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Matroids with at least two regular elements

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A B S T R A C T

For a matroid M, an element e such that both $M \setminus e$ and M/e are regular is called a regular element of M. We determine completely the structure of non-regular matroids with at least two regular elements. Besides four small size matroids, all 3-connected matroids in the class can be pieced together from F_7 or S_8 and a regular matroid using 3-sums. This result takes a step toward solving a problem posed by Paul Seymour: find all 3-connected non-regular matroids with at least one regular element Oxley (1992) [5, 14.8.8].

1. Introduction

The matroid terminology follows Oxley [5]. Let M be a matroid and X be a subset of the ground set E. The *connectivity function* λ is defined as $\lambda(X) = r(X) + r(E - X) - r(M)$. Observe that $\lambda(X) = \lambda(E - X)$. For $j \geq 1$, a partition (X_1, X_2) of E is called a j-separation if $|X_1|, |X_2| \geq j$, and $\lambda(X_1) \leq j - 1$. When $\lambda(X_1) = j - 1$, we call (X_1, X_2) an *exact j-separation*. When $\lambda(X_1) = j - 1$ and $|X_1| = j$ or $|X_2| = j$ we call (X_1, X_2) a *minimal exact j-separation*. For $k \geq 2$, we say M is *k-connected* if M has no j-separation for $j \leq k - 1$. A matroid is *internally k-connected* if it is k-connected and has no non-minimal exact k-separations. In particular, a simple matroid is 3-connected if $\lambda(X_1) \geq 2$ for all partitions (X_1, X_2) with $|X_1|, |X_2| \geq 3$. A 3-connected matroid is *internally 4-connected* if $\lambda(X_1) \geq 3$ for all partitions (X_1, X_2) with $|X_1|, |X_2| \geq 4$.

The 1-sum, 2-sum, and 3-sum of binary matroids are defined in [6]. A *cycle* of a binary matroid is a disjoint union of circuits. Let M_1 and M_2 be binary matroids with non-empty ground sets E_1 and E_2, respectively. We define a new binary matroid $M_1 \triangle M_2$ to be the matroid with ground set $E_1 \triangle E_2$ and with cycles having the form $C_1 \triangle C_2$ where C_i is a cycle of M_i for $i = 1, 2$. When $E_1 \cup E_2 = \phi$, then $M_1 \triangle M_2$ is a 1-*sum* of M_1 and M_2. When $|E_1|, |E_2| \geq 3$, $E_1 \cap E_2 = \{z\}$ and z is not a loop or coloop of

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M_1 or M_2, then $M_1 \triangle M_2$ is a 2-*sum* of M_1 and M_2. When $|E_1|, |E_2| \geq 7$, $E_1 \cap E_2 = T$ and T is a triangle in M_1 and M_2, then $M_1 \triangle M_2$ is a 3-*sum* of M_1 and M_2.

An element e in a non-regular matroid M is called a *regular element* if both $M \backslash e$ and M/e are regular. Seymour posed the following problem that appears in Oxley's book *Matroid Theory* [5, 14.8.8]: find all 3-connected non-regular matroids with at least one regular element. In this paper, we take a step toward solving this problem by determining the class of 3-connected non-regular matroids with at least two regular elements.

We denote the 4-point line as $U_{2,4}$ and the Fano matroid as F_7. We denote by S_8 the following single-element extension of F_7. It is self-dual. A single-element extension of S_8 that will play a role is P_9 shown below.

$$F_7 = \left[\begin{array}{c|cccc} & 1 & 0 & 1 & 1 \\ I_3 & 1 & 1 & 0 & 1 \\ & 0 & 1 & 1 & 1 \end{array}\right] \qquad S_8 = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 \\ I_4 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \end{array}\right]$$

$$P_9 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ I_4 & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \end{array}\right].$$

For this paper, it helps to think of F_7 as the single-element extension of the 3-wheel with spokes labeled $\{1, 2, 3\}$ where the new element forms a circuit with $\{1, 2, 3\}$. The matroid P_9 is the single-element extension of the 4-wheel with spokes $\{1, 2, 3, 4\}$ where the new element forms a circuit with any three consecutive spokes, say $\{1, 2, 3\}$. Then $P_9 \backslash 1 \cong S_8$ and $P_9 \backslash 3 \cong S_8$. Moreover, $P_9 \backslash \{1, 3\} \cong F_7^*$.

Let F_7^p and S_8^p be the matroids obtained from F_7 and S_8, respectively, by adding an element in parallel with an element belonging to at least two triangles. Note that every element of F_7 is in at least two triangles, but only one element of S_8 is in two triangles. The main result of this paper gives a complete characterization of the matroids with at least two regular elements.

Theorem 1.1. *A* 3*-connected non-regular matroid M has at least two regular elements if and only if*

(i) *M is $U_{2,4}$, F_7, F_7^* or S_8; or*
(ii) *M is the* 3*-sum of F_7 or S_8 with a* 3*-connected regular matroid (with the possible exception of elements in parallel with the* 3*-sum triangle); or*
(iii) *M is the* 3*-sum of F_7^p or S_8^p with two* 3*-connected regular matroids (with the possible exception of elements in parallel with the* 3*-sum triangle). These two* 3*-sums are made along two disjoint triangles of F_7^p or S_8^p.*

In order to prove this result we use the following theorems. The first is by Oxley and appears in [4, 3.9].

Theorem 1.2. *Let M be a non-binary* 3*-connected matroid having an element e such that $M \backslash e$ and M/e are both regular. Then $M \cong U_{2,4}$.* □

The next result by Zhou appears in [7, 1.2]. The matroid S_{10}, shown below, is the first matroid in the internally 4-connected infinite family of almost-graphic matroids S_{3n+1} [3]. The matroid $M[E_5]$ appears in [1] where Kingan characterized the class of matroids with no minors isomorphic to $M(K_5 \backslash e)$, $M^*(K_5 \backslash e)$ and $AG(3, 2)$. $M[E_5]$ is a splitter for this class. It is self dual and internally 4-connected. The self-dual 4-connected matroid T_{12} appears in [2].

$$S_{10} = \left[\begin{array}{c|cccccc} & 1 & 0 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 1 & 1 \\ I_4 & 0 & 1 & 1 & 0 & 1 & 1 \\ & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \qquad E_5 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$T_{12} = \left[\begin{array}{c|cccccc} & 1 & 1 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 & 0 & 1 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 1 \\ I_6 & 0 & 0 & 1 & 1 & 1 & 0 \\ & 0 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right].$$

Theorem 1.3. *A non-regular internally 4-connected binary matroid other than F_7 and F_7^* contains one of the following matroids as a minor: $M(E_5)$, S_{10}, S_{10}^*, $T_{12} \setminus e$, and T_{12}/e.* □

It can be checked that S_{10}, $S_{10}^* T_{12} \setminus e$, and T_{12}/e each have one regular element and $M(E_5)$ has zero regular elements. Moreover, the number of regular elements in a non-regular matroid is bounded above by the number of regular elements in any non-regular minor. We use this fact throughout the paper. The next result follows from Theorem 1.2 and the above discussion.

Corollary 1.4. *If M is an internally 4-connected binary non-regular matroid having at least two regular elements, then M is isomorphic to F_7 or F_7^*.*

Finally, we use the following results by Seymour that appear in [6, 2.9 and 4.1].

Theorem 1.5. *If (X_1, X_2) is an exact 3-separation of a binary matroid M, with $|X_1|, |X_2| \geq 4$, then there are binary matroids M_1, M_2 on $X_1 \cup T, X_2 \cup T$, respectively (where T contains three new elements), such that M is the 3-sum of M_1 and M_2. Conversely if M is the 3-sum of M_1 and M_2, then $(E(M_1) - E(M_2), E(M_2) - E(M_1))$ is an exact 3-separation of M, and $|E(M_1) - E(M_2)|, |E(M_2) - E(M_1)| \geq 4$.* □

Theorem 1.6. *If M is binary and is the 3-sum of M_1 and M_2, and M is 3-connected, then M_1 and M_2 are isomorphic to minors of M.* □

In the next section, we give several separation lemmas that are used in the proof of the main theorem. In the third section, we give results on the number of regular elements in a matroid. Finally, in the fourth section, we prove Theorem 1.1. The difficulty in completely finishing off Seymour's problem lies in determining the structure of the non-regular internally 4-connected matroids with one regular element.

2. Understanding 3-separations in the context of regular elements

We will denote by $si(M)$ and $co(M)$ the simple and cosimple matroid, respectively, associated with M. Let M be a 3-connected non-regular binary matroid such that M is the 3-sum of matroids M_1 and M_2 where $|E(M_1)|, |E(M_2)| \geq 7$, $E(M_1) \cap E(M_2) = T$ and T is a triangle in M_1 and M_2. Assume that $e \in E(M_1) - E(M_2)$ is a regular element of M.

Lemma 2.1. *The element e is not spanned by $E(M_2) - E(M_1)$ in M.*

Proof. Suppose e is spanned by $E(M_2) - E(M_1)$ in M. Then e is spanned by T in M_1 and so e is in parallel to some element $t \in T$. By hypothesis, $M \setminus e$ is regular. Observe that $M_1 \setminus e$ and M_2 are regular because:

(i) when $|E(M_1)| > 7$, $M \setminus e$ is the 3-sum of $M_1 \setminus e$ with M_2;
(ii) when $|E(M_1)| = 7$, $M_1 \setminus e$ has 6 elements and is isomorphic to $M(K_4)$. So $M \setminus e$ is obtained from M_2 after a $\Delta - Y$ operation along the triangle T.

But M_1 is obtained from $M_1 \setminus e$ by adding e in parallel with t. Therefore M_1 and M_2 are regular; a contradiction because the class of regular matroids is closed under 3-sums. Thus e is not spanned by $E(M_2) - E(M_1)$ in M. □

Lemma 2.2. *The element e is not spanned by $E(M_2) - E(M_1)$ in M^*.*

Proof. If N_i is obtained from M_i by a $\Delta - Y$ operation along the triangle T, then M^* is the 3-sum of N_1^* and N_2^*. Applying Lemma 2.1, we conclude that e is not spanned by $E(M_2) - E(M_1)$ in M^*. □

In the next result, we describe how the presence of a regular element in M_1 impacts the structure of M. We prove that one of two situations must occur: either M_1 is non-regular with e as a regular element and M_2 is regular or M_2 is non-regular and M_1 is a small matroid with a specific structure. In the latter situation we prove that $E(M_1) - T = T' \cup T^*$ where T' is a triangle and T^* is a triad such that $e \in T' \cap T^*$ and $E(M_1) - E(M_2)$ is closed in M. Since M is binary, a triangle and triad must intersect in an even number of elements. This means M_1 has just 7 elements, one of which is parallel with an element of T.

Lemma 2.3. (i) *M_2 is a regular matroid; or*
(ii) *there is a triangle T' and a triad T^* of M such that $e \in T' \cap T^*$ and $E(M_1) - T = T' \cup T^*$.*

Moreover,

(iii) *when (i) happens, M_1 is a non-regular matroid having e as a regular element;*
(iv) *when (ii) happens, $E(M_1) - E(M_2)$ is closed in M.*

Proof. Assume that (i) does not hold, that is,

$$M_2 \text{ is non-regular.} \tag{1}$$

First, we establish that

$$r(M_1) = 3 \text{ or } \mathrm{si}(M/e) \text{ is not 3-connected.} \tag{2}$$

Suppose that $r(M_1) \geq 4$ and $\mathrm{si}(M/e)$ is 3-connected. If T' is a triangle of M containing e, then, by Lemma 2.1, $|E(M_2) \cap T'| \leq 1$. Therefore we may assume that $\mathrm{si}(M/e) = M/e \setminus X$, for $X \subseteq E(M_1) - T$. If $M_1/e \setminus X \simeq M(K_4)$, then M_2 is obtained from $\mathrm{si}(M/e)$ after a $Y - \Delta$ operation along the triad $E(M_1) - (e \cup X \cup T)$. So M_2 is regular; a contradiction to (1). If $M_1/e \setminus X \not\simeq M(K_4)$, then $\mathrm{si}(M/e)$ is the 3-sum of $M_1/e \setminus X$ and M_2. As $\mathrm{si}(M/e)$ is regular, it follows that M_2 is regular; a contradiction to (1). We have (2).

If N_i is obtained from M_i by a $\Delta - Y$ operation along the triangle T, then M^* is the 3-sum of N_1^* and N_2^*. Note that Lemma 2.3(i) holds for the decomposition $M = M_1 \triangle M_2$ if and only if Lemma 2.3(i) holds for the decomposition $M^* = N_1^* \triangle N_2^*$. The analogous statement occurs when we replace (i) by (ii). Therefore, the dual of (2) becomes

$$r(N_1^*) = 3 \quad \text{or} \quad [\mathrm{co}(M \setminus e)]^* = \mathrm{si}(M^*/e) \text{ is not 3-connected.} \tag{3}$$

By Bixby's Theorem [5, 8.4.6], $\mathrm{si}(M/e)$ or $\mathrm{co}(M \setminus e)$ is 3-connected. By (2) and (3), $r(M_1) = 3$ or $r(N_1^*) = 3$. Taking the dual when necessary, we may assume that

$$r(M_1) = 3. \tag{4}$$

Next, we prove the following claim.

Claim: M_1 does not have a minor N such that T and $T' = E(N) - T$ are triangles of N, $e \notin E(N) = T \cup T'$ and $r(N) = 2$.

Suppose that N exists, say $N = M_1 \setminus X/Y$. By hypothesis, $e \in X \cup Y$ and so $M \setminus X/Y$ is regular. Moreover, $M \setminus X/Y$ is isomorphic to M_2. Thus M_2 is regular; a contradiction to (1). Therefore the claim holds.

If $\mathrm{si}(M_1) \simeq F_7$, then M_1/e is a rank-2 matroid. By Lemma 2.1, M_1/e has T as a triangle. We have a contradiction by the claim because every parallel class of M_1/e is non-trivial. Hence, by (4), $\mathrm{si}(M_1) \simeq M(K_4)$. In particular, $T^* = E(M_1) - \mathrm{cl}_{M_1}(T)$ is a triad of M_1. By Lemma 2.1, $e \in T^*$, say $T^* = \{e, e_1, e_2\}$. Let $f_1, \ldots, f_k$ be the elements of $\mathrm{cl}_{M_1}(T) - T$. For each i, there is $t_i \in T$ such that $\{f_i, t_i\}$ is a parallel class of M_1. By the claim, $k \leq 2$. Next, we establish that

$$k = 1. \tag{5}$$

As $|E(M_1)| \geq 7$ and $|E(M_1) - \mathrm{cl}_{M_1}(T)| = 3$, it follows that $k \geq 1$. If (5) does not hold, then $k = 2$. In M_1/e, by the claim, e_i is in parallel with f_j, say e_i is in parallel with f_i, for both i. Therefore $T_i = \{e, e_i, f_i\}$ is a triangle of M, for both i, and so $T_1 \triangle T_2 \triangle \{f_1, f_2, t_3\} = \{e_1, e_2, t_3\}$, where $T = \{t_1, t_2, t_3\}$ is a triangle of M_1. Thus $N = M_1 \setminus e/e_1$ is a minor of M_1 contrary to the claim. Thus (5) holds. By the claim e_1 or e_2 is in parallel with f_1 in M_1/e, say e_1. That is, $T' = \{e, e_1, f_1\}$ is a triangle of M_1 and so of M. We have (ii).

Assume that (i) happens, that is, M_2 is regular. Thus M_1 is non-regular because M is non-regular. To conclude (iii) we need to prove only that e is a regular element of M_1. By the proof of Theorem 1.6, there are disjoint subsets Y and Z of $E(M_2) - E(M_1)$ such that $N = M_2 \setminus Y/Z$ is a 6-element matroid such that $T'' = E(N) - T$ is a triangle of N and, for each $f \in T$, there is an $f'' \in T''$ such that $\{f, f''\}$ is a circuit of N. So $M \setminus Y/Z$ is isomorphic to M_1—this isomorphism fix each element of $E(M_1) - E(M_2)$ and sends f'' into f, for each $f'' \in T''$. As both $M \setminus e$ and M/e are regular, it follows that $(M \setminus e) \setminus Y/Z \simeq M_1 \setminus e$ and $(M/e) \setminus Y/Z \simeq M_1/e$ are regular. That is, e is a regular element of M_1. We have (iii).

Assume that (ii) happens. If $E(M_1) - E(M_2)$ spans an element g of $E(M_2) - E(M_1)$ in M, then $[E(M_1) - E(M_2)] \cup g$ is a 3-separating set for M. Using the 3-separation induced by this set, we can decompose M as the 3-sum of matroids M_1' and M_2' such that $E(M_1') = [E(M_1) - E(M_2)] \cup g \cup T''$ and $T'' = E(M_1') \cap E(M_2')$. Note that, in M_1', the element g is in parallel with some element of T''. In particular, $M_1' \setminus g \simeq M_1$ is regular. So M_1' is regular; a contradiction to this lemma. Thus $E(M_1) - E(M_2)$ is closed in M. □

Now that we have shown M has a clearly defined structure, we want to say more about the second situation. Recall that $R(M)$ is the set of regular elements. For a triangle T' and triad T^* of M, we say that T', T^* is an *undesired fan* if $T' \cap T^* \cap R(M) \neq \emptyset$. Note that $\{T' \cup T^*, E(M) - (T' \cup T^*)\}$ is an exact 3-separation for M and by Theorem 1.5, it is possible to decompose M as a 3-sum using it. In the next lemma we show that the presence of an undesired fan implies the existence of two regular elements.

Lemma 2.4. *If T', T^* is an undesired fan in M such that $E(M_1) - E(M_2) = T' \cup T^*$, then $T' \cap T^* \subseteq R(M)$. Moreover, if $T^* - T' = \{f\}$, then M/f is a 3-connected non-regular matroid such that $T' \cap T^* \subseteq R(M/f)$.*

Proof. Suppose that $T' = \{e, e', t\}$, $T^* = \{e, e', f\}$ and $e \in R(M)$. In M/e', t and e are in parallel. As $M \setminus e$ and so $M/e' \setminus e$ is regular, it follows that M/e' is regular because M/e' is obtained from $M/e' \setminus e$ by adding e in parallel with t. Using duality, we conclude that $M \setminus e'$ is regular. Hence e' is a regular element of M and so $T' \cap T^* \subseteq R(M)$.

Next, observe that $E(M_1) = T' \cup T^* \cup T$ and $E(M_2) = [E(M) - (T' \cup T^*)] \cup T$. As M_1 is regular, it follows that M_2 is non-regular. By Lemma 2.3, f does not belong to a triangle of M. So M/f is 3-connected because $\mathrm{si}(M/f)$ is 3-connected. But $M/f \simeq M_2$ because M_1/f has three non-trivial parallel classes each containing one element of T' and another of T. The result follows because $R(M) \subseteq R(M/f)$. □

In the next lemma, we prove that, when this happens, it is possible to uncontract f keeping the property of these two regular elements.

Lemma 2.5. *Let N be a 3-connected non-regular binary matroid having different regular elements e and e'. Suppose that T' is a triangle of N such that $e, e' \in T'$ and $\{e, e'\}$ is not contained in a triad of N. If M is a one-element binary lift of N, say $M/f = N$, such that $\{e, e', f\}$ is a triad of M, then e and e' are regular elements of M (and M is 3-connected).*

Proof. Observe that $\mathrm{si}(M/e) = M/e \setminus e'$. But, in $M \setminus e'$, e and f are in series. So $M/e \setminus e' \simeq M/f \setminus e' = N \setminus e'$ and $\mathrm{si}(M/e)$ is regular. Thus M/e is regular. As $M \setminus e/f = N \setminus e$, it follows that $M \setminus e/f$ is regular and so $M \setminus e$ is regular. That is, e is a regular element of M. A similar argument holds with e'. □

3. The number of regular elements in a matroid

Next, we prove a result on the number of regular elements in a binary non-regular matroid. Observe that, F_7^* has two single-element extensions S_8 and $AG(3, 2)$. The matroid $AG(3, 2)$ has one single-element extension Z_4. The matroid S_8 has two single-element extensions, Z_4 and P_9. Observe further that F_7 and F_7^* have seven regular elements, S_8 has six regular elements, and P_9 has four regular elements. $AG(3, 2)$ has zero regular elements and consequently, so do all its 3-connected extensions and coextensions.

Lemma 3.1. *Let M be a 3-connected non-regular binary matroid. If $|E(M)| \geq 9$, then $|R(M)| = 0, 1, 2$ or 4. Moreover, if $|R(M)| = 4$, then $R(M)$ is both a circuit and a cocircuit of M.*

Proof. Assume this result fails. Choose a minimal counter-example M. We have four possibilities: $|R(M)| = 3$; or $|R(M)| = 4$ and $R(M)$ is not a circuit of M; or $|R(M)| = 4$ and $R(M)$ is not a cocircuit of M; or $|R(M)| \geq 5$. In all four cases, $R(M) \neq \emptyset$. In particular, $AG(3, 2)$ is not a minor of M because $R(AG(3, 2)) = \emptyset$. Thus S_8 is a minor of M. But the only 3-connected binary single-element extension of S_8 without a minor isomorphic to $AG(3, 2)$ is P_9. Therefore M has P_9 or P_9^* as a minor. But $|R(P_9)| = 4$ and $R(P_9) = R(P_9^*)$ is both a circuit and a cocircuit of P_9. Hence $|E(M)| \geq 10$. Moreover, $|R(M)| \leq |R(P_9)| = 4$ and by Corollary 1.4, M is not internally 4-connected.

Suppose $|R(M)| = 3$. By Theorem 1.5, we can decompose M as the 3-sum of matroids M_1 and M_2 such that $E(M_1) \cap E(M_2) = T$ and $E(M_1) \cap R(M) \neq \phi$. If Lemma 2.3(ii) occurs and $f \in T^* - T'$, then by Lemma 2.4 and the choice of M, the results holds for M/f. Moreover, $T' \cap T^* \subseteq R(M)$. As $R(M) \subseteq R(M/f)$ and $|R(M)| = 3$, it follows that $|R(M/f)| = 4$ and $R(M/f)$ is both a circuit and a cocircuit of M/f. Thus $R(M) \cup g$ is a cocircuit of M, where $\{g\} = R(M/f) - R(M)$. If $R(M) \cup g$ is not a circuit of M, then $R(M) \cup \{f, g\}$ is a circuit of M containing T^*; a contradiction. Hence $R(M) \cup g$ is both a circuit and a cocircuit of M. Note that $[R(M) \cup g] \triangle T^*$ is a triad of M and $[R(M) \cup g] \triangle T'$ is a triangle of M whose intersection contains a regular element. Therefore, by Lemma 2.4 the intersection has two regular elements (g is the other regular element); a contradiction. Thus Lemma 2.3(i) occurs. Observe that $R(M)$ is contained in a circuit–cocircuit of M_1 consisting of regular elements avoiding T. Thus every element in this circuit–cocircuit is also a regular element of M; a contradiction. Thus we proved that M cannot have exactly three regular elements.

Next, suppose $|R(M)| = 4$, but $R(M)$ is not a circuit and cocircuit. By Theorem 1.5, we can decompose M as the 3-sum of matroids M_1 and M_2 such that $E(M_1) \cap E(M_2) = T$ and $E(M_1) \cap R(M) \neq \phi$. If Lemma 2.3(ii) occurs, $f \in T^* - T'$, then, by Lemma 2.4, M/f has the same regular elements as M. By the choice of M, $R(M)$ is a circuit–cocircuit of M/f. As $R(M) \cup f$ contains a triad of M, it follows that $R(M) \cup f$ is not a circuit of M. Thus $R(M)$ is a circuit–cocircuit of M.

We may assume by Lemma 2.3(i) that M_2 is regular, M_1 is non-regular, and $|R(M)| \subseteq E(M_1)$. By the choice of M if $|E(M_1)| \geq 9$, $R(M)$ is a circuit–cocircuit of $\mathrm{si}(M_1)$ and therefore of M; a contradiction. Thus M_1 has at most 8 elements. Since $\mathrm{si}(M_1)$ is non-regular, $\mathrm{si}(M_1)$ is isomorphic to F_7 or S_8. In both cases, $R(M)$ is a circuit–cocircuit of this matroid. □

Using the previous lemma, we can refine the second part of Lemma 2.4.

Lemma 3.2. *Let M be a 3-connected non-regular binary matroid with $|E(M)| \geq 10$ and suppose T, T^* is an undesired fan of M such that $T^* - T = \{f\}$. Then M/f is a non-regular 3-connected matroid such that $R(M/f) = R(M)$.*

Proof. We argue by contradiction. Since $T \cap T^* \subseteq R(M)$, it follows from Lemma 2.4 that $|R(M)| \geq 2$. Lemma 3.1 implies that $|R(M/f)|$ is 2 or 4. If $|R(M/f)| = |R(M)|$, then $R(M/f) = R(M)$ because $R(M) \subseteq R(M/f)$; a contradiction. By Lemma 3.1, $|R(M/f)| = 4$ and $|R(M)| = 2$. Moreover, $R(M/f)$ is a circuit–cocircuit of M/f.

Since $T^* \subseteq R(M/f) \cup f$, it follows that $R(M/f)$ is also a circuit–cocircuit of M. Therefore $T' = T \triangle R(M/f)$ is a triangle of M and $T'^* = T^* \triangle R(M/f)$ is a triad of M. But T' is a triangle of M/f containing two regular elements of M/f such that no triad of M/f contains these two elements. By Lemma 2.5 these two elements are also regular in M. Hence $R(M/f) = R(M)$; a contradiction. □

A 3-separation $\{X, Y\}$ for a 3-connected matroid is said to be trivial provided $|X| = 3$ or $|Y| = 3$.

Lemma 3.3. *Let M be a 3-connected non-regular binary matroid such that $|R(M)| \geq 1$. If any non-trivial 3-separation for M has the union of a triangle and a triad of a undesired fan as one of its sets, then M is isomorphic to S_8, F_7 or F_7^*.*

Proof. If $|E(M)| \leq 8$, then the result holds. Therefore, suppose that $|E(M)| \geq 9$. First assume that M has just one non-trivial 3-separation. By Theorem 1.5, M is the 3-sum of matroids M_1 and M_2 such that $E(M_1) - E(M_2)$ is the union of the triangle and the triad of the undesired fan. Thus $E(M_1) \cap R(M) \neq \phi$. Observe that Lemma 2.3(ii) holds in this case. By the uniqueness of the 3-separation for M, M_2 is internally 4-connected. By Theorem 1.3, M_2 is isomorphic to F_7. Thus $|E(M)| = 8$; a contradiction. Hence M has at least two non-trivial 3-separations.

Let T_1, T_1^* and T_2, T_2^* be different undesired fans of M. For $i \in \{1, 2\}$, set $Z_i = T_i \cap T_i^*$. By Lemmas 2.4 and 3.1, and orthogonality, $R(M) = Z_1 \cup Z_2$ is a circuit–cocircuit of M. In particular, Z_1 and Z_2 are unique and these are the unique undesired fans of M. If $T_1 - T_1^* = \{t\}$ and $T_1^* - T_1 = \{f\}$, then $T_2 = Z_2 \cup t$ and $T_2^* = Z_2 \cup f$ because $T_1 \triangle T_2 = T_1^* \triangle T_2^* = R(M)$. Observe that $Z_1 \cup Z_2 \cup \{f, t\}$ is a 2-separating set for M; a contradiction. □

4. The main result

In this section, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. First, we prove the "only if" part. If M is non-binary, then by Theorem 1.2 we may conclude that $M \cong U_{2,4}$. Therefore suppose M is binary and non-regular. Assume that M is binary non-regular and $|R(M)| \geq 2$. If M is an internally 4-connected matroid, then by Corollary 1.4, M is isomorphic to F_7 or F_7^*.

Thus we may assume that M is not internally 4-connected. By Lemma 3.3, S_8 is the unique matroid having all non-trivial 3-separations induced by the union of a triangle and a triad of some undesired fan. The result follows in this case. Therefore, we can assume that M has a 3-separation such that none of its sets is the union of a triangle and a triad in a undesired fan, say $\{X_1, X_2\}$. By Theorem 1.5 there are 3-connected matroids (up to parallel elements with the common triangle) M_1 and M_2 such that M is the 3-sum of M_1 and M_2 and, for $i \in \{1, 2\}$, $E(M_i) = X_i \cup T$. By definition, T is the common triangle between M_1 and M_2. By Lemma 2.3 we may assume that M_1 is non-regular and M_2 is regular. Moreover, $R(M) \subseteq X_1$. We may assume that M_1 is also 3-connected (the elements in parallel with elements of T, if they exist, are in M_2). By Lemmas 2.1 and 2.2, T does not span any element of $R(M)$ in M_1 or M_1^*. Thus by induction we have three possibilities.

First, suppose M_1 is isomorphic to F_7 or S_8. The result follows because M is the 3-sum of a matroid isomorphic to F_7 or S_8 (that is M_1) with a regular matroid (that is M_2).

Second, suppose M_1 is the 3-sum of matroids N_1 and N_2 along a triangle T' such that $R(M) \subseteq E(N_1)$; T' does not span any element of $R(M)$ in N_1; N_1 is isomorphic to F_7 or S_8 and N_2 is regular (We may assume that $T' \cap E(M_2) = \emptyset$.). If $|E(N_2) \cap T| \geq 2$, then $T \subseteq E(N_2)$ because an element of $E(N_1) - E(N_2)$ spanned by $E(N_2) - E(N_1)$ in M_1 must be in parallel with some element of T' in N_1. In this subcase, M is the 3-sum of N_1 and the regular matroid obtained by doing the 3-sum of N_2 and M_2 along the triangle T. The result follows in this case. Thus we may assume that $|E(N_2) \cap T| \leq 1$. As any two triangles of N_1 meet (recall that N_1 is isomorphic to F_7 or S_8), it follows that $E(N_2) \cap T = \{t\}$. Thus t is in parallel with an element t' of T' in N_2. Let N_1' be the matroid obtained from N_1 by adding t in parallel with t'. Note that T is a triangle of N_1'. Thus N_1' is isomorphic to F_7^p or S_8^p. Moreover, M is the 3-sum of N_1' with $N_2 \setminus t$ and M_2. The result also follows in this case.

Third, suppose there are matroids N, N_1, and N_2 such that:

(1) M_1 is the 3-sum of N, N_1 and N_2;
(2) N has elements t_1 and t_2 in parallel;
(3) $N \setminus t_1$ is isomorphic to F_7 or S_8;
(4) $E(N_1)$ and $E(N_2)$ are disjoint;
(5) $T_i = E(N) \cap E(N_i)$ is a triangle in both N and N_i, for both $i \in \{1, 2\}$;
(6) $t_i \in T_i$, for both $i \in \{1, 2\}$;
(7) N_1 and N_2 are regular and 3-connected (up to some parallel elements with elements of T_1 and T_2 respectively);
(8) $(T_1 \cup T_2) \cap E(M_2) = \emptyset$.

We begin by showing that $|E(N_i) \cap T| \leq 1$, for both $i \in \{1, 2\}$. If $|E(N_i) \cap T| \geq 2$, say $i = 2$, then $E(N_2) - T_2$ spans T in M_1. As t_1 and t_2 are the only elements of N in parallel, it follows that $T \subseteq E(N_2) - T_2$, otherwise the unique element belonging to $E(N_2) - T_2$ would be in parallel in N with some element of T_2 and this element is not t_1. Hence M is the 3-sum of N, N_1 and N_2', where N_2' is the 3-sum of N_2 and M_2 along T. The result follows, by induction. Thus we may assume that $|E(N_i) \cap T| \leq 1$, for both $i \in \{1, 2\}$. Moreover, when $|E(N_i) \cap T| = 1$, say $E(N_i) \cap T = \{a_i\}$, a_i is in parallel with some element $a_i' \in T_i$ in N_i. If $A_i = \{a_i\}$, when this happens, and $A_i = \emptyset$ otherwise, then M_1 is the 3-sum or $N' \setminus [\{a_1, a_2\} - (A_1 \cup A_2)]$ with $N_1 \setminus A_1$ and $N_2 \setminus A_2$, where N' is obtained from N by adding, for both $i \in \{1, 2\}$, a_i in parallel with a_i'. As T does not span any element of $R(M)$ in N', by Lemma 2.1, and

$|R(M)| \geq 2$, it follows that T spans T_1 or T_2, say T_2. That is, each element of T is in parallel with some element of T_2 in N'. We can transfer these elements for N_2 and we arrive at the previous case.

Finally, to see the "if" part, we use Lemmas 2.5 and 3.2 to reduce the S_8 case to the F_7 case in the 3-sums. The F_7 case is easy to verify. □

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