1$ *conditions does not hold true*:

$$\rho_k \leqq 1 \ (k=1, ..., n) \qquad \textit{and} \qquad \sigma < n.$$

THEOREM 5. *Let* $\Delta_k = 0$ $(k=1, ..., n)$, $A < C$, *and*

$$|x_1| = \cdots = |x_n| = 1. \tag{3.4}$$

Then the series (1.4)

(i) *converges absolutely if and only if* $\rho_k < 0$ $(k=1, ..., n)$;

(ii) *converges conditionally when* $x_k \neq 1$ $(k=1, ..., n)$, *if*

$$\rho_k < 1 \qquad (k=1, ..., n),$$

ρ_k *being defined by* (3.2).

THEOREM 6. *Let* $\Delta_k = 0$ $(k=1, ..., n)$ *and* $A > C$. *Then the series* (1.4) *converges absolutely when*

$$|x_1|^{1/(A-C)} + \cdots + |x_n|^{1/(A-C)} = 1 \qquad (x_k \neq 0; k=1, ..., n), \tag{3.5}$$

if

$$\sigma + A - C < 1, \tag{3.6}$$

where σ *is defined by* (3.3).

We conclude by remarking that the convergence conditions, given by Theorems 4, 5, and 6 above, can readily be specialized to deduce the corresponding results for the familiar Lauricella hypergeometric series $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$, and $F_D^{(n)}$ in n variables.

Acknowledgments

The present investigation was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

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