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Explicit Formulas for the Szegö Kernel for Some Domains in C^2

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We study the Szegö kernel for a class of strictly pseudoconvex domains in C^2. An explicit algorithm is given to compute the complete asymptotic expansion for the symbol of the Szegö kernel for these domains. It is then easy to compute the first three terms explicitly in terms of the defining function and its derivatives. We give an example where the first three terms (including the logarithmic term) are all non zero. Finally, we show that if the second term vanishes identically, then the boundary is locally biholomorphic to the surface Im $w = |z|^2$. © 1990 Academic Press, Inc.

0. INTRODUCTION AND STATEMENTS OF RESULTS

Our goal here is to obtain explicit formulas for the Szegö kernel for some domains in C^2. If we let (z, w) be coordinates near $0 \in C^2$ then we shall study strictly pseudoconvex domains whose boundary (near 0) is given by the equation

$$\operatorname{Im} w = \varphi(z, \bar{z}). \tag{0.1}$$

Here φ is a real valued, real analytic function of $(z, \bar{z})$ with $\varphi(0, 0) = 0$.

It is well known from the work of Boutet de Monvel and Sjöstrand [2] that the Szegö kernel for a bounded strictly pseudoconvex domain can be written as a Fourier integral operator with complex phase. Such an operator is determined completely by a phase function which we denote by ψ and a classical symbol which we denote by h. It follows from [2] that in the case under discussion here

$$\psi(z, w, \bar{z}', \bar{w}') = w - \bar{w}' - 2i\varphi(z, \bar{z}'), \tag{0.2}$$

where $(z, w, \bar{z}', \bar{w}')$ varies near $0 \in C^4$.

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It is also possible to compute the principal symbol of h. This is done in [2] by using the fact that the principal symbol is invariant under canonical transformation. The lower order terms however are not invariant and are *not discussed in* [2]. *One of our goals here is to compute the complete* asymptotic expansion for h. Indeed we give a recursive algorithm for computing all the lower order terms for h. In particular we compute the first three terms explicitly in terms of φ and its derivatives. This gives a complete description of the logarithmic term. We call this term h_{-1} in what follows. In fact we compute the Szegö kernel explicitly modulo a bounded function. See the corollary below.

Now we state our results. Let Ω be a smoothly bounded strictly pseudoconvex domain in C^2. Assume that near 0 the boundary of Ω is given by (0.1). We will denote by S the Szegö kernel of Ω. Our main result is

THEOREM. *Let Ω and S be as described above. Then there exists a symbol h of order* 1 *such that $S(z, w, \bar{z}', \bar{w}')$ is (modulo a smooth function) equal to*

$$\int_0^{\infty} e^{i\psi(z, w, \bar{z}', \bar{w}')\tau} h(z, \bar{z}', \tau)\, d\tau \tag{0.3}$$

near $(0, 0) \in \partial\Omega \times \partial\Omega$. Here h is smooth in all arguments. Also h is holomorphic in z and antiholomorphic in z' for each fixed τ. Furthermore h has an asymptotic expansion of the form

$$h(z, \bar{z}', \tau) \sim \sum_{k=-1}^{\infty} h_{-k}(z, \bar{z}')\tau^{-k}, \tag{0.4}$$

where the terms $h_1, h_0, h_{-1}, \ldots$ can be computed recursively using formulas (2.24), (2.25), *and* (2.26). *The first three terms are given explicitly in terms of φ and its derivatives in* (2.28), (2.32), *and* (2.38).

Remark 0.1. We interpret the expression in (0.3) as an oscillatory integral defining a distribution near $(0, 0) \in \partial\Omega \times \partial\Omega$. The formula is well defined because of the estimate (2.9). We refer the reader to either Melin and Sjöstrand [7] or Treves [9] for the general theory of Fourier integral operators with complex phase.

Remark 0.2. We make the assumption that φ is real analytic primarily to simplify the exposition. In this way we obtain formulas for the h_j as quickly as possible. The arguments we give in Sections 2 and 3 should go through to yield the C^∞ analog of the theorem. Indeed, if $\varphi(z, \bar{z})$ is merely smooth, then ψ can be defined using almost analytic extensions, as in [2]. Then the transport equations can be derived using the (complex) stationary phaıse method. We leave these details to the reader.

The next result is an immediate consequence of the theorem.

COROLLARY. *Let Ω and S be as described above. Then we have*

$$S = h_1/(-i\psi)^2 + h_0/(-i\psi) - h_{-1}\log(-i\psi) + O(1) \tag{0.5}$$

near $(0,0) \in \partial\Omega \times \partial\Omega$. *Here* ψ *is given by* (0.2), h_1 *is given by* (2.28), h_0 *is given by* (2.32), *and* h_{-1} *is given by* (2.38).

The paper proceeds as follows. In Section 1 we describe the three dimensional abstract CR structure determined by (0.1). We denote by L a particular non vanishing holomorphic vector field generating this structure.

In Section 2 we construct operators H and K such that $LK = I - H$. This is valid locally and the construction depends only on the abstract CR structure. We impose conditions on H and K so that they are uniquely determined by "transport equations" (2.24), (2.25), and (2.26). This section is motivated by the classical geometrical optics construction from hyperbolic theory and the article of Greiner, Kohn, and Stein [5]. We use only the most elementary facts from the theory of Fourier integral operators with complex phase. In particular, we do not reduce the problem to normal form via canonical transformation as was done in [2].

In Section 3 we realize the abstract CR structure as a piece of hypersurface in C^2. Then we show that if Ω is a strictly pseudoconvex domain whose boundary (near 0) is given by (0.1), then H is essentially the Szegö kernel for Ω. The work in Sections 1, 2, and 3 is enough to prove the theorem.

In Section 4 we indicate how the corollary follows from the theorem and conclude with some examples. In particular we show that our results are consistent with the classical formula in the case when $\varphi(z, \bar{z}) = |z|^2$. We also give an example where h_1, h_0, and h_{-1} are all non zero. In this example φ is a polynomial of degree 3.

We conclude the article by pointing out the fact that if h_0 vanishes identically, then the surface (0.1) is biholomorphnically equivalent to the surface $\operatorname{Im} w = |z|^2$.

Remark 0.3. In [4] Fefferman studied the asymptotic behavior of the Bergman kernel for strictly pseudoconvex domains and indicated how similar results could be obtained for the Szegö kernel. The results of Boutet de Monvel and Sjöstrand [2] came later using different methods.

In [4] an example is given of a domain where the logarithmic term for the Bergman kernel is present. The defining function for this domain is a polynomial of degree 8.

1. The CR Structure

We begin by describing the CR structure under consideration. Let (x, y, t) be coordinates near $0 \in R^3$. Throughout this article z will denote the function

$$z = x + iy \tag{1.1}$$

and $\bar{z}$ will be its complex conjugate. Let φ be a real valued, real analytic function defined near $0 \in R^2$. In other words we will assume that φ has a convergent power series expansion near 0 in the variables $(z, \bar{z})$ and we will write $\varphi = \varphi(z, \bar{z})$. We also assume that

$$\varphi(0, 0) = 0. \tag{1.2}$$

We define the function w by

$$w = t + i\varphi(z, \bar{z}). \tag{1.3}$$

The functions z and w determine a unique CR structure near $0 \in R^3$. If we define the vector field

$$\bar{L} = \partial/\partial\bar{z} - i\varphi_{\bar{z}}(z, \bar{z})\, \partial/\partial t \tag{1.4}$$

then it follows that

$$\bar{L}z = 0 = \bar{L}w. \tag{1.5}$$

We will use the standard notation

$$\partial/\partial z = \tfrac{1}{2}(\partial/\partial x - i\partial/\partial y) \tag{1.6}$$

and $\partial/\partial\bar{z}$ will denote the complex conjugate of $\partial/\partial z$. Also φ_z will be the same as $\partial\varphi/\partial z$.

L will denote the complex conjugate of $\bar{L}$, that is

$$L = \partial/\partial z + i\varphi_z(z, \bar{z})\, \partial/\partial t. \tag{1.7}$$

We see then that L and $\bar{L}$ are linearly independent and hence generate a CR structure near $0 \in R^3$. Formula (1.5) tells us that z and w are first integrals for this structure.

The final assumption that we will make is that the structure is strictly pseudoconvex at 0. In other words we will assume that

$$\varphi_{z\bar{z}}(0, 0) > 0. \tag{1.8}$$

Remark 1.1. Structures where $\operatorname{Im} w$ is independent of t are called rigid in Baouendi, Rothschild, and Treves [1]. Before this, rigid structures were studied by Tanaka [8]. This paper of Tanaka was the precursor to the article of Chern and Moser [3].

2. An Approximate Inverse for L

Our goal in this section is to construct operators K and H such that

$$LK = I - H, \tag{2.1}$$

where I is the identity operator. K and H are by no means uniquely determined. We will impose further conditions on these operators so that H will resemble a Szegö projector.

Remark 2.1. Let U be a neighborhood of $0 \in R^3$. All operators constructed in this section will be continuous linear operators from $C_0^\infty(U) \to C^\infty(U)$, for appropriate U. If A is such an operator, we will denote its distribution kernel by $A(x, y, t; x', y', t')$. Here (x', y', t) is another set of coordinates near $0 \in R^3$. If $u \in C_0^\infty(U)$ then we write formally

$$Au(x, y, t) = \int_{R^3} A(x, y, t; x', y', t')\, u(x', y', t')\, dx'\, dy'\, dt', \tag{2.2}$$

where $dx'\, dy'\, dt'$ is Lebesgue measure.

From the work of Boutet de Monvel and Sjöstrand [2] we know the Szegö projector (for a strictly pseudoconvex domain) can be written as a Fourier integral operator with complex phase. Taking this as motivation, we will construct K and H as Fourier integral operators. See Melin and Sjöstrand [7] or Treves [9] for the basic facts about the theory.

To begin the construction we introduce the phase function

$$\psi(z, w, \bar{z}', \bar{w}') = w - \bar{w}' - 2i\varphi(z, \bar{z}'), \tag{2.3}$$

where z and w are defined as before and

$$z' = x' + iy' \tag{2.4}$$

$$w' = t' + i\varphi(z', \bar{z}'). \tag{2.5}$$

We pause to discuss some of the properties of ψ. First observe that

$$\psi(z, w, \bar{z}, t' - i\varphi(z, \bar{z})) = t - t'. \tag{2.6}$$

Since $\varphi(z, \bar{z})$ is real it follows that

$$\overline{\varphi(z, \bar{z}')} = \varphi(z', \bar{z}). \tag{2.7}$$

Hence we see that

$$\operatorname{Im} \psi(z, w, \bar{z}', \bar{w}') = \varphi(z, \bar{z}) - \varphi(z, \bar{z}') + \varphi(z', \bar{z}') - \varphi(z', \bar{z}). \tag{2.8}$$

It follows from (2.8) and (1.8) that there exists a constant $C > 0$ such that

$$\operatorname{Im} \psi(z, w, \bar{z}', \bar{w}') \geqslant C\, |z - z'|^2 \tag{2.9}$$

for (x, y, t, x', y', t') near 0.

A straightforward computation shows that

$$L\psi = 2i(\varphi_z(z, \bar{z}) - \varphi_z(z, \bar{z}')). \tag{2.10}$$

If we define the function g as follows

$$g(z, \bar{z}, \bar{z}') = \int_0^1 \varphi_{z\bar{z}}(z, \rho\bar{z} + (1 - \rho)\bar{z}')\, d\rho \tag{2.11}$$

we see that (2.10) becomes

$$L\psi = 2i(\bar{z} - \bar{z}')\, g(z, \bar{z}, \bar{z}'). \tag{2.12}$$

Also observe that

$$g(z, \bar{z}, \bar{z}) = \varphi_{z\bar{z}}(z, \bar{z}). \tag{2.13}$$

Again a simple computation shows that

$$L\bar{\psi} = 0. \tag{2.14}$$

These are the main facts we will need about ψ. Note that ψ is essentially the phase used by Boutet de Monvel and Sjöstrand [2].

We now begin our construction of K and H. H will be a Fourier integral operator with ψ as phase. The operator K will be the sum of two operators, K^+ and K^- which we define now.

We define the kernel of K^- by

$$K^-(x, y, t; x', y', t') = 1/\pi(\bar{z} - \bar{z}') \int_{-\infty}^0 e^{i\bar{\psi}(z, w, \bar{z}', \bar{w}')\tau}\, d\tau/2\pi. \tag{2.15}$$

We define the kernel of K^+ by

$$K^+(x, y, t; x', y', t')$$
$$= 1/\pi(\bar{z} - \bar{z}') \int_0^\infty e^{i\psi(z, w, \bar{z}', \bar{w}')\tau} s(z, \bar{z}, \bar{z}', \tau)\, d\tau/2\pi \tag{2.16}$$

We define the kernel of H by

$$H(x, y, t; x', y', t') = \int_0^\infty e^{i\psi(z, w, \bar{z}', \bar{w}')\tau} h(z, \bar{z}', \tau)\, d\tau. \tag{2.17}$$

Our goal now is to find symbols s and h so that (2.1) is satisfied. We assume s to be of order 0 and h to be of order 1. Indeed we assume that s and h have asymptotic expansions of the following type:

$$s(z, \bar{z}, \bar{z}', \tau) \sim \sum_{k=0}^{\infty} s_{-k}(z, \bar{z}, \bar{z}')\tau^{-k} \tag{2.18}$$

$$h(z, \bar{z}', \tau) \sim \sum_{k=-1}^{\infty} h_{-k}(z, \bar{z}')\tau^{-k} \tag{2.19}$$

We will see that (2.1) will determine K and H uniquely if we impose the following conditions on s:

$$s_0(z, \bar{z}, \bar{z}) = 1 \tag{2.20}$$

$$s_{-k}(z, \bar{z}, \bar{z}) = 0, \; k \geqslant 1 \tag{2.21}$$

for all z near 0. We shall see that each s_{-k} is real analytic in the variables $(z, \bar{z}, \bar{z}')$. We will also assume that each h_{-k} is real analytic in the variables $(z, \bar{z}')$. Hence h_{-k} will be completely determined once we know its value when $\bar{z}' = \bar{z}$.

If we let $K = K^+ + K^-$ and compute LK we will see that $LK = I - H$ where the symbol of h is given by

$$h(z, \bar{z}', \tau) = 1/2\pi^2(2\tau g(z, \bar{z}, \bar{z}')\, s(z, \bar{z}, \bar{z}', \tau)$$
$$- s_z(z, \bar{z}, \bar{z}', \tau)/(\bar{z} - \bar{z}')). \tag{2.22}$$

To arrive at (2.22) we have used (2.20), (2.21), (2.12), (2.14), and (2.6). We have also used the fact that $1/\pi\bar{z}$ is a fundamental solution for $\partial/\partial z$. Note that (2.6) was used in conjunction with the Fourier inversion formula.

We may write for small $(z, \bar{z}, \bar{z}')$

$$s_z(z, \bar{z}, \bar{z}')/(\bar{z} - \bar{z}') = -\int_0^1 s_{z\bar{z}'}(z, \bar{z}, \rho\bar{z} + (1-\rho)\bar{z}')\, d\rho. \tag{2.23}$$

If we substitute this formula into (2.22) and take (2.18) and (2.19) into account we obtain the following "transport equations"

$$h_1(z, \bar{z}') = 1/\pi^2 g(z, \bar{z}, \bar{z}')\, s_0(z, \bar{z}, \bar{z}') \tag{2.24}$$

$$\begin{aligned} h_{-k}(z, \bar{z}') = 1/2\pi^2 \Big(& 2g(z, \bar{z}, \bar{z}')\, s_{-k-1}(z, \bar{z}, \bar{z}') \\ & + \int_0^1 (s_{-k})_{z\bar{z}'}\, (z, \bar{z}, \rho\bar{z} + (1-\rho)\bar{z}')\, d\rho \Big) \end{aligned} \tag{2.25}$$

for $k \geqslant 0$. We obtain these formulas by equating terms of like homogeneity (in τ) in (2.22).

Note that by (2.21) and (2.25) we also have for $k \geqslant 0$

$$h_{-k}(z, \bar{z}) = (1/2\pi^2)(s_{-k})_{z\bar{z}'}\, (z, \bar{z}, \bar{z}) \tag{2.26}$$

$$= (-1/2\pi^2)(s_{-k})_{z\bar{z}}\, (z, \bar{z}, \bar{z}). \tag{2.26$'$}$$

We see that (2.24), (2.20), and (2.13) show that

$$h_1(z, \bar{z}) = (1/\pi^2)\, \varphi_{z\bar{z}}(z, \bar{z}). \tag{2.27}$$

Since we assume h_1 is analytic we have

$$h_1(z, \bar{z}') = (1/\pi^2)\, \varphi_{z\bar{z}}(z, \bar{z}'). \tag{2.28}$$

Now that h_1 is completely determined we substitute into (2.24) to obtain

$$s_0(z, \bar{z}, \bar{z}') = \varphi_{z\bar{z}}(z, \bar{z}')/g(z, \bar{z}, \bar{z}'). \tag{2.29}$$

Now that h_1 and s_0 have been determined we can compute h_0 from (2.26) and hence s_{-1} from (2.25). Continuing in this manner all the terms in the asymptotic expansions for s and h can be computed in the following order

$$h_1, s_0, h_0, s_{-1}, h_{-1}, s_{-2}, \ldots . \tag{2.30}$$

by alternating between (2.25) and (2.26).

Now the theory [7], [9] allows us to choose smooth (i.e., C^∞, symbols s and h with the given asymptotic expansions (2.18) and (2.19). These symbols are uniquely determined up to an error of order $-\infty$. We will assume then that $s = s(z, \bar{z}, \bar{z}', \tau)$ is smooth for (z, z') near 0 and $\tau > 0$. Furthermore we may assume that s is real analytic in $(z, \bar{z}, \bar{z}')$ for each fixed $\tau > 0$. We may also assume that s vanishes for τ near 0. We pick a similar realization for h.

We now have achieved our goal of constructing operators K and H such that $LK = I - H$. Furthermore we have an algorithm for computing the full symbols of K and H. One of our goals here is to compute h_0 and h_{-1} explicitly in terms of φ and its derivatives. Before doing so we introduce some notation.

For non negative integers j and k we define φ_{jk} by

$$\varphi_{jk} = (\partial^{j+k}\varphi/\partial z^j\,\partial \bar{z}^k)(z, \bar{z}). \tag{2.31}$$

To compute h_0 we use the fact that s_0 is known from (2.29) and use (2.26)′. It then follows that

$$h_0(z, \bar{z}) = (1/4\pi^2)(\varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21})/(\varphi_{11})^2. \tag{2.32}$$

The computation of h_{-1} is more complicated. First note that we have for $k \geqslant 0$

$$h_{-k-1}(z, \bar{z}) = (-1/8\pi^2)(\partial/\partial z)((s_{-k})_{z\bar{z}\bar{z}'}\,\varphi_{11}), \tag{2.33}$$

where the right hand side of (2.33) is evaluated at $\bar{z}' = \bar{z}$. To prove this first differentiate both sides of (2.25) with respect to $\bar{z}$. Evaluate this expression when $\bar{z}' = \bar{z}$ and take (2.21) into account. It follows that (2.33) is true by differentiating again with respect to z and using (2.26′).

Letting $k=0$ in (2.33) we may compute h_{-1} since s_0 is known. First some notation. We define M_{22}, M_{23}, M_{33} as follows:

$$M_{22} = \varphi_{11}\varphi_{33} - \varphi_{13}\varphi_{31} \tag{2.34}$$

$$M_{23} = \varphi_{11}\varphi_{32} - \varphi_{12}\varphi_{31} \tag{2.35}$$

$$M_{33} = \varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21}. \tag{2.36}$$

Also define Q by

$$\begin{aligned} Q = {} & 3M_{33}(M_{33} - 3\varphi_{12}\varphi_{21}) - M_{22}(\varphi_{11})^2 \\ & + 6\varphi_{11}\,\mathrm{Re}(\varphi_{12}M_{23}). \end{aligned} \tag{2.37}$$

It now follows from (2.33) that

$$h_{-1}(z, \bar{z}) = -Q/24\pi^2(\varphi_{11})^5. \tag{2.38}$$

Remark 2.2. It follows from the work of this section that if f is a smooth function near $0 \in R^3$ such that Hf is real analytic then there exists u such that $Lu = f$. Indeed, since L has analytic coefficients we can find v such that $Lv = Hf$ by the Cauchy–Kovalevska Theorem. Then we may let $u = Kf + v$. On the other hand, one would like to prove the converse of this statement as was done in Greiner, Kohn, and Stein [5]. That is we would like to show that if there exists u such that $Lu = f$ then Hf is real analytic.

This does not seem to be directly possible using our approach. This is because of the estimate (2.9). Usually one needs an estimate of the kind

$$\operatorname{Im} \psi \geqslant C(|z-z'|^2 + (t-t')^2). \tag{2.39}$$

We can obtain (2.39) if we wish to use as phase $\psi + i\psi^2$ to define H and K. Using this approach we can obtain the correct solvability results. However, this method greatly complicates the transport equations (2.24) and (2.25). Since our aim is to provide explicit formulas we have chosen to use ψ as phase.

3. H and the Szegö Kernel

In this section we construct a small dommain $\Omega' \subset C^2$ and show that the Szegö kernel for Ω' differs from H by a smooth function (near the origin). We then show how the Theorem follows from this fact. We use only the most elementary facts about the Szegö kernel. For those not familiar with this topic, see for example Krantz [6].

From Section 2 we have operators H and K defined by (2.15), (2.16), and (2.17) such that (2.1) holds. To be precise there exists an open neighborhood U of $0 \in R^3$ such that H and K are continuous linear operators from $C_0^\infty(U) \to C^\infty(U)$ and $LK = I - H$.

We shall assume that U is small enough so that the map from U into C^2 given by

$$(x, y, t) \to (x + iy,\ t + i\varphi(z, \bar{z})) \tag{3.1}$$

is a real analytic diffeomorphism. If we denote coordinates near 0 in C^2 by (z, w) then we see that the image of the map (3.1) is a piece of hypersurface given by the equation

$$\operatorname{Im} w = \varphi(z, \bar{z}). \tag{3.2}$$

We denote this hypersurface by M. Now we may think of H and K as operators mapping $C_0^\infty(M) \to C^\infty(M)$.

If $\varepsilon > 0$ we define the open set $O_\varepsilon \subset C^2$ by the following

$$O_\varepsilon = \{(z, w) \colon \operatorname{Im} w > \varphi(z, \bar{z}) \text{ and } |z|^2 + |w|^2 < \varepsilon^2\}. \tag{3.3}$$

Note that we may define ψ as in (2.3) where now we may think of (z, w, z', w') as an arbitrary point near $0 \in C^4$. Hence it follows that there exists an $\varepsilon > 0$ such that the estimate (2.9) holds for $(z, w, z', w') \in \bar{O}_\varepsilon \times M$. This is enough to ensure that H and K can be extended as bounded operators mapping $C_0^\infty(M) \to C^\infty(\bar{O}_\varepsilon)$ provided that $\varepsilon > 0$ and U are taken small enough. Note that because h depends only on z and $\bar{z}'$ it follows that if $f \in C_0^\infty(M)$ then Hf will be holomorphic on O_ε.

Now let Ω' be a sxtrictly pseudoconvex open subset of C^2 with smooth boundary such that $\Omega' \subset O_\varepsilon$. We also assume that near the origin the boundary of Ω' is given by Eq. (3.2). Such an Ω' is easily constructed. Let S' be the Szegö projector associated with Ω'. In other words S' is the orthogonal projection from $L^2(\partial\Omega')$ onto $H^2(\partial\Omega')$. Here H^2 is the closed subspace of L^2 consisting of the "boundary values" of holomorphic functions on Ω'. For the precise definition of H^2, see for example [6].

Recall that the kernel of S' is holomorphic in (z, w) and antiholomorphic in (z', w'). Hence we write $S' = S'(z, w; \bar{z}', \bar{w}')$. Also we have $S'f = f$ for all $f \in H^2$.

If A and B are bounded operators from $C_0^\infty(M) \to C^\infty(\partial\Omega')$ we say that $A \sim B$ if the kernel of $A - B$ is a smooth function near $(0, 0)$ in $M \times M$.

We now will show that $S' \sim H$. Observe that if we denote the formal transpose of L by ${}^tL$, we have ${}^tL = -L$. Let $\chi \in C_0^\infty(M)$ have small support and be equal to 1 near the origin. It follows from (2.1) that

$$S'\chi LK = S'\chi - S'\chi H. \tag{3.4}$$

In (3.4) the symbol χ indicates "multiply by χ." Since S' is anti holomorphic in (z', w') and ${}^tL = -L$ we have $S'\chi LK \sim 0$. Hence it follows that $S' \sim S'H$. We also have $S'H \sim H$. Indeed this follows since $Hf \in H^2(\partial\Omega')$ for all $f \in C_0^\infty(M)$. Hence $S' \sim H$.

Now to conclude the proof of the Theorem. Let $\Omega \subset C^2$ be a bounded strictly pseudoconvex domain with smooth boundary. Assume that near 0 the boundary of Ω is given by (0.1). Let S be the Szegö kernel for Ω. We know (from [2]) that near 0, S can be written as a Fourier integral operator with phase ψ. Furthermore the full symbol of S near 0 depends only on the defining function of Ω near 0. This is pointed out in [2, p. 162]. Hence near 0, S and S' have the same full symbol. We conclude that S differs from H by a smooth function near 0.

4. Concluding Remarks

We would like to conclude with some remarks and examples.

Remark 4.1. From formula (2.19) we see that there exists a constant $C > 0$ such that

$$|h(z, \bar{z}', \tau) - h_1(z, \bar{z}')\tau - h_0(z, \bar{z}') - h_{-1}(z, \bar{z}')/\tau| \leqslant C/\tau^2 \tag{4.1}$$

for z and $\bar{z}'$ near 0 and τ large. Hence we see that the contribution to the kernel of S from the remaining terms (i.e., those of order $\leqslant -2$) is a bounded function.

Now the contribution from the first three terms can be computed by interpreting the oscillatory interals as "finite part" integrals. For example we have the classical formula

$$pf \int_0^\infty e^{i\psi\tau}\, d\tau/\tau = -\log(-i\psi) - \gamma, \tag{4.2}$$

where γ is Euler's constant. Hence it follows that the Szegö kernel can be written modulo a bounded function as

$$S = h_1/(-i\psi)^2 + h_0/(-i\psi) - h_{-1}\log(-i\psi) + O(1) \tag{4.3}$$

near the origin. Note that ψ is given explicitly in (2.3), h_1 is given by (2.28), h_0 is given by (2.32) and h_{-1} is given by (2.38). Please note that similar comments have been made in [2].

EXAMPLE 4.2. Suppose that $\varphi(z, \bar{z}) = |z|^2$. It follows from our computations that in this case we have

$$\psi(z, w, \bar{z}', \bar{w}') = w - \bar{w}' - 2iz\bar{z}' \tag{4.4}$$

$$h_1(z, \bar{z}') = 1/\pi^2 \tag{4.5}$$

$$h_{-k}(z, \bar{z}') = 0 \qquad \text{for all} \quad k \geqslant 0. \tag{4.6}$$

Hence the results we obtain from (4.3) are consistent with the classical formula

$$S(z, w, \bar{z}', \bar{w}') = -1/\pi^2(w - \bar{w}' - 2iz\bar{z}')^2. \tag{4.7}$$

EXAMPLE 4.3. Suppose that $\varphi(z, \bar{z}) = |z|^2 + az\bar{z}^2 + \bar{a}z^2\bar{z}$ with $a \in C$. A simple application of our formulas shows that

$$h_1(0, 0) = 1/\pi^2 \tag{4.8}$$

$$h_0(0, 0) = -|a|^2/\pi^2 \tag{4.9}$$

$$h_{-1}(0, 0) = -8\,|a|^2/\pi^2. \tag{4.10}$$

So we see that if the constant a is not zero then the first three terms in the asymptotic expansion for S are really there.

Remark 4.4. If h_0 vanishes identically. then the surface Im $w = \varphi(z, \bar{z})$ is biholomorphically equivalent to the surface Im $w = |z|^2$.

Indeed, the fact that h_0 vanishes is equivalent to the fact that

$$\Delta \log \Delta\varphi = 0. \tag{4.11}$$

This follows immediately from (2.32). Now (4.11) implies that there exist functions f and g holomorphic near 0 such that

$$\varphi(z, \bar{z}) = |f(z)|^2 + g(z) + \overline{g(z)} \tag{4.12}$$

with $f_z(0) \neq 0$. We introduce new coordinates near $(0, 0) \in C^2$

$$\tilde{z} = f(z) \tag{4.13}$$

$$\tilde{w} = w - 2ig(z). \tag{4.14}$$

Now we see that $\varphi(z, \bar{z}) - \operatorname{Im} w = |\tilde{z}|^2 - \operatorname{Im} \tilde{w}$.

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