A numerical technique for solving a class of fractional variational problems

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ABSTRACT

This paper presents a numerical method for solving a class of fractional variational problems (FVPs) with multiple dependent variables, multi order fractional derivatives and a group of boundary conditions. The fractional derivative in the problem is in the Caputo sense. In the presented method, the given optimization problem reduces to a system of algebraic equations using polynomial basis functions. An approximate solution for the FVP is achieved by solving the system. The choice of polynomial basis functions provides the method with such a flexibility that initial and boundary conditions can be easily imposed. We extensively discuss the convergence of the method and finally present illustrative examples to demonstrate validity and applicability of the new technique.

1. Introduction

Fractional order dynamics appear in several problems in science and engineering such as, viscoelasticity [1,2], bioengineering [3], dynamics of interfaces between nanoparticles and substrates [4], etc. It is also shown that materials with memory and hereditary effects and dynamical processes including gas diffusion and heat conduction in fractal porous media can be modeled by fractional order models better than integer models [5]. Interested reader in fractional calculus and fractional differential equations can refer to [6–8]. Although the calculus of variation has been under development for years, the fractional variational theory is a new area in mathematics. A FVP can be defined with respect to different definitions of fractional derivatives but the most important types are Riemann–Liouville and Caputo derivatives. General conditions of optimality have been developed for FVPs. For instance in [9] the author has achieved the necessary conditions of optimization for FVPs with Riemann–Liouville derivatives. In [10] the authors present necessary and sufficient conditions of optimization for a class of FVPs with respect to the Caputo fractional derivative. There also exist conditions of optimization for FVPs with functionals containing both fractional derivatives and integrals [11] or FVPs with other definitions of fractional derivatives [12]. In [13] the author discussed the general form of FVPs. The author also claimed that the derived equations are the general form of Euler–Lagrange equations for problems with fractional Riemann–Liouville, Caputo, Riesz–Caputo and Riesz–Riemann–Liouville derivatives. There also exist other generalizations of Euler–Lagrange equations which have free boundary conditions [14–16]. Except some special cases [17], it is hard to find an exact solution of FVPs, specially when the problem has boundary conditions which should be satisfied. To remove this difficulty, we present our numerical technique in this article. There also exist other numerical methods for solving FVPs. For instance, the finite element method in [18] and fractional variational integrator in [19] are developed and applied for a class of FVPs. In [20] a numerical scheme is proposed for solving a class of parametric FVPs. In this paper, the variational problem contains a group of boundary conditions while the problems considered in [18–20] are much simpler in the boundary conditions. The approximate solutions, achieved in [18–20] are discrete while the approximate

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solutions in our method are continuous smooth functions and satisfy all boundary conditions. The numerical examples in Section 5 show that satisfactory accuracy can be obtained using only a few basis functions.

In the current paper we focus on the following FVP

\[
\begin{align*}
\text{Min}(\text{Max}) \quad & f[y_1, \ldots, y_m] = \int_{t_0}^{t_1} F(t, y_1, \ldots, y_m, \sum_{i=1}^{n} \alpha_i D^\alpha_i y_1, \ldots, \sum_{i=1}^{n} \alpha_i D^\alpha_i y_m) dt \\
y_j(t_0) &= y_{j0}, \quad y_j(t_1) = y_{j1}, \quad 0 \leq j \leq m.
\end{align*}
\]

(1)

where \(1 \leq j \leq m, i - 1 < \alpha_i \leq i, i = 1, \ldots, n\), and the function \(F\) is continuously differentiable with respect to all its arguments. The fractional derivatives are defined in the Caputo sense,

\[
\begin{align*}
\sum_{i=1}^{n} \alpha_i D^\alpha_i y(t) &= \frac{1}{\Gamma(n - \alpha)} \int_{t_0}^{t} (t - \tau)^{n - \alpha - 1} \frac{d^n}{d\tau^n} y(\tau) d\tau, \quad n - 1 < \alpha < n, \\
&= \frac{d^n}{dt^n} y(t), \quad \alpha = n.
\end{align*}
\]

(3)

For the above problem, the Euler–Lagrange equations are as follows [9,10,15]

\[
\frac{\partial F}{\partial y_i} + \sum_{j=1}^{n} \alpha_j D^\alpha_j \frac{\partial F}{\partial D^\alpha_j y_i} = 0, \quad 1 \leq i \leq m,
\]

(4)

which should satisfy boundary conditions (2). Here \(D^\alpha_j\) denotes the right Riemann–Liouville fractional derivative. The optimal solution of the problem (1)–(2) should satisfy the system (4) and boundary conditions (2) [9,10,15]. So to find extremals of the problem (1)–(2) we should solve the system (4) with the boundary conditions (2). Since the system (4) combines the right Riemann–Liouville operator with the left Caputo differential operator, it is hard to find the exact solution of the system. Even if we try to solve the system (4) approximately we will face with complexity in computations. To overcome these difficulties, we solve directly the problem without using the Euler–Lagrange equations (4). Our tool for this aim is polynomial basis.

The used method is to reduce the given variational problem to the problem of finding an optimal solution of a real value function. Unknown functions are expanded with polynomial basis and unknown coefficients. Then an algebraic function in terms of unknown coefficients is achieved which should be optimized with respect to its variables. We study the convergence of our method in Section 4 and present numerical examples to illustrate the applicability of the new approach.

This paper is organized as follows. In Section 2 we present some preliminaries needed for our subsequent developments. Section 3 is devoted to the numerical method of solving the FVP. In Section 4 we discuss the convergence of the method and finally in Section 5 we report our numerical findings and demonstrate the accuracy of our numerical method. Section 6 consists of a brief summary.

2. Some preliminaries

Without loss of generality, we consider \(t_0 = 0, t_1 = 1\) and \(t \in [0, 1]\) in problem (1)–(2). Here we state a very important theorem in approximation theory which plays a key role in our discussion in Section 4. Beforehand we state the following definition and lemmas.

**Definition 1.** Define the modulus of continuity \(\omega(f, \delta)\) of a general function \(f\) on \([a, b]\) by

\[
\omega(f, \delta) = \sup_{x, y \in [a, b], |x - y| \leq \delta} |f(x) - f(y)|.
\]

**Lemma 1.** \(f(x)\) is continuous on \([a, b]\) if and only if \(\lim_{\delta \to 0} \omega(f, \delta) = 0\).

**Proof** ([21]). \(\square\)

**Theorem 1.** If \(f(x)\) is bounded on \([0, 1]\), then

\[
\|f - p(f, m)\|_\infty \leq \frac{3}{2} \omega \left( f, \frac{1}{\sqrt{m}} \right)
\]
where
\[ p(f, m) = \sum_{k=0}^{m} f \left( \frac{k}{m} \right) B_{k,m}, \]
\[ B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad 0 \leq i \leq m, \]
\[ \binom{m}{i} = \frac{m!}{i!(m-i)!}, \]
and \( \|f\|_\infty = \sup \{|f(x)| \mid x \in [0, 1]\} \). Let \( f \) also satisfies the Lipschitz condition of order \( \alpha \) on \([0, 1]\), then
\[ \|f - p(f, m)\|_\infty \leq \frac{3}{2} km^{-\frac{\alpha}{2}}, \]
where \( k \) is the Lipschitz constant.

**Proof** ([21]). Since in our subsequent development we use Legendre polynomials for approximating functions, here we state some basic properties of these polynomials. Of course it is possible to use other types of polynomials for approximations such as Taylor, Bernstein, etc.

The Legendre polynomials are orthogonal polynomials on the interval \([-1, 1]\) and can be determined with the following recurrence formula:
\[ L_{i+1}(y) = \frac{2i+1}{i+1} y L_i(y) - \frac{i}{i+1} L_{i-1}(y), \quad i = 1, 2, \ldots \]
where \( L_0(y) = 1 \) and \( L_1(y) = y \). By the change of variable \( y = 2t - 1 \) we will have the well-known shifted Legendre polynomials. Let \( p_m(t) \) be the shifted Legendre polynomials of order \( m \) which are defined on the interval \([0, 1]\) and can be determined with the following recurrence formula
\[ p_{m+1}(t) = \frac{2m+1}{m+1} (2t-1)p_m(t) - \frac{m}{m+1} p_{m-1}(t), \quad m = 1, 2, 3, \ldots \]
\[ p_0(t) = 1, \quad p_1(t) = 2t - 1. \]

We also have the analytical form of the shifted Legendre polynomial of degree \( i \), \( p_i(t) \) as follows
\[ p_i(t) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!t^k}{(i-k)!(k)!}, \quad i = 0, 1, 2, \ldots. \]

In the following lemma we prove a simple fact which is used in approximating the method presented in Section 3. □

**Lemma 2.** Let \( p(t) \) be a polynomial which satisfies the following conditions
\[ p^{(l)}(0) = y_0^l, \quad p^{(l)}(1) = y_1^l, \quad 0 \leq l \leq n, \]
then \( p(t) \) has the following form
\[ p(t) = \sum_{j=0}^{k} c_j t^{n+1}(t-1)^{n+1} p_j(t) + w(t), \]
where, \( k \in \mathbb{Z}^+, c_j \in \mathbb{R} \) and \( w(t) \) is the Hermite interpolating polynomial of degree at most \( 2n + 1 \) which satisfies the above conditions.

**Proof.** Obviously we have
\[ p(t) = p(t) - w(t) + w(t), \]
where
\[ p^{(l)}(0) - w^{(l)}(0) = 0, \]
\[ p^{(l)}(1) - w^{(l)}(1) = 0, \]
\[ 0 \leq l \leq n. \]

So we have
\[ p(t) - w(t) = t^{n+1}(t-1)^{n+1}s(t), \]
where
\[ s(t) = \sum_{j=0}^{k} c_j p_j(t). \]

### 3. Solving the fractional variational problem

Consider the following fractional variational problem

\[
\text{Min}(\text{Max}) \quad J[y_1, \ldots, y_m] = \int_0^1 F(t, y_1, \ldots, y_m, \frac{\partial}{\partial t} y_1, \ldots, \frac{\partial}{\partial t} y_m, \ldots) \, dt
\]

with boundary conditions (6). The method presented here is based on the Ritz method. We refer the interested reader to [22] for more information.

#### 4. On the convergence of the method

In this section we discuss the convergence of the method presented in Section 3. We will show that the approximate value of minimum (maximum) tends to the exact value with the increase of \( k \) in (7). We show this fact in Theorem 3. Now we define our function space and provide some needed lemmas.

Consider the Banach space \( C^n[0, 1], \| \cdot \|_n \) as follows

\[
C^n[0, 1] = \{ f(t) : f^{(n)}(t) \in C[0, 1] \},
\]

\[
\| f \|_n = \| f \|_{\infty} + \| f' \|_{\infty} + \cdots + \| f^{(n)} \|_{\infty}.
\]
Now let
\[ E[0, 1] = \{ f(t) \in C^n[0, 1] \mid f^{(0)}(0) = f_j^0, f^{(1)}(1) = f_j^1, j = 0, 1, \ldots, n - 1 \}, \]
where \( f_j^0, f_j^1 \) are given constant values. Now we state a lemma which plays an important role in our discussion. The lemma shows that polynomial functions of the metric space \( E[0, 1] \) are dense in that space.

**Lemma 3.** Let \( f(t) \in E[0, 1] \). There exists a sequence of polynomial functions \( \{ s(t) \}_{t \in N} \subset E[0, 1] \) such that \( s_t \to f \) with respect to \( \| \cdot \|_n \).

**Proof.** Let \( f(t) \in E[0, 1] \), then \( f^{(n)}(t) \in C[0, 1] \). According to Lemma 1 and Theorem 1 there exists a sequence of polynomials \( \{ h_t \}_{t \in N} \) such that \( h_t \to f^{(n)} \) with respect to \( \| \cdot \|_\infty \). Now consider
\[
\begin{align*}
    f(t) &= \sum_{j=0}^{n-1} \frac{t^j}{j!} f_j^0 + \int_0^t \cdots \int_0^t f^{(n)}(s) \, ds \cdots ds, \\
    f(t) &= \sum_{j=0}^{n-1} \frac{(t-1)^j}{j!} f_j^1 + \int_1^t \cdots \int_1^t f^{(n)}(s) \, ds \cdots ds.
\end{align*}
\]
Let
\[
\begin{align*}
    p_l(t) := \sum_{j=0}^{n-1} \frac{t^j}{j!} f_j^0 + \int_0^t \cdots \int_0^t h_l(s) \, ds \cdots ds, \\
    q_l(t) := \sum_{j=0}^{n-1} \frac{(t-1)^j}{j!} f_j^1 + \int_1^t \cdots \int_1^t h_l(s) \, ds \cdots ds.
\end{align*}
\]
Here \( p_l(t) \) and \( q_l(t) \) are two polynomials with degree depend on \( l \) and the following properties
\[
\begin{align*}
    p_l^{(j)}(0) &= f_j^0, & 0 &\leq j &\leq n - 1, \quad (10) \\
    q_l^{(j)}(1) &= f_j^1, & 0 &\leq j &\leq n - 1, \quad (11) \\
    |p_l^{(j)}(t) - f^{(j)}(t)| &\leq \| h_l - f^{(n)} \|_\infty, & 0 &\leq j &\leq n, \\
    |q_l^{(j)}(t) - f^{(j)}(t)| &\leq \| h_l - f^{(n)} \|_\infty, & 0 &\leq j &\leq n.
\end{align*}
\]
So far we have achieved two sequences of polynomials \( \{ p_l \} \) and \( \{ q_l \} \) such that \( p_l^{(j)} \to f^{(j)} \) and \( q_l^{(j)} \to f^{(j)} \) with respect to \( \| \cdot \|_\infty \) for \( 0 \leq j \leq n \); hence \( p_l \to f \) and \( q_l \to f \) with respect to \( \| \cdot \|_n \). Now consider polynomial \( H(t) \) with the following properties:
\[
\begin{align*}
    H^{(0)}(0) &= 0, & 0 &\leq j &\leq n - 1, \\
    H(1) &= 1, \\
    H^{(j)}(1) &= 0, & 1 &\leq j &\leq n - 1. 
\end{align*}
\]
According to the Hermite interpolation method we can achieve such a polynomial. Obviously we have
\[
f(t) = (1 - H(t))f(t) + H(t)f(t).
\]
Now consider the polynomial \( s_l(t) \) as follows
\[
s_l(t) := (1 - H(t))p_l(t) + H(t)q_l(t).
\]
We show in the rest of the proof that \( \{ s_l \} \) is the sequence which we look for. For this polynomial we have
\[
s_l^{(m)}(t) = \sum_{j=0}^{m} \binom{m}{j} [(1 - H)^{(j)}(t) p_l^{(m-j)}(t) + H^{(j)}(t) q_l^{(m-j)}(t)],
\]
\[
0 \leq m \leq n.
\]
Considering (10)-(12), we have
\[
\begin{align*}
    s_l^{(m)}(0) &= f_j^m, & 0 &\leq m &\leq n - 1, \\
    s_l^{(m)}(1) &= f_j^m, & 0 &\leq m &\leq n - 1.
\end{align*}
\]
so \( \{s_i(t)\}_{i \in \mathbb{N}} \subset E[0, 1]. \) On the other hand we have
\[
f^{(m)}(t) = \sum_{j=0}^{m} \binom{m}{j} [(1 - H)^{(j)}(t) f^{(m-j)}(t) + H^{(j)}(t) f^{(m-j)}(t)].
\]
and therefore
\[
|s_t^{(m)}(t) - f^{(m)}(t)| \leq \sum_{j=0}^{m} \binom{m}{j} \left[ \| (1 - H)^{(j)} \|_\infty \| p_t^{(m-j)} - f^{(m-j)} \|_\infty + \| H^{(j)} \|_\infty \| q_t^{(m-j)} - f^{(m-j)} \|_\infty \right]
\]
\[
\leq \| p_t - f \|_n \sum_{j=0}^{n} \binom{n}{j} \| (1 - H)^{(j)} \|_\infty + \| q_t - f \|_n \sum_{j=0}^{n} \binom{n}{j} \| H^{(j)} \|_\infty,
\]
\[0 \leq m \leq n.\]

So we have
\[
\| s_t^{(m)} - f^{(m)} \|_\infty \leq M_1 \| p_t - f \|_n + M_2 \| q_t - f \|_n, \quad 0 \leq m \leq n.
\]
Obviously for \( 0 \leq m \leq n, s_t^{(m)} \to f^{(m)} \) with norm \( \| \cdot \|_\infty \) and hence \( s_t \to f \) with respect to \( \| \cdot \|_n. \) \( \square \)

Consider Banach space \( (D^n_1[0, 1], \| \cdot \|) \) and metric subspace \( F_m[0, 1] \) of \( D^n_1[0, 1] \) as follows
\[
D^n_1[0, 1] = C^n[0, 1] \times \cdots \times C^n[0, 1],
\]
\[
F_m[0, 1] = E_1[0, 1] \times \cdots \times E_m[0, 1],
\]
where
\[
E_j[0, 1] = \{ f(t) \in C^n[0, 1] | f(0) = y_0, f(1) = y_1, f^{(i)}(0) = y_0^{(i)}, f^{(i)}(1) = y_1^{(i)}, i = 1, \ldots, n-1 \},
\]
and \( \| \cdot \| \) is the product norm which can be considered as follows
\[
\| (y_1, \ldots, y_m) \| = \sum_{j=1}^{m} \| y_j \|_n.
\]

Note that \( y_0, y_1, y_0^{(i)}, y_1^{(i)} \) are boundary conditions \( (6). \) Consider \( G^k_m[0, 1] \) as follows
\[
G^k_m[0, 1] = E_1[0, 1] \bigcap \{ (p_j^k)_{j=0}^k \} \times \cdots \times E_m[0, 1] \bigcap \{ (p_j^k)_{j=0}^k \},
\]
where \( \{ (p_j^k)_{j=0}^k \} \) is the Banach subspace of \( C^n[0, 1] \) generated by the Legendre polynomials of degree at most \( k. \) Of course \( G^k_m[0, 1] \) is a metric subspace of \( F_m[0, 1]. \)

Let \( y \in C^n[0, 1]. \) For the Caputo fractional derivative of order \( \alpha, n - 1 < \alpha < n \) we have \( C_0 D_t^\alpha y(t) \in C[0, 1] \) \( [6–8]. \) We also have
\[
C_0 D_t^\alpha y(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} y^{(n)}(s) ds,
\]
\[
\| C_0 D_t^\alpha y(t) \| \leq \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} \| y^{(n)}(s) \| ds
\]
\[
\leq \| y^{(n)} \|_\infty \int_0^t (t - s)^{n-\alpha-1} ds = \frac{\| y^{(n)} \|_\infty t^{n-\alpha}}{\Gamma(n - \alpha)(n - \alpha)} \leq \frac{\| y^{(n)} \|_\infty}{\Gamma(n - \alpha + 1)}.
\]
So
\[
\| C_0 D_t^\alpha y(t) \|_\infty \leq \frac{\| y^{(n)} \|_\infty}{\Gamma(n - \alpha + 1)}, \quad n - 1 < \alpha \leq n.
\]

Now consider the functional \( J \) in \( (5) \) as an operator \( J : (D^n_0[0, 1], \| \cdot \|) \to \mathbb{R}. \) Lemma 4 shows that the functional \( J \) is continuous on its domain. We use this important property later in Theorem 3 and state a theorem from real analysis which we need in the proof of Lemma 4.

**Theorem 2.** Let \( f \) be a continuous mapping of a compact metric space \( X \) into a metric space \( Y, \) then \( f \) is uniformly continuous.

**Proof** ([23]). \( \square \)
Lemma 4. The functional $J$ is continuous on the Banach space $(D^{\mu}_{m}[0, 1], \| \cdot \|)$. 

Proof. Let $(y^*_1, \ldots, y^*_m) \in D^{\mu}_{m}[0, 1]$, $\epsilon > 0$ is given. Consider $r > 0$ and 

$$I = [0, 1] \times \prod_{j=0}^{n} \prod_{i=1}^{m} [-L^j_i - r, L^j_i + r],$$

where

$$L^0_i = \|y^*_i\|_\infty, \quad L^j_i = \|C^0 D^j_i y^*_i\|_\infty, \quad 1 \leq j \leq n.$$ 

Obviously for $t \in [0, 1]$ we have

$$Y = (t, y^*_1, \ldots, y^*_m, \sum C^0 D^1 t y^*_1, \ldots, \sum C^0 D^m t y^*_m, \ldots, \sum C^0 D^{\mu} t y^*_1, \ldots, \sum C^0 D^{\mu} t y^*_m) \in I.$$ 

Let $\delta > 0$ and $\| (y_1, \ldots, y_m) - (y^*_1, \ldots, y^*_m) \| < \delta$; hence we have $\| y_j - y^*_j \|_n < \delta, 1 \leq j \leq m$, and according to (13)

$$\| C^0 D^j t y_j(t) - C^0 D^j t y^*_j(t) \|_\infty \leq \frac{1}{\Gamma(i - \alpha_i + 1)} \| y_j - y^*_j \|_n < \frac{\delta}{\Gamma(i - \alpha_i + 1)},$$

$$1 \leq i \leq n, 1 \leq j \leq m.$$ 

So for small enough value of $\delta$ we have

$$Y = (t, y_1, \ldots, y_m, C^0 D^1 t y_1, \ldots, C^0 D^m t y_m, \ldots, C^0 D^{\mu} t y_1, \ldots, C^0 D^{\mu} t y_m) \in I, \quad t \in [0, 1].$$

Since $F$ is continuous on $I$ and $I$ is a compact set, according to Theorem 2, $F$ is uniformly continuous on $I$. So if $\delta > 0$ be sufficiently small, then $|Y - Y^*| < \delta$ implies that $|F(Y) - F(Y^*)| < \epsilon$ and

$$\| J[y_1, \ldots, y_m] - J[y^*_1, \ldots, y^*_m] \| < \epsilon.$$ 

Now we can show the convergence of the approximating method.

Theorem 3. Let $\mu$ be the minimum (maximum) of the functional $J$ on the set $F_m[0, 1]$ and also let $\mu_k$ be the minimum (maximum) of the functional $J$ on $G_k^m[0, 1]$; then we have

$$\lim_{k \to \infty} \mu_k = \mu.$$ 

Proof. Here we prove the theorem in the minimum case. Of course for the maximum case the procedure is similar. For any given $\epsilon > 0$, let $(y^*_1, \ldots, y^*_m) \in F_m[0, 1]$ such that $J[y^*_1, \ldots, y^*_m] < \mu + \epsilon$ (such $(y^*_1, \ldots, y^*_m)$ exist by the properties of minimum). According to Lemma 4, $J$ is continuous on $(D^{\mu}_{m}[0, 1], \| \cdot \|)$ so we have

$$J[y_1, \ldots, y_m] < J[y^*_1, \ldots, y^*_m] + \epsilon;$$

provided that $\| (y_1, \ldots, y_m) - (y^*_1, \ldots, y^*_m) \| < \delta$. According to Lemma 3 for large enough value of $k$ there exist $(\eta^*_1, \ldots, \eta^*_m) \in C^k_m[0, 1]$ such that $\| (\eta^*_1, \ldots, \eta^*_m) - (y^*_1, \ldots, y^*_m) \| < \delta$. Moreover let $(y^*_1, \ldots, y^*_m)$ be the element of $C^0_m[0, 1]$ such that $J[y^*_1, \ldots, y^*_m] = \mu_k$, then using (14) we have

$$\mu \leq J[y^*_1, \ldots, y^*_m] \leq J[\eta^*_1, \ldots, \eta^*_m] < \mu + 2\epsilon.$$ 

Since the $\epsilon > 0$ is arbitrary, it follows that

$$\lim_{k \to \infty} \mu_k = \lim_{k \to \infty} J[y^*_1, \ldots, y^*_m] = \mu.$$ 

5. Illustrative test problems

In this section we apply the method presented in Section 3 to solve the following test examples.

5.1. Example 1

Consider the following problem

$$\text{Min} J[y] = \int_0^1 \left( \frac{1}{4} C^1_0 D^2_0 y + \frac{1}{4} C^1_0 D^2_1 y - g(t) \right)^2 dt,$$

where

$$g(t) = \frac{5}{2} \sqrt{\pi} \Gamma \left( \frac{3}{4} \right) t^{\frac{3}{4}} + \frac{15}{8} \sqrt{\pi} t^{\frac{3}{4}} \Gamma \left( \frac{13}{4} \right).$$
Fig. 1. Exact (−) and approximate solutions of $y(t)$, (***) for $k = 3$, (ooo) for $k = 5$ and (•••) for $k = 10$ for Example 1.

with the following boundary conditions

$$y(0) = 0, \quad y'(0) = 0,$$
$$y(1) = 1, \quad y'(1) = \frac{5}{2}.$$  

For the above problem there exist $y(t) = \frac{t^5}{2}$ which satisfies all boundary conditions and $J[y] = 0$. Using the method presented in Section 3 we determine $w(t) = \frac{1}{2}(t^3 + t^2)$ and achieve the following values of $c_j$ for different values of $k$ in approximation (7)

$k = 3$: $c_0 = -0.186261$, $c_1 = 0.092195$, $c_2 = -0.032149$, $c_3 = 0.015588$,

$k = 5$: $c_0 = -0.190235$, $c_1 = 0.0975335$, $c_2 = -0.0436693$, $c_3 = 0.0231088$,
\quad $c_4 = -0.008106$, $c_5 = 0.00465174$,

$k = 10$: $c_0 = -0.192108$, $c_1 = 0.0965389$, $c_2 = -0.0502571$, $c_3 = 0.0211055$,
\quad $c_4 = -0.015319$, $c_5 = 0.00208785$, $c_6 = -0.00465174$,
\quad $c_7 = -0.00203425$, $c_8 = -0.00203425$, $c_9 = -0.00203425$, $c_{10} = -0.000595373$.

In the following table the values of minimum $\mu_k$ for different values of approximations are demonstrated. It is obvious that with increase in the number of basis functions $k$, the approximate value $\mu_k$ converges to the exact value.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\mu_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$1.38821 \times 10^{-6}$</td>
</tr>
<tr>
<td>5</td>
<td>$2.14559 \times 10^{-7}$</td>
</tr>
<tr>
<td>10</td>
<td>$1.53822 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

In the following table the absolute errors for $k = 3, k = 5$ and $k = 10$ are demonstrated. Also the approximate solutions of $y(t)$ obtained with $k = 3, k = 5$ and $k = 10$ are plotted in Fig. 1 in comparison with the exact solution $y(t) = \frac{t^5}{2}$. Since the error is very low, the numerical solutions coincide to the exact solution. From this figure it can be seen that the numerical solutions are convergent to the exact solution.

| Absolute error with $k = 3, 5, 10$ for example 1. |
|----------------|----------------|----------------|
| $t$        | Absolute error, $k = 3$ | Absolute error, $k = 5$ | Absolute error, $k = 10$ |
| 0.0       | 0.0               | 0.0               | 0.0               |
| 0.1       | 1.01696E−4        | 1.74675E−5        | 2.53136E−6        |
| 0.2       | 3.77978E−5        | 1.44218E−5        | 1.65597E−5        |
| 0.3       | 3.56299E−5        | 2.91059E−5        | 7.26854E−7        |
| 0.4       | 8.25824E−5        | 1.47271E−5        | 7.49977E−6        |
| 0.5       | 8.66485E−5        | 8.29798E−6        | 7.09086E−6        |
| 0.6       | 4.20329E−5        | 1.93796E−5        | 5.75885E−6        |
| 0.7       | 1.49476E−5        | 1.09378E−5        | 6.00485E−6        |
| 0.8       | 3.78518E−5        | 3.49695E−6        | 7.68284E−7        |
| 0.9       | 4.55519E−5        | 1.01498E−5        | 9.08911E−7        |
| 1.0       | 0.0               | 0.0               | 0.0               |
5.2. Example 2

Consider the following problem

\[
\text{Min } J[y_1, y_2] = \int_0^1 \left( c_0 D^2_t y_1 + c_1 D^2_t y_2 - g(t) \right)^2 dt,
\]

where

\[
g(t) = \frac{8t^2}{3\sqrt{\pi}} + \frac{15\sqrt{\pi} t^2}{16},
\]

with the boundary conditions

\[
y_1(0) = 1, \quad y_1(1) = 2,
\]
\[
y_2(0) = 0, \quad y_2(1) = 1.
\]

For the given problem we have \( y_1(t) = t^2 + 1 \) and \( y_2(t) = t^{\frac{5}{2}} \) as minimizing functions with \( J[y_1, y_2] = 0 \). Using the method presented in Section 3 we achieve \( w_1(t) = 1 + t, w_2(t) = t \) and the following values of \( c_i \)'s, for different values of \( k \) in approximation (7)

\[
\begin{align*}
    k = 2: & \quad c_0^1 = 1.14038, \quad c_1^1 = 0.118211, \quad c_2^1 = -0.0123663, \\
    & \quad c_0^2 = 1.14038, \quad c_1^2 = 0.118211, \quad c_2^2 = -0.0123663; \\
    k = 4: & \quad c_0^1 = 1.14023, \quad c_1^1 = 0.120251, \quad c_2^1 = -0.0131621, \\
    & \quad c_0^3 = 0.00296862, \quad c_4^1 = -0.00101809, \\
    & \quad c_0^2 = 1.14023, \quad c_1^2 = 0.120251, \quad c_2^2 = -0.0131621, \quad c_3^2 = 0.00296862, \quad c_4^2 = -0.00101809, \\
    k = 6: & \quad c_0^1 = 1.1402, \quad c_1^1 = 0.120485, \quad c_2^1 = -0.013298, \quad c_3^1 = 0.00338138, \\
    & \quad c_0^4 = -0.0012204, \quad c_4^3 = 0.000414134, \quad c_6^1 = 0.000197209, \\
    & \quad c_0^2 = 1.1402, \quad c_1^2 = 0.120485, \quad c_2^2 = -0.013298, \quad c_3^2 = 0.00338138, \\
    & \quad c_4^2 = -0.0012204, \quad c_5^2 = 0.000414134, \quad c_6^2 = 0.000197209.
\end{align*}
\]

The values of approximate minimum \( \mu_k \), for different number of basis functions \( k \), are demonstrated in the following table.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \mu_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( 5.75479 \times 10^{-7} )</td>
</tr>
<tr>
<td>4</td>
<td>( 1.05904 \times 10^{-8} )</td>
</tr>
<tr>
<td>6</td>
<td>( 6.46315 \times 10^{-10} )</td>
</tr>
</tbody>
</table>

5.3. Example 3

Consider the following problem

\[
\text{Min } J[y_1, y_2, y_3] = \int_0^1 \left( c_0 D^2_t y_1 + \left( c_2 D^2_t y_2 \right)^2 + c_3 D^2_t y_3 + y_1 + y_2 + y_3 - g(t) \right)^2 dt,
\]

where

\[
g(t) = 2 + t^2 + t^2 + t^2 + t^3 + \frac{9t^4}{4 \Gamma(\frac{1}{3})} + \frac{81t^7}{8 \Gamma(\frac{7}{6})} + \frac{3\sqrt{\pi} t^8}{4 \Gamma(\frac{9}{4})} + \frac{15\sqrt{\pi} t^{11}}{8 \Gamma(\frac{17}{6})},
\]

with the following boundary conditions

\[
y_1(0) = 0, \quad y_1(1) = 1, \\
y_2(0) = 1, \quad y_2(1) = 3, \\
y_3(0) = 1, \quad y_3(1) = 2.
\]
Functions $y_1(t) = t^2$, $y_2(t) = 1 + t^{3/2} + t^3$ and $y_3(t) = t^{3/2} + 1$ are minimizing functions for the problem and $\{y_1, y_2, y_3\} = 0$. For this problem we achieve $w_1(t) = t$, $w_2(t) = 2t + 1$, $w_3(t) = t + 1$ and the following values of $c_i^k$ for different values of $k$ in approximation (7)

\[
\begin{align*}
    c_0 = 1.13167, & \quad c_0^2 = 2.253557, & \quad c_0^3 = 1.12219, \\
    c_1^0 = 1.26902, & \quad c_1^1 = -0.836793, & \quad c_1^2 = 2.08643, & \quad c_1^3 = 0.419413, \\
    c_2^0 = 1, & \quad c_2^1 = 1, & \quad c_2^2 = 2.37027, & \quad c_2^3 = 0.506811, & \quad c_2^4 = 0.169214, \\
    c_3^0 = 2.13793, & \quad c_3^1 = 0.245104, & \quad c_3^2 = 0.0343934, \\
    c_4^0 = 0, & \quad c_4^1 = 0, & \quad c_4^2 = 0.
\end{align*}
\]

for $k = 2$. The following table shows the values of minimum $\mu_k$ for different values of approximations.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\mu_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0937144</td>
</tr>
<tr>
<td>1</td>
<td>0.000025677</td>
</tr>
<tr>
<td>2</td>
<td>$2.41603 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

6. Conclusion

In this paper an efficient and accurate method for solving a class of fractional variational problems is developed. Utilizing special type of polynomial basis functions, we reduce the problem to the problem of solving a system of algebraic equations. The proposed polynomial functions have the great flexibility in satisfying initial and boundary conditions. The convergence of the method has been extensively discussed and illustrative test examples to demonstrate validity and applicability of the new technique are included.

Acknowledgments

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References