A Characterization of graphs $G$ with $G \cong K^2(G)$

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Abstract

A graph $G$ is called a D-graph if for every set of cliques of $G$ whose pairwise intersections are nonempty there is a vertex of $G$ common to all the cliques of the set. A D-graph $G$ is called a D$_1$-graph if it has the T$_1$ property: for any two distinct vertices $x$ and $y$ of $G$, there exist cliques $C$ and $D$ of $G$ such that $x \in C$ but $y \notin C$ and $y \in D$ but $x \notin D$.

Lim proved that if $G$ is a D$_1$-graph, then $G \cong K^2(G)$. Motivated by this result of Lim, we ask the following question:

Can one characterize those graphs $G$ with $G \cong K^2(G)$?

In this paper, we prove that in the class of D-graphs,

$G \cong K^2(G)$ if and only if $G$ has the T$_1$ property.

1. Introduction

All graphs considered in this paper are simple, undirected, connected, finite, loopless and without multiple edges. Undefined terms and notations can be found in [1] or [4].

A graph $G$ is called a D-graph if for every set of cliques (maximal complete subgraphs) of $G$ whose pairwise intersections are nonempty there is a vertex of $G$ common to all cliques of the set. D-graphs have been studied by Escalante [2], Hamelink [3] and others.

A D-graph $G$ is called a D$_0$-graph if it has the T$_0$ property: for any two distinct vertices $x$ and $y$ of $G$, there exists a clique $C$ of $G$ such that $x \in C$ but $y \notin C$ or $y \in C$ but $x \notin C$.

A D-graph $G$ is called a D$_1$-graph if it has the T$_1$ property: for any two distinct vertices $x$ and $y$ of $G$, there exist cliques $C$ and $D$ of $G$ such that $x \in C$ but $y \notin C$ and $y \in D$ but $x \notin D$.

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From the definitions, it is easily seen that a $D_1$-graph is a $D_0$-graph but a $D_0$-graph is not necessarily a $D_1$-graph. Furthermore, every vertex of a $D_1$-graph has degree at least two.

In [6, Theorem 2.1], Lim proved that if $G$ is a $D_1$-graph, then

$$G \cong K^2(G).$$

We remark that our definition of the $T_1$ property is slightly different from that of [6]. However, the converse is not true as can be seen from the example of a non-$D_1$-graph $G$ (Fig. 1) satisfying $G \cong K^2(G)$ [2].

The cliques of $G$ are given as follows:

- $C_1 = \{1, 2, 7\}$
- $C_2 = \{3, 4, 9\}$
- $C_3 = \{4, 5, 10\}$
- $C_4 = \{5, 6, 11\}$
- $C_5 = \{1, 6, 12\}$
- $C_6 = \{1, 7, 12, 13\}$
- $C_7 = \{2, 7, 8, 14\}$
- $C_8 = \{7, 13, 14\}$
- $C_9 = \{4, 9, 10, 16\}$
- $C_{10} = \{9, 15, 16\}$
- $C_{11} = \{5, 10, 11, 17\}$
- $C_{12} = \{10, 16, 17\}$
- $C_{13} = \{6, 11, 12, 18\}$
- $C_{14} = \{12, 13, 18\}$
- $C_{15} = \{11, 17, 18\}$
- $C_{16} = \{13, 19\}$
- $C_{17} = \{14, 20\}$
- $C_{18} = \{15, 21\}$
- $C_{19} = \{16, 22\}$
- $C_{20} = \{19, 22\}$
- $C_{21} = \{17, 23\}$
- $C_{22} = \{20, 23\}$
- $C_{23} = \{18, 24\}$
- $C_{24} = \{21, 24\}$
- $C_{25} = \{2, 3, 8, 25\}$
- $C_{26} = \{2, 8, 14, 25\}$
- $C_{27} = \{3, 8, 9, 15, 25\}$
- $C_{28} = \{8, 14, 15, 25\}$

$C_1, C_7, C_8$ and $C_{26}$ are pairwise nondisjoint cliques of $G$. However, $C_1 \cap C_7 \cap C_8 \cap C_{26} = \emptyset$. Hence, $G$ is not a $D$-graph.

From the above list of cliques of $G$, we see that every clique containing the vertex 25 also contains the vertex 8. Hence, there does not exist a clique containing vertex 25 but not vertex 8. Therefore, $G$ does not have the $T_1$ property. However, $G \cong K^2(G)$.

Motivated by [6, Theorem 2.1], we ask the following question:

Can one characterize those graphs $G$ with $G \cong K^2(G)$?

In what follows, among other things, we prove that in the class of $D$-graphs, $G \cong K^2(G)$ if and only if $G$ has the $T_1$ property.
2. The main theorem

The main purpose of this paper is to prove the following theorem:

**Theorem 2.1.** Let $G$ be a $D$-graph. Then $G \cong K^2(G)$ if and only if $G$ has the $T_1$ property.

Before we prove Theorem 2.1, we shall first prove some results which are necessary for the proof of this theorem.

**Theorem 2.2.** Let $G$ be a graph and $x \in G$. For all $y \in \bigcap \{C : C \subseteq K(x)\}$, the following conditions are equivalent:

(a) $\overline{N(x)} = \overline{N(y)}$,
(b) $\overline{K(x)} = \overline{K(y)}$,

where $N(x) = \{x\} \cup \{y \in G : (x, y) \in E(G)\}$ and $K(x) = \{C \subseteq K(G) : x \in C\}$.
Proof. Let \( y \in \bigcap \{ C : C \in K(x) \} \).

(a) \( \Rightarrow \) (b): If \( y = x \), the result is obvious. Suppose that \( y \neq x \). Then either \( K(x) \subseteq K(y) \) or \( K(x) = K(y) \). If \( K(x) \subseteq K(y) \), then there exists \( D \in K(y) \) with \( D \notin K(x) \), that is, \( y \in D \) but \( x \notin D \). It follows that \( \overline{N(x)} \neq \overline{N(y)} \), which is a contradiction. Hence, \( K(x) = K(y) \).

(b) \( \Rightarrow \) (a): Suppose on the contrary that \( \overline{N(x)} \neq \overline{N(y)} \).

Case 1: Suppose that there exists \( w \) such that \( w \in \overline{N(x)} \) but \( w \notin \overline{N(y)} \). Then there exists \( D \in K(G) \) with \( x, w \in D \) but \( y \notin D \). In other words, \( D \in K(x) \) but \( D \notin K(y) \). Hence \( K(x) \neq K(y) \), which is a contradiction.

Case 2: Suppose that there exists \( w \) such that \( w \in \overline{N(y)} \) but \( w \notin \overline{N(x)} \). Then there exists \( D \in K(G) \) with \( y, w \in D \) but \( x \notin D \). In other words, \( D \in K(y) \) but \( D \notin K(x) \). Hence, \( K(x) \neq K(y) \), which is a contradiction.

Cases 1 and 2 imply that \( N(x) = N(y) \). \( \square \)

Lemma 2.3. Let \( G \) be a graph and \( x \in G \). If \( K(x) \in K^2(G) \), then \( \overline{N(x)} = \overline{N(y)} \) for all \( y \in \bigcap \{ C : C \in K(x) \} \).

Proof. Let \( K(x) \in K^2(G) \). If \( |\bigcap \{ C : C \in K(x) \}| = 1 \), then the result is obvious. Suppose that there exists \( y \in \bigcap \{ C : C \in K(x) \} \), \( y \neq x \). Suppose on the contrary that \( \overline{N(x)} \neq \overline{N(y)} \).

Case 1: Suppose that there exists \( w \) such that \( w \in \overline{N(x)} \) but \( w \notin \overline{N(y)} \). Then there exists \( D \in K(G) \) with \( x, w \in D \) but \( y \notin D \), which is impossible since \( y \in \bigcap \{ C : C \in K(x) \} \).

Case 2: Suppose that there exists \( w \) such that \( w \in \overline{N(y)} \) but \( w \notin \overline{N(x)} \). Then there exists \( D \in K(G) \) with \( y, w \in D \) but \( x \notin D \). In other words, \( D \in K(y) \) but \( D \notin K(x) \). It follows that \( K(x) \not\subseteq K(y) \). Hence, \( K(x) \notin K^2(G) \), which is a contradiction.

Cases 1 and 2 imply that \( \overline{N(x)} = \overline{N(y)} \) for all \( y \in \bigcap \{ C : C \in K(x) \} \). \( \square \)

Remark: The converse of Lemma 2.3 is not true as can be seen from the graph \( G \) as shown in Fig. 2.

Clique of \( G \):

\[
C_1 = \{ a, b, c \}, \\
C_2 = \{ b, d, e \}, \\
C_3 = \{ b, c, e \}, \\
C_4 = \{ c, e, f \}.
\]

Clique of \( K(G) \) (Fig. 2):

\[
C_1^* = \{ C_1, C_2, C_3, C_4 \}.
\]

Furthermore,

\[
K(a) = \{ C_1 \}, \\
K(b) = \{ C_1, C_2, C_3 \}, \\
K(c) = \{ C_1, C_3, C_4 \}, \\
K(d) = \{ C_2 \}, \\
K(e) = \{ C_2, C_3, C_4 \}, \\
K(f) = \{ C_4 \}.
\]
For every vertex $x \in G$, $\overline{N(x)} = \overline{N(y)}$ for all $y \in \bigcap\{C : C \in K(x)\}$. However, $K(x) \notin K^2(G)$.

**Lemma 2.4.** Let $G$ be a D-graph and $x \in G$. If $\overline{N(x)} = \overline{N(y)}$ for all $y \in \bigcap\{C : C \in K(x)\}$, then $K(x) \in K^2(G)$.

**Proof.** Suppose on the contrary that $K(x) \notin K^2(G)$ and let $\{C : C \in K(x)\} = \{C_i : i \in I\}$. Note that $K(x)$ is necessarily a complete subgraph of $K(G)$, so if it is not a clique, there exists some $D \in K(G)$ such that

(i) $D \cap C_i \neq \emptyset$ for all $i \in I$,
(ii) $x \notin D$.

Let $S = \bigcap\{C_i : i \in I\}$. Since $G$ is a D-graph, $S \cap D \neq \emptyset$. Let $y \in S \cap D$. Then $y \in D$ but on the other hand, from (ii) above, $x \notin D$. Hence, $\overline{N(x)} \neq \overline{N(y)}$, which is a contradiction. Hence, the result follows. \qed

Combining Lemmas 2.3 and 2.4, we obtain the following result.

**Theorem 2.5.** Let $G$ be a D-graph and $x \in G$. $K(x) \in K^2(G)$ if and only if $\overline{N(x)} = \overline{N(y)}$ for all $y \in \bigcap\{C : C \in K(x)\}$.

Combining Theorems 2.2 and 2.5, we obtain Theorem 2.6 which is stated as follows:

**Theorem 2.6.** Let $G$ be a D-graph and $x \in G$. Then the following conditions are equivalent:

(a) $K(x) \in K^2(G)$,
(b) $\overline{N(x)} = \overline{N(y)}$ for all $y \in \bigcap\{C : C \in K(x)\}$,
(c) $K(x) = K(y)$ for all $y \in \bigcap\{C : C \in K(x)\}$. 
Corollary 2.7. Let $G$ be a D-graph and $x \in G$. If $|\bigcap \{C: C \in K(x)\}| = 1$, then $K(x) \in K^2(G)$.

Proof. The result follows immediately from Theorem 2.6. □

Lemma 2.8. Let $G$ be a graph. Then $G$ has the $T_1$ property if and only if $|\bigcap \{C: C \in K(x)\}| = 1$ for all $x$ in $G$.

Proof. Necessity: Suppose that $y \in \bigcap \{C: C \in K(x)\}$ for some $y \in G, y \neq x$. Then there does not exist $D \in K(G)$ such that $x \in D$ but $y \notin D$. In other words, $G$ does not have the $T_1$ property.

Sufficiency: Let $x$ and $y$ be two distinct vertices of $G$. Since $\bigcap \{C: C \in K(x)\} = \{x\}$, there exists $C \in K(G)$ such that $x \in C$ but $y \notin C$. Similarly, there exists $D \in K(G)$ such that $y \in D$ but $x \notin D$. Hence, $G$ has the $T_1$ property. □

Corollary 2.9. Let $G$ be a D-graph. If $G$ has the $T_1$ property, then $K(x) \in K^2(G)$ for all $x \in G$.

Proof. If $G$ has the $T_1$ property, then from Lemma 2.8,

$$|\bigcap \{C: C \in K(x)\}| = 1 \quad \text{for all } x \in G.$$ 

Furthermore, since $G$ is a D-graph, from Corollary 2.7, $K(x) \in K^2(G)$ for all $x \in G$. □

Remark. The converse of Corollaries 2.7 and 2.9 is not true as can be seen from the graph $G$ as shown in Fig. 3.
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Cliques of $G$:

$C_1 = \{a, b\}, \quad C_2 = \{b, c, d\},$

$C_3 = \{c, d, e\}, \quad C_4 = \{a, e\}.$

Cliques of $K(G)$ (Fig. 3):

$C^* = \{C_1, C_2\}, \quad C^* = \{C_2, C_3\},$

$C^* = \{C_3, C_4\}, \quad C^* = \{C_1, C_4\}.$

Moreover

$K(a) = \{C_1, C_4\} = C^*,$ \quad $K(b) = \{C_1, C_2\} = C^*,$

$K(c) = \{C_2, C_3\} = C^*,$ \quad $K(d) = \{C_2, C_3\} = C^*,$

$K(e) = \{C_3, C_4\} = C^*.$

$G$ is a D-graph with $K(x) \in K^2(G)$ for all $x \in G$. However, $C_2 \cap C_3 = \{c, d\}$, that is,

$|\bigcap \{C : C \in K(c)\}| = |\bigcap \{C : C \in K(d)\}| = 2.$

Furthermore, $G$ does not have the $T_1$ property.

**Theorem 2.10.** Let $G$ be a D-graph. Then the following conditions are equivalent:

(a) $G$ has the $T_1$ property.

(b) $G$ has the $T_0$ property and $K(x) \in K^2(G)$ for all $x \in G$.

(c) $K(x) \neq K(y)$ and $K(x) \in K^2(G)$ for all $x \neq y$, $x, y \in G$.

**Proof.** Let $G$ be a D-graph.

(a) $\Rightarrow$ (b): Since $G$ has the $T_1$ property, $G$ has the $T_0$ property as well. Furthermore, from Corollary 2.9, $K(x) \in K^2(G)$ for all $x \in G$.

(b) $\Rightarrow$ (c): Since $G$ has the $T_0$ property, from [5, Theorem 2.3], $K(x) \neq K(y)$ for all $x \neq y$, $x, y \in G$.

(c) $\Rightarrow$ (a): Let $x$ and $y$ be any two distinct vertices of $G$. We shall prove that $G$ has the $T_1$ property. If $x$ and $y$ are nonadjacent vertices, the result is obvious. Suppose $x$ and $y$ are adjacent vertices. Since $K(x) \neq K(y)$ and $K(x) \in K^2(G)$ for all $x \neq y$, $x, y \in G$, $K(x) \notin K(y)$ and $K(y) \notin K(x)$. In other words, there exist cliques $C, D \in K(G)$ such that $C \in K(x) \setminus K(y), \quad D \in K(y) \setminus K(x).$ Hence, $x \in C$ but $y \notin C$ and $y \in D$ but $x \notin D$. Hence, $G$ has the $T_1$ property. \[\square\]

**Remark.** The two conditions below do not imply each other:

(a) $G$ is a $D_0$-graph,

(b) $K(x) \in K^2(G)$ for all $x \in G$. 
Fig. 4.

For example, consider the graphs shown in Fig. 4. $G$ satisfies condition (b) but not condition (a). On the other hand, $H$ satisfies condition (a) but not condition (b).

We are now in a position to prove Theorem 2.1.

**Proof of Theorem 2.1.** Necessity: Since $G$ is a D-graph, from [6, Lemma 2.2], for any clique $A$ of $K(G)$, there is an $x \in G$ such that $A = K(x)$. In other words, $K^2(G) \subseteq \{K(x): x \in G\}$. Furthermore, since $G \cong K^2(G)$, there are exactly $n$ distinct cliques in $K(G)$, where $n$ is the number of vertices in $G$. Hence,

$$n = |K^2(G)| \leq |\{K(x): x \in G\}| \leq n.$$ 

It follows that

$$|K^2(G)| = |\{K(x): x \in G\}| = n$$

and hence $K(x) \in K^2(G)$ with $K(x) \neq K(y)$ for all $x \neq y, x, y \in G$. From Theorem 2.10, $G$ has the $T_1$ property.

Sufficiency: Let $G$ be a D-graph with the $T_1$ property, Claim that $G \cong K^2(G)$. To see this, define a function

$$f: V(G) \to V(K^2(G))$$

by $f(x) = K(x)$. We shall show that $f$ is a graph isomorphism. From Theorem 2.10, $K(x) \in K^2(G)$ for all $x$ and $K(x) \neq K(y)$ for all $x \neq y, x, y \in G$. Hence, $f$ is well-defined and one-to-one. From [6, Lemma 2.2], any clique of $K(G)$ is of the form $K(x)$ for some $x \in G$ and hence $f$ is onto. Furthermore, $x$ and $y$ are adjacent if and only if $K(x) \cap K(y) \neq \emptyset$. Hence, $f$ preserves adjacency. Therefore, $f$ is an isomorphism and $G \cong K^2(G)$. 

Remark. For Theorem 2.1 to be true, the condition that $G$ being a D-graph cannot be dropped as can be seen from the following examples.
The graph $G$ shown in Fig. 5 is a non-D-graph satisfying the $T_1$ property but $G \ncong K^2(G)$.

Moreover, the graph $G$ in Fig. 1 is a non-D-graph satisfying $G \cong K^2(G)$ but not the $T_1$ property.

Combining Lemma 2.8 and Theorems 2.10 and 2.1, we have the following result:

**Theorem 2.11.** Let $G$ be a D-graph. The following conditions are equivalent:

(a) $G \cong K^2(G)$,

(b) $G$ has the $T_1$ property,

(c) $G$ has the $T_0$ property and $K(x) \in K^2(G)$ for all $x \in G$,

(d) $K(x) \neq K(y)$ and $K(x) \in K^2(G)$ for all $x \neq y, x, y \in G$,

(e) $|\bigcap \{C: C \in K(x)\}| = 1$ for all $x \in G$.

**References**


