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Note

On Polynomials with a Prescribed Zero

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Let $p(z)=\sum_{v=0}^{n} a_v z^v$ be a polynomial of degree at most n vanishing at $z=\zeta$ ($\zeta^{n+1}\neq 1$). It has been proved that for every complex λ and $k=0, 1, 2, ..., n$,

$$a_k=\frac{1}{k!}\frac{1}{(n-k+1)}\sum_{v=0}^{n-k} p^{(k)}(e^{2v\pi i/(n-k+1)})-\frac{\lambda}{(n+1)}\sum_{v=0}^{n}\frac{p(e^{2v\pi i/(n+1)})}{(\zeta e^{-2v\pi i/(n+1)}-1)}.$$

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Let $p(z)=\sum_{v=0}^{n} a_v z^v$ be a polynomial of degree n. Then clearly

$$|a_0|\leqslant \max_{|z|=1}|p(z)|, \tag{1.1}$$

with equality holding if and only if $p(z)$ is of constant modulus on $|z|=1$. In case $p(z)$ has a zero on $|z|=1$, Boas [1] sharpened the above inequality (1.1) and proved

$$|a_0|\leqslant\left(\frac{n}{n+1}\right)\max_{|z|=1}|p(z)|. \tag{1.2}$$

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Rahman and Schmeisser [3] generalized the above result of Boas [1] for polynomials having a zero on $|z|=\rho$ $(0<\rho<\infty)$ by proving the following.

THEOREM A [3, Theorem 2]. *Let* $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ *be a polynomial of degree at most n vanishing at*

$$\zeta = \rho e^{i\phi} \neq e^{2v\pi i/(n+1)} \qquad (1 \leqslant v \leqslant n,\ \rho \geqslant 0,\ 0 \leqslant \phi < 2\pi).$$

Then

$$|a_0| \leqslant \frac{2\rho}{n+1} \sum_{v=1}^{n} \frac{\sin(\pi v/(n+1))}{\sqrt{\rho^2 - 2\rho\cos(2\pi v/(n+1) - \phi) + 1}} \max_{1 \leqslant k \leqslant n} |p(e^{2k\pi i/(n+1)})|. \tag{1.3}$$

This inequality is best possible for every admissible $\rho e^{i\phi}$.

The above theorem has been deduced from the following lemma which itself is of independent interest.

LEMMA A [3, Lemma 1, p. 95]. *If* $p(z)=a_0+a_1z+\cdots+a_nz^n$ *is a polynomial of degree at most n vanishing at* $z=\zeta$ $(\zeta^{n+1}\neq 1)$, *then for every complex number* λ,

$$a_0 = \frac{1}{n+1} \sum_{v=0}^{n} \left(1 - \frac{\lambda}{(\zeta e^{-2v\pi i/(n+1)} - 1)}\right) p(e^{2v\pi i/(n+1)}). \tag{1.4}$$

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In this note we prove a generalization of the above lemma by obtaining a representation of a_k for $k=0, 1, 2, \ldots, n$, which for $k=0$ reduces to the above lemma. Besides, we believe our proof is much simpler. We prove

THEOREM. *If* $p(z)=a_0+a_1z+\cdots+a_nz^n$ *is a polynomial of degree at most n vanishing at* $z=\zeta$ $(\zeta^{n+1}\neq 1)$, *then for every complex* λ *and* $k=0, 1, 2, \ldots, n$,

$$a_k = \frac{1}{k!}\,\frac{1}{(n-k+1)} \sum_{v=0}^{n-k} p^{(k)}(e^{2v\pi i/(n-k+1)}) - \frac{\lambda}{(n+1)} \sum_{v=0}^{n} \frac{p(e^{2v\pi i/(n+1)})}{(\zeta e^{-2v\pi i/(n+1)} - 1)}. \tag{2.1}$$

Proof. Using Lagrange's interpolation formula [2, p. 62] with z_0,

$z_1, \ldots, z_{n-k}$ as interpolation nodes, where $z_0, z_1, \ldots, z_{n-k}$ are the zeros of $z^{n-k+1}-1$, we get

$$p^{(k)}(z)=\sum_{v=0}^{n-k} \frac{p^{(k)}(e^{2v\pi i/(n-k+1)})\, e^{2v\pi i/(n-k+1)}(z^{n-k+1}-1)}{(z-e^{2v\pi i/(n-k+1)})(n-k+1)}$$
$$=\frac{(z^{n-k+1}-1)}{(n-k+1)} \sum_{v=0}^{n-k} \frac{p^{(k)}(e^{2v\pi i/(n+1)})}{(ze^{-2v\pi i/(n-k+1)}-1)}.$$

In particular,

$$p(z)=\frac{(z^{n+1}-1)}{(n+1)} \sum_{v=0}^{n} \frac{p(e^{2v\pi i/(n+1)})}{(ze^{-2v\pi i/(n+1)}-1)}.$$

Since by assumption $p(z)$ vanishes at $z=\zeta$ $(\zeta^{n+1} \neq 1)$, we get for every complex λ,

$$a_k=\frac{p^{(k)}(0)}{k!}=\frac{p^{(k)}(0)}{k!}-\frac{\lambda p(\zeta)}{\zeta^{n+1}-1},$$

which implies

$$a_k=\frac{1}{k!} \frac{1}{(n-k+1)} \sum_{v=0}^{n-k} p^{(k)}(e^{2v\pi i/(n-k+1)})$$
$$-\frac{\lambda}{(n+1)} \sum_{v=0}^{n} \frac{p(e^{2v\pi i/(n+1)})}{(\zeta e^{-2v\pi i/(n+1)}-1)},$$

and the proof of the theorem is complete.

It is clear that for $k=0$, the above theorem reduces to Lemma A.

References

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