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Asymptotic behavior of a viscous liquid-gas model with mass-dependent viscosity and vacuum

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In this paper, we consider two classes of free boundary value problems of a viscous two-phase liquid-gas model relevant to the flow in wells and pipelines with mass-dependent viscosity coefficient. The liquid is treated as an incompressible fluid whereas the gas is assumed to be polytropic. We obtain the asymptotic behavior and decay rates of the mass functions $n(x,t)$, $m(x,t)$ when the initial masses are assumed to be connected to vacuum both discontinuously and continuously, which improves the corresponding result about Navier–Stokes equations in Zhu (2010) [23].

Contents

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1. Introduction

Consider the one-dimensional liquid-gas two-phase model with viscosity which can be written in Eulerian coordinates as (cf. [4,5]):

$$\begin{cases} \partial_t[\alpha_g\rho_g]+\partial_x[\alpha_g\rho_g u_g]=0, \\ \partial_t[\alpha_l\rho_l]+\partial_x[\alpha_l\rho_l u_l]=0, \\ \partial_t[\alpha_g\rho_g u_g+\alpha_l\rho_l u_l]+\partial_x\big[\alpha_g\rho_g u_g^2+\alpha_l\rho_l u_l^2+P\big]=-q+\partial_x[\varepsilon\partial_x u_{mix}], \end{cases} \tag{1.1}$$

where $u_{mix}=\alpha_g u_g+\alpha_l u_l$ and the unknown variables $\alpha_g,\alpha_l\in[0,1]$ denote volume fractions satisfying the fundamental relation:

$$\alpha_g+\alpha_l=1. \tag{1.2}$$

Furthermore, the other unknown variables ρ_g, ρ_l, u_g, u_l denote gas density, liquid density, velocities of gas and liquid respectively, whereas P is the common pressure for both phases, q presents external forces, like gravity and friction, and $\varepsilon>0$ denotes viscosity.

We focus on a simplified model as in [4,5] obtained by assuming that fluid velocities are equal, i.e., $u_g=u_l=u$ and neglecting the external forces, i.e., $q=0$. In addition, we neglect the gas phase effects in the mixture momentum conservation equation $(1.1)_3$. This motivation is from the fact that liquid density is much higher than the gas density, generality speaking, $\rho_l/\rho_g=O(10^3)$. Thus, we can obtain the following simplified model:

$$\begin{cases} \partial_t[\alpha_g\rho_g]+\partial_x[\alpha_g\rho_g u]=0, \\ \partial_t[\alpha_l\rho_l]+\partial_x[\alpha_l\rho_l u]=0, \\ \partial_t[\alpha_l\rho_l u]+\partial_x\big[\alpha_l\rho_l u^2+P\big]=\partial_x[\varepsilon\partial_x u]. \end{cases} \tag{1.3}$$

Let

$$n=\alpha_g\rho_g, \qquad m=\alpha_l\rho_l. \tag{1.4}$$

Then, we get a model of the form

$$\begin{cases} \partial_t n+\partial_x(nu)=0, \\ \partial_t m+\partial_x(mu)=0, \\ \partial_t(mu)+\partial_x\big(mu^2+P\big)=\partial_x(\varepsilon\partial_x u). \end{cases} \tag{1.5}$$

As in [4], we assume the liquid is incompressible, i.e., $\rho_l=constant$ and the gas is polytropic

$$P=C\rho_g^\gamma, \quad \gamma>1,\ C>0, \tag{1.6}$$

i.e.,

$$P(n,m)=C\rho_l^\gamma\left(\frac{n}{\rho_l-m}\right)^\gamma=A\left(\frac{n}{\rho_l-m}\right)^\gamma, \tag{1.7}$$

where $A=C\rho_l^\gamma$. Moreover the viscosity coefficient is taken as the form (cf. [4,5]):

$$\varepsilon=\varepsilon(n,m)=B\frac{n^{\beta}}{(\rho_l-m)^{\beta+1}},\quad B>0,\ \beta>0, \tag{1.8}$$

or

$$\varepsilon=\varepsilon(m)=B\frac{m^{\beta}}{(\rho_l-m)^{\beta+1}},\quad B>0,\ \beta>0. \tag{1.9}$$

In the following, without loss of generality, we consider only the form (1.8).

It is worth noting that we consider the simplified model (1.5) rather than the full two-phase model (1.1). In [4], Evje, Karlsen demonstrate that the simplified model can give a good approximation to the original two-phase model by numerical experiments. For the detailed description and the motivation, see Section 2 in [4]. For some relevant physical background, please refer to [1,15].

In this paper, we will consider the following two classes of the free boundary value problems of the system (1.5):

(1) The initial masses connect to vacuum discontinuously:

The boundary conditions are given as

$$\begin{cases} \big(-P(m,n)+\varepsilon(n,m)\partial_x u\big)\big(a(t)^+,t\big)=0,\\ \big(-P(m,n)+\varepsilon(n,m)\partial_x u\big)\big(b(t)^-,t\big)=0, \quad t\geqslant 0, \end{cases} \tag{1.10}$$

and the initial data are given as

$$n(x,0)=n_0(x)>0,\qquad m(x,0)=m_0(x)>0,\qquad u(x,0)=u_0(x),\quad x\in[a,b]. \tag{1.11}$$

(2) The initial masses connect to vacuum continuously:

The boundary conditions are given as

$$n\big(a(t),t\big)=n\big(b(t),t\big)=0,\qquad m\big(a(t),t\big)=m\big(b(t),t\big)=0,\quad t\geqslant 0, \tag{1.12}$$

and the initial data are given as

$$n(x,0)=n_0(x)>0,\qquad m(x,0)=m_0(x)>0,\qquad u(x,0)=u_0(x),\quad x\in(a,b), \tag{1.13}$$

and $n_0(a)=n_0(b)=m_0(a)=m_0(b)=0$.

Here $-\infty<a<b<\infty$, $a(t)$ and $b(t)$ are the free boundaries defined by

$$\begin{cases} \dfrac{da(t)}{dt}=u\big(a(t),t\big), \quad t>0,\\ a(0)=a, \end{cases} \tag{1.14}$$

and

$$\begin{cases} \dfrac{db(t)}{dt}=u\big(b(t),t\big), \quad t>0,\\ b(0)=b, \end{cases} \tag{1.15}$$

which are the interfaces separating the gas from the vacuum.

Let's first review some of the previous works in this direction. When the viscosity coefficient ε was taken as the form (1.9) and the initial masses connected to vacuum discontinuously, Evje and Karlsen in [4] got the global existence and uniqueness of weak solutions when $\beta\in(0,\frac{1}{3})$ by energy method. This result was later generalized to the case when $\beta \in (0, 1]$ by Yao and Zhu in [17]. When the viscosity coefficient ε was taken as the form (1.8) and the initial masses connected to vacuum continuously, Evje, Flatten and Friis in [5] proved the global existence of the weak solutions under some approximate assumptions on $n_0(x)$, $m_0(x)$, $u_0(x)$, and $\frac{n_0(x)}{m_0(x)} = c_0(x)$ when $\beta \in (0, \frac{1}{3})$. Recently, the existence of global weak solutions was obtained by Yao and Zhu in [16] when the viscosity was a constant and the initial masses connected to vacuum continuously.

Few results concerning asymptotic behavior and decay rate estimates on the mass functions $m(x,t)$, $n(x,t)$ for the free boundary value problem (1.5), (1.10), (1.11) and (1.5), (1.12), (1.13). The main purpose of this paper is to answer this question. More precisely, we show that the masses n and m tend to zero as time goes to infinity. Moreover, we can obtain a stabilization rate estimates of the mass functions for any $\beta > 0$ as $t \to \infty$.

It is necessary for us to illustrate that the main methods used to obtain our results are similar to those in [9,23], because we have used the variable transformations as in [4,5], by which we can rewrite our problem into (2.7)–(2.12) similar to the model in single-phase Navier–Stokes equations. In view of this, let's review some of the relevant works about single-phase Navier–Stokes equations with density-dependent viscosity and vacuum. When the initial density connected to vacuum discontinuously, the global existence of weak solutions for isentropic flow was obtained by Okada, Matušů-Nečasová, Makino in [14] for $\mu(\rho) = \rho^\theta$, $0 < \theta < \frac{1}{3}$, by Yang, Yao and Zhu in [19] for $0 < \theta < \frac{1}{2}$ and by Jiang, Xin and Zhang in [10] for $0 < \theta < 1$. Qin, Yao and Zhao in [18] extended the results in [10,14,19] to the case $0 < \theta \leqslant 1$. Guo and Jiang in [8] proved the global existence of weak solutions when $0 < \theta < \max\{3 - \gamma, \frac{3}{2}\}$ where θ can be greater than 1. Recently, Zhu in [23] investigated the asymptotic behavior and decay rate estimates about the density function $\rho(x,t)$ by overcoming some new difficulties which came from the appearance of the boundary layers. When the initial density connected to vacuum continuously, the local existence of weak solutions was obtained in [20] by Yang and Zhao. The global existence of weak solutions was given in [21] by Yang and Zhu for $0 < \theta < \frac{2}{9}$, and later was improved in [22] for $0 < \theta < \frac{1}{3}$ by Vong, Yang and Zhu, in [6] for $0 < \theta < \frac{1}{2}$ and in [7] for $0 < \theta < 1$ by Fang and Zhang. Guo and Zhu in [9] obtained a global existence result when $0 < \theta < \max\{3 - \gamma, \frac{3}{2}\}$ and firstly studied the asymptotic behavior and the decay rate of the density function $\rho(x,t)$ with respect to the time t for any $\theta > 0$ based on the following new mathematical entropy inequality, which was obtained first by Kanel in [11] for one-dimensional case and Bresch, Desjardins, Lin and Mellet, Vasseur for multi-dimensional case, cf. [2,3,12]:

$$\int_{a(t)}^{b(t)} \left\{ \frac{1}{2}\rho u^2 + u\big(\rho^\theta\big)_x + \frac{1}{2}\rho^{2\theta-3}\rho_x^2 + \frac{\rho^{\gamma-1}}{\gamma-1} \right\} dx + \int_0^t \int_{a(t)}^{b(t)} \frac{4\theta\gamma}{(\gamma+\theta)^2}\rho^{\gamma+\theta-3}\rho_x^2 \, dx\, dt \leqslant C,$$

where C is a uniform constant independent of t. The rest of this paper is organized as follows. In Section 2, we reformulate the two free boundary value problems (1.5), (1.10), (1.11) and (1.5), (1.12), (1.13) into the two fixed boundary value problems by introducing the Lagrangian coordinates and using the variable transformations. Then we state the main theorems of this paper. In Section 3, we derive some crucial uniform estimates for studying the asymptotic behavior and the decay rate estimates about the mass functions. In Section 4, the decay rate estimates on the mass functions will be given by introducing a new function $w(x,t)$ in [13] by Nagasawa.

2. Reformulation of the problems and the main results

To solve the two free boundary problems above, it is convenient to convert the free boundaries to the fixed boundaries by using Lagrangian coordinates. To do this, let

$$\xi = \int_{a(t)}^{x} m(y,t)\, dy, \quad \tau = t.$$

Then the free boundaries $x = a(t)$ and $x = b(t)$ become $\xi = 0$ and $\xi = \int_{a(t)}^{b(t)} m(y,t)\,dy = \int_a^b m_0(y)\,dy$ by the conservation of mass, where $\int_a^b m_0(y)\,dy$ is the total liquid mass initially. We normalize $\int_a^b m_0(y)\,dy$ to 1. Hence in the Lagrangian coordinates, the two free boundary problems (1.5), (1.10), (1.11) and (1.5), (1.12), (1.13) become

$$\begin{cases} n_\tau + (nm)u_\xi = 0, \\ m_\tau + m^2 u_\xi = 0, \\ u_\tau + \big(P(n,m)\big)_\xi = \big(\varepsilon(n,m)mu_\xi\big)_\xi, \end{cases} \tag{2.1}$$

with the boundary conditions (corresponding to the initial masses connect to vacuum discontinuously)

$$P(n,m) = E(n,m)u_\xi, \quad \text{at } \xi = 0, 1,\ \tau \geqslant 0, \tag{2.2}$$

and the initial data

$$n(\xi,0) = n_0(\xi) > 0, \qquad m(\xi,0) = m_0(\xi) > 0, \qquad u(\xi,0) = u_0(\xi), \quad \xi \in [0,1], \tag{2.3}$$

or with the boundary conditions (corresponding to the initial masses connect to vacuum continuously)

$$n(0,\tau) = n(1,\tau) = 0, \qquad m(0,\tau) = m(1,\tau) = 0, \quad \tau \geqslant 0, \tag{2.4}$$

and the initial data

$$n(\xi,0) = n_0(\xi) > 0, \qquad m(\xi,0) = m_0(\xi) > 0, \qquad u(\xi,0) = u_0(\xi), \quad \xi \in (0,1), \tag{2.5}$$

and $n_0(0) = n_0(1) = m_0(0) = m_0(1) = 0$.

Here

$$P(n,m) = \left(\frac{n}{\rho_l - m}\right)^\gamma, \qquad E(n,m) = \varepsilon(n,m)m = \frac{n^\beta m}{(\rho_l - m)^{\beta+1}}, \quad \beta > 0. \tag{2.6}$$

Here we have assumed $A = B = 1$ in (2.6) for simplicity.

In the following, we replace the coordinates (ξ,τ) by (x,t). Introduce the variables (cf. [4,5]):

$$c = \frac{n}{m}, \qquad Q(m) = \frac{m}{\rho_l - m} = \frac{\alpha_l}{1-\alpha_l} \geqslant 0.$$

Form the first two equations of (2.1), we get

$$c_t = \frac{n_t}{m} - \frac{n}{m^2}m_t = -\frac{mnu_x}{m} + \frac{nm^2}{m^2}u_x = 0,$$

and

$$\begin{aligned} Q(m)_t &= \left(\frac{m}{\rho_l - m}\right)_t = \left(\frac{1}{\rho_l - m} + \frac{m}{(\rho_l - m)^2}\right) m_t \\ &= \frac{\rho_l}{(\rho_l - m)^2} m_t = -\frac{\rho_l m^2}{(\rho_l - m)^2} u_x \\ &= -\rho_l Q(m)^2 u_x. \end{aligned}$$

Then we can rewrite the initial boundary problems (2.1), (2.2), (2.3) and (2.1), (2.4), (2.5) into the following forms:

$$\begin{cases} \partial_t c = 0, \\ \partial_t Q(m) + \rho_l Q(m)^2 \partial_x u = 0, \\ \partial_t u + \partial_x \big(P(c,m)\big) = \partial_x \big(E(c,m)\partial_x u\big), \end{cases} \tag{2.7}$$

with the boundary conditions (corresponding to the initial masses connect to vacuum discontinuously)

$$P(c,m) = E(c,m) u_x, \quad \text{at } x = 0, 1,\ t \geqslant 0, \tag{2.8}$$

and the initial data

$$c(x,0) = c_0(x) > 0, \qquad Q(m)(x,0) = Q(m_0)(x) > 0, \qquad u(x,0) = u_0(x), \quad x \in [0,1], \tag{2.9}$$

or with the boundary conditions (corresponding to the initial masses connect to vacuum continuously)

$$c(0,t) = c(1,t) = 0, \qquad Q(m)(0,t) = Q(m)(1,t) = 0, \quad t \geqslant 0, \tag{2.10}$$

and the initial data

$$c(x,0) = c_0(x) > 0, \qquad Q(m)(x,0) = Q(m_0)(x) > 0, \qquad u(x,0) = u_0(x), \quad x \in (0,1), \tag{2.11}$$

and $c_0(0) = c_0(1) = Q(m_0)(0) = Q(m_0)(1) = 0$.

Here

$$P(c,m) = \left(\frac{n}{\rho_l - m}\right)^{\gamma} = c^{\gamma} Q(m)^{\gamma}, \qquad E(c,m) = m\varepsilon(n,m) = c^{\beta} Q(m)^{\beta+1}, \quad \beta > 0. \tag{2.12}$$

Throughout this paper, our assumptions on the initial data and β, γ are stated as follows:

- (A_1) $\inf_{x\in[0,1]} n_0(x) > 0$, $\sup_{x\in[0,1]} n_0(x) < \infty$, $\inf_{x\in[0,1]} m_0(x) > 0$, $\sup_{x\in[0,1]} m_0(x) < \rho_l$;
- $(A_1)'$ There are positive constants K_1, K_2, K_3 and K_4 such that $K_1\phi(x)^{\frac{\alpha}{2}} \leqslant m_0(x) \leqslant K_2\phi(x)^{\frac{\alpha}{2}} < \rho_l$, $K_3\phi(x)^{\alpha} \leqslant n_0(x) \leqslant K_4\phi(x)^{\alpha}$, where $\phi(x) = x(1-x)$, $0 < \alpha < 1$. In particular, this implies that there exist positive constants C_1, C_2, such that $C_1\phi(x)^{\frac{\alpha}{2}} \leqslant c_0(x) = \frac{n_0(x)}{m_0(x)} \leqslant C_2\phi(x)^{\frac{\alpha}{2}}$;
- (A_2) $u_0(x) \in L^{2n}([0,1])$ for any given positive integer n satisfying $n \geqslant \frac{2\gamma+\beta}{2\beta}$;
- (A_3) $((c_0 Q(m_0))^{\beta})_x \in L^2([0,1])$;
- (A_4) $\beta > 0$, $\gamma \geqslant 1 + \beta$.

Now we give the following definition of weak solution:

Definition 2.1 *(Weak solution).* We call $(n(x,t), m(x,t), u(x,t))$ a global weak solution to the initial boundary value problems (2.1), (2.2), (2.3) or (2.1), (2.4) (2.5), if the following estimates hold for any $t > 0$,

$$n, m, u \in L^\infty\big([0,1]\times[0,+\infty)\big) \cap C^1\big([0,+\infty); H^1\big([0,1]\big)\big),$$

$$E(n,m)u_x \in L^\infty\big([0,1]\times[0,+\infty)\big) \cap C^{\frac{1}{2}}\big([0,+\infty); L^2\big([0,1]\big)\big),$$

$$0 \leqslant n(x,t) < \rho_l \sup_{x\in[0,1]} c_0,$$

and

$$0 \leqslant m(x,t) < \rho_l.$$

Furthermore, the following equations hold:

$$n_t + mnu_x = 0, \qquad m_t + m^2 u_x = 0,$$

$$(n,m)(x,0) = \big(n_0(x), m_0(x)\big), \quad \text{for a.e. } x \in [0,1] \text{ and any } t \geqslant 0,$$

and

$$\int_0^\infty \int_0^1 \big(u\varphi_t + \big(P(n,m) - E(n,m)u_x\big)\varphi_x\big)\,dx\,dt + \int_0^1 u_0(x)\varphi(x,0)\,dx = 0,$$

for any test functions $\varphi \in C_0^\infty(\Omega)$ with $\Omega = \{(x,t)\colon\ 0 \leqslant x \leqslant 1,\ t \geqslant 0\}$.

In what follows, we always use C (and C_n) to denote a generic positive constant depending only on the initial data (and the given positive integer n), but independent of t.

We now state the main theorems in this paper as follows:

Theorem 2.2 *(The asymptotic behavior of the mass functions). Under the assumptions* (A_1), (A_2), (A_3) *and* (A_4) (*corresponding to the boundary conditions* (2.8)) *or* $(A_1)'$, (A_2), (A_3) *and* (A_4) (*corresponding to the boundary conditions* (2.10)), *let* $(n(x,t), m(x,t), u(x,t))$ *be a global weak solution to the initial boundary value problem* (2.1), (2.2), (2.3) *or* (2.1), (2.4), (2.5). *Then we have the following asymptotic behavior of the mass functions* $n(x,t)$, $m(x,t)$

$$\lim_{t\to\infty} \sup_{x\in[0,1]} n(x,t) = 0, \tag{2.13}$$

$$\lim_{t\to\infty} \sup_{x\in[0,1]} m(x,t) = 0. \tag{2.14}$$

Furthermore, we can get the decay rate estimates of the mass functions $n(x,t)$, $m(x,t)$ as follows:

Theorem 2.3 *(The decay rate of the mass functions). Under the assumptions of Theorem* 2.2, *let* $(n(x,t), m(x,t), u(x,t))$ *be a global weak solution to the initial boundary value problem* (2.1), (2.2), (2.3) *or* (2.1), (2.4), (2.5). *Then the following decay rate estimates on the mass functions* $n(x,t)$, $m(x,t)$ *hold*:

(i) *Under the boundary conditions* (2.8), *if* $0 < \beta < 1$ *or* $\beta > 1$, $\frac{\gamma-1}{\gamma-\beta} > 2$, *then*

$$n(x,t), m(x,t) \leqslant C(1+t)^{-\frac{\theta}{\gamma-1+2\beta}}, \tag{2.15}$$

for any $x \in [0,1]$.

If $\beta = 1$ *or* $\beta > 1$, $\frac{\gamma-1}{\gamma-\beta} \leqslant 2$, *we have*

$$n(x,t), m(x,t) \leqslant C(1+t)^{-\frac{\theta}{\gamma-1+2\beta}} \big(\ln(1+t)\big)^{\frac{1}{\gamma-1+2\beta}}, \tag{2.16}$$

for any $x \in [0,1]$.

(ii) *Under the boundary conditions* (2.10), *if* $0 < \beta < 1$ *or* $\beta > 1$, $\frac{\gamma-1}{\gamma-\beta} > 2$, *then*

$$n(x,t), m(x,t) \leqslant C(1+t)^{-\frac{\theta}{\gamma-1+4\beta}}, \tag{2.17}$$

for any $x \in [0,1]$.

If $\beta = 1$ *or* $\beta > 1$, $\frac{\gamma-1}{\gamma-\beta} \leqslant 2$, *we have*

$$n(x,t), m(x,t) \leqslant C(1+t)^{-\frac{\theta}{\gamma-1+4\beta}} \big(\ln(1+t)\big)^{\frac{1}{\gamma-1+4\beta}}, \tag{2.18}$$

for any $x \in [0,1]$, *where* θ *is defined by Lemma* 4.1.

3. A priori estimates and the asymptotic behavior of the mass functions

In this section, we will give some useful uniform a priori estimates of the solutions with respect to the time t. Then we study the asymptotic behavior of the mass functions $n(x,t)$ and $m(x,t)$ by using these uniform a priori estimates.

3.1. Uniform a priori estimates

Lemma 3.1 *(Some identities).*

$$c(x,t) = c_0(x), \tag{3.1}$$

$$\frac{d}{dt}\int_0^x u(y,t)\,dy = -\frac{d}{dt}\int_x^1 u(y,t)\,dy, \tag{3.2}$$

$$\big(c^\beta Q(m)^{\beta+1}u_x\big)(x,t) = \big(c^\gamma Q(m)^\gamma\big)(x,t) + \int_0^x u_t(y,t)\,dy = \big(c^\gamma Q(m)^\gamma\big)(x,t) - \int_x^1 u_t(y,t)\,dy, \tag{3.3}$$

$$\frac{1}{\beta\rho_l}\big(c^\beta Q(m)^\beta\big)(x,t) + \int_0^t c^\gamma Q(m)^\gamma(x,s)\,ds = \frac{1}{\beta\rho_l}c_0^\beta Q(m_0)^\beta - \int_0^x\int_0^t u_t(y,s)\,dy\,ds. \tag{3.4}$$

Proof. These identities can be obtained directly from (2.7). □

Lemma 3.2 *(Basic energy estimate). Under the conditions in Theorem* 2.2, *the following energy estimate holds*:

$$\int_0^1 \left(\frac{1}{2}u^2 + \frac{c^\gamma}{\rho_l(\gamma-1)} Q(m)^{\gamma-1}(x,t)\right) dx + \int_0^t \int_0^1 c^\beta Q(m)^{\beta+1} u_x^2 \, dx\, ds$$
$$= \int_0^1 \left(\frac{1}{2}u_0^2 + \frac{c_0^\gamma}{\rho_l(\gamma-1)} Q(m_0)^{\gamma-1}\right) dx \leqslant C. \tag{3.5}$$

Proof. Multiplying the second and the third equations of (2.7) by $c^\gamma Q(m)^{\gamma-2}$ and u, and integrating the resulting equations with respect to x over $[0,1]$, we get

$$\frac{d}{dt}\int_0^1 \left(\frac{u^2}{2} + \frac{c^\gamma}{\rho_l(\gamma-1)} Q(m)^{\gamma-1}\right) dx + \big(uP(c,m)\big)\big|_0^1 = \big(E(c,m)u_x u\big)\big|_0^1 - \int_0^1 E(c,m)u_x^2 \, dx. \tag{3.6}$$

Using the boundary conditions (2.8) or (2.10) we get

$$\frac{d}{dt}\int_0^1 \left(\frac{u^2}{2} + \frac{c^\gamma}{\rho_l(\gamma-1)} Q(m)^{\gamma-1}\right) dx + \int_0^1 c^\beta Q(m)^{\beta+1} u_x^2 \, dx = 0. \tag{3.7}$$

Then integrating it with respect to t over $[0,t]$, we get (3.5).

The proof of Lemma 3.2 is completed. □

Lemma 3.3 *(The uniform upper bound for the $cQ(m)$). Under the conditions of Theorem* 2.2, *we have for any* $x \in [0,1]$, $t > 0$,

$$0 \leqslant \big(cQ(m)\big)(x,t) \leqslant C. \tag{3.8}$$

Proof. By (3.4), we have

$$\begin{aligned}\frac{1}{\beta\rho_l}\big(c^\beta Q(m)^\beta\big)(x,t) + \int_0^t c^\gamma Q(m)^\gamma(x,s)\, ds &= \frac{1}{\beta\rho_l} c_0^\beta Q(m_0)^\beta - \int_0^x \big(u(y,t) - u_0(y)\big)\, dy \\ &\leqslant C + \int_0^1 \big|u(y,t)\big|\, dy + \int_0^1 \big|u_0(y)\big|\, dy \\ &\leqslant C + \left(\int_0^1 u^2\, dy\right)^{\frac{1}{2}} + \left(\int_0^1 u_0^2\, dy\right)^{\frac{1}{2}} \\ &\leqslant C.\end{aligned}$$

Here we have used Lemma 3.2 and the assumptions (A_1), $(A_1)'$, (A_2).

The proof of Lemma 3.3 is completed. □

Corollary 3.4. *For any $x \in [0,1]$ and $t > 0$,*

$$\int_0^t c^\gamma Q(m)^\gamma(x,s)\,ds \leqslant C. \tag{3.9}$$

Lemma 3.5. *Under the boundary conditions* (2.8), *we have for any $t > 0$,*

$$Q(m)(d,t) = Q(m_0)(d)\left(\frac{1}{(\gamma-\beta)\rho_l c_0^{\gamma-\beta}(d)Q(m_0)^{\gamma-\beta}t+1}\right)^{\frac{1}{\gamma-\beta}}, \quad d = 0,1. \tag{3.10}$$

Proof. By (3.2), we have

$$\frac{d}{dt}\int_0^1 u(y,t)\,dy = 0.$$

By taking $x = 1$ or $x = 0$ in (3.4), we have

$$\frac{1}{\beta\rho_l}c_0^\beta(d)Q(m)^\beta(d,t) + \int_0^t c_0^\gamma(d)Q(m)^\gamma(d,s)\,ds = \frac{1}{\beta\rho_l}c_0^\beta(d)Q(m_0)^\beta(d), \quad d = 0,1. \tag{3.11}$$

Since $\gamma > \beta$, the integral equation (3.11) yields

$$\frac{1}{\rho_l}c_0^\beta(d)Q(m)^{\beta-1}(d,t)Q(m)(d,t)_t + c_0^\gamma(d)Q(m)^\gamma(d,t) = 0, \tag{3.12}$$

which implies (3.10) by solving the ordinary differential equation (3.12).

The proof of Lemma 3.5 is completed. □

Corollary 3.6. *There exist positive constants C_1 and C_2 such that for any $t > 0$,*

$$C_1(1+t)^{-\frac{1}{\gamma-\beta}} \leqslant Q(m)(d,t) \leqslant C_2(1+t)^{-\frac{1}{\gamma-\beta}}.$$

Lemma 3.7. *For any positive integer n in (A_2), we have for any $t > 0$*

$$\int_0^1 u^{2n}\,dx + n(2n-1)\int_0^t\int_0^1 c^\beta Q(m)^{\beta+1}u^{2n-2}u_x^2\,dx\,ds \leqslant C_n, \tag{3.13}$$

where C_n is a positive constant depending on n, but independent of t.

Proof. Multiplying the third equation of (2.7) by $2nu^{2n-1}$ and integrating the resulting equation with respect to x over $[0,1]$, we have

$$\frac{d}{dt}\int_0^1 u^{2n}\,dx + 2n\big(u^{2n-1}P(c,m)\big)\big|_0^1 - 2n(2n-1)\int_0^1 c^{\gamma}\,Q(m)^{\gamma}u^{2n-2}u_x\,dx$$
$$= 2n\big(u^{2n-1}E(c,m)u_x\big)\big|_0^1 - 2n(2n-1)\int_0^1 c^{\beta}\,Q(m)^{\beta+1}u^{2n-2}u_x^2\,dx. \tag{3.14}$$

Using the boundary conditions (2.8) or (2.10), we have

$$\frac{d}{dt}\int_0^1 u^{2n}\,dx + 2n(2n-1)\int_0^1 c^{\beta}\,Q(m)^{\beta+1}u^{2n-2}u_x^2\,dx = 2n(2n-1)\int_0^1 c^{\gamma}\,Q(m)^{\gamma}u^{2n-2}u_x\,dx. \tag{3.15}$$

Integrating (3.15) with respect to t over $[0,t]$, we get

$$\int_0^1 u^{2n}\,dx + 2n(2n-1)\int_0^t\int_0^1 c^{\beta}\,Q(m)^{\beta+1}u^{2n-2}u_x^2\,dx\,ds$$
$$= \int_0^1 u_0^{2n}\,dx + 2n(2n-1)\int_0^t\int_0^1 c^{\gamma}\,Q(m)^{\gamma}u^{2n-2}u_x\,dx\,ds. \tag{3.16}$$

Applying Cauchy–Schwarz inequality to the last term in (3.16) yields

$$\int_0^1 u^{2n}\,dx + n(2n-1)\int_0^t\int_0^1 c^{\beta}\,Q(m)^{\beta+1}u^{2n-2}u_x^2\,dx\,ds$$
$$\leqslant \int_0^1 u_0^{2n}\,dx + n(2n-1)\int_0^t\int_0^1 c^{2\gamma-\beta}\,Q(m)^{2\gamma-\beta-1}u^{2n-2}\,dx\,ds. \tag{3.17}$$

Now we estimate the last term on the right-hand side in (3.17) as follows:

$$n(2n-1)\int_0^t\int_0^1 c^{2\gamma-\beta}\,Q(m)^{2\gamma-\beta-1}u^{2n-2}\,dx\,ds$$
$$= n(2n-1)\int_0^t\int_0^1 c^{\frac{\gamma}{n}+\gamma-\beta}\,Q(m)^{\frac{\gamma}{n}+\gamma-\beta-1}c^{\frac{n-1}{n}\gamma}\,Q(m)^{\frac{n-1}{n}\gamma}u^{2n-2}\,dx\,ds$$
$$\leqslant (2n-1)\int_0^t\int_0^1 c^{\gamma+n(\gamma-\beta)}\,Q(m)^{\gamma+n(\gamma-\beta-1)}\,dx\,ds + (n-1)(2n-1)\int_0^t\int_0^1 c^{\gamma}\,Q(m)^{\gamma}u^{2n}\,dx\,ds$$
$$= (2n-1)\int_0^t\int_0^1 \big(c\,Q(m)\big)^{n(\gamma-\beta-1)}c^{n}\big(c\,Q(m)\big)^{\gamma}\,dx\,ds + (n-1)(2n-1)\int_0^t\int_0^1 c^{\gamma}\,Q(m)^{\gamma}u^{2n}\,dx\,ds$$

$$
\begin{aligned}
&\leqslant C\int_0^t \max_{[0,1]}\big(cQ(m)\big)^{\gamma}\,ds + C\int_0^t \max_{[0,1]}\big(cQ(m)\big)^{\gamma}\int_0^1 u^{2n}\,dx\,ds\\
&\leqslant C + C\int_0^t \max_{[0,1]}\big(cQ(m)\big)^{\gamma}\int_0^1 u^{2n}\,dx\,ds. \qquad (3.18)
\end{aligned}
$$

Here we have used the Young inequality $ab \leqslant \frac{a^p}{p}+\frac{b^q}{q}$, where $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$, $a,b\geqslant 0$, the assumption (A_4), (3.8) and (3.9).

Substituting (3.18) into (3.17), we have

$$
\begin{aligned}
&\int_0^1 u^{2n}\,dx + n(2n-1)\int_0^t\int_0^1 c^{\beta}Q(m)^{\beta+1}u^{2n-2}u_x^2\,dx\,ds\\
&\quad\leqslant C + C\int_0^t \max_{[0,1]}\big(cQ(m)\big)^{\gamma}\int_0^1 u^{2n}\,dx\,ds. \qquad (3.19)
\end{aligned}
$$

By (3.19), (3.9) and Gronwall's inequality, we have

$$
\int_0^1 u^{2n}\,dx \leqslant C_n, \qquad (3.20)
$$

where C_n is a positive constant depending on n, but independent of t.

(3.19) and (3.20) show that (3.13) holds and this completes the proof of Lemma 3.7. □

Lemma 3.8. *We have the following uniform estimate on the derivative of the function $cQ(m)$,*

$$
\int_0^1 \big(\big(cQ(m)\big)^{\beta}\big)_x^2\,dx + \int_0^1\int_0^t \big(\big(cQ(m)\big)^{\frac{\beta+\gamma}{2}}\big)_x^2\,ds\,dx \leqslant C. \qquad (3.21)
$$

Proof. From (2.7), we have

$$
\begin{aligned}
\big(\big(cQ(m)\big)^{\beta}\big)_{xt} &= \big(\beta\big(cQ(m)\big)^{\beta-1}\big(cQ(m)\big)_t\big)_x\\
&= \big(\beta c^{\beta}Q(m)^{\beta-1}Q(m)_t\big)_x\\
&= -\beta\rho_l\big(c^{\beta}Q(m)^{\beta+1}u_x\big)_x\\
&= -\beta\rho_l\big(u_t+P(c,m)_x\big). \qquad (3.22)
\end{aligned}
$$

Multiplying (3.22) by $((cQ(m))^{\beta})_x$, integrating the resulting equation over $[0,1]\times[0,t]$ and integrating by parts, we get

$$
\begin{aligned}
\frac{1}{2}\int_0^1 \big(\big(cQ(m)\big)^\beta\big)_x^2\,dx = {}& \frac{1}{2}\int_0^1 \big(\big(c_0Q(m_0)\big)^\beta\big)_x^2\,dx - \beta\rho_l\int_0^1 u\big(\big(cQ(m)\big)^\beta\big)_x\,dx \\
& + \beta\rho_l\int_0^1 u_0\big(\big(c_0Q(m_0)\big)^\beta\big)_x\,dx + \beta\rho_l\int_0^1\int_0^t u\big(\big(cQ(m)\big)^\beta\big)_{xt}\,ds\,dx \\
& - \frac{4\gamma\beta^2\rho_l}{(\beta+\gamma)^2}\int_0^1\int_0^t \big(\big(cQ(m)\big)^{\frac{\beta+\gamma}{2}}\big)_x^2\,ds\,dx. \qquad (3.23)
\end{aligned}
$$

Substituting (3.22) into (3.23), we have

$$
\begin{aligned}
& \frac{1}{2}\int_0^1 \big(\big(cQ(m)\big)^\beta\big)_x^2\,dx + \frac{4\gamma\beta^2\rho_l}{(\beta+\gamma)^2}\int_0^1\int_0^t \big(\big(cQ(m)\big)^{\frac{\beta+\gamma}{2}}\big)_x^2\,ds\,dx \\
& = \frac{1}{2}\int_0^1 \big(\big(c_0Q(m_0)\big)^\beta\big)_x^2\,dx - \beta\rho_l\int_0^1 u\big(\big(cQ(m)\big)^\beta\big)_x\,dx \\
& + \beta\rho_l\int_0^1 u_0\big(\big(c_0Q(m_0)\big)^\beta\big)_x\,dx - (\beta\rho_l)^2\int_0^1\int_0^t uu_t\,ds\,dx \\
& - (\beta\rho_l)^2\int_0^1\int_0^t u\big(\big(cQ(m)\big)^\gamma\big)_x\,ds\,dx \\
& = \frac{1}{2}\int_0^1 \big(\big(c_0Q(m_0)\big)^\beta\big)_x^2\,dx - \beta\rho_l\int_0^1 u\big(\big(cQ(m)\big)^\beta\big)_x\,dx \\
& + \beta\rho_l\int_0^1 u_0\big(\big(c_0Q(m_0)\big)^\beta\big)_x\,dx - \frac{(\beta\rho_l)^2}{2}\int_0^1 u^2\,dx + \frac{(\beta\rho_l)^2}{2}\int_0^1 u_0^2\,dx \\
& - (\beta\rho_l)^2\int_0^t \big\{\big(cQ(m)\big)^\gamma(1,s)u(1,s) - \big(cQ(m)\big)^\gamma(0,s)u(0,s)\big\}\,ds \\
& + (\beta\rho_l)^2\int_0^1\int_0^t \big(cQ(m)\big)^\gamma u_x\,ds\,dx = \sum_{i=1}^{i=7} J_i. \qquad (3.24)
\end{aligned}
$$

Now we estimate J_1–J_7 as follows:

First, by the assumptions (A_1)–(A_3), or $(A_1)'$–(A_3), Lemma 3.2, Lemma 3.3, Corollary 3.4 and Cauchy–Schwarz inequality, we have

$$\begin{cases} J_1 \leqslant C, \\ J_2 \leqslant \dfrac{1}{4}\displaystyle\int_0^1 \left(\left(cQ(m)\right)^\beta\right)_x^2 dx + C\int_0^1 u^2\,dx \leqslant C + \dfrac{1}{4}\int_0^1 \left(\left(cQ(m)\right)^\beta\right)_x^2 dx, \\ J_3 \leqslant C\displaystyle\int_0^1 \left(\left(c_0Q(m_0)\right)^\beta\right)_x^2 dx + C\int_0^1 u_0^2\,dx \leqslant C, \\ J_4 \leqslant C, \\ J_5 \leqslant C, \\ J_7 \leqslant C\displaystyle\int_0^t\int_0^1 c_0(x)c^\beta Q(m)^{\beta+1}u_x^2\,dx\,ds + C\int_0^t\int_0^1 \left(cQ(m)\right)^{2\gamma-\beta-1}dx\,ds \\ \quad \leqslant C\max\limits_{[0,1]} c_0(x)\displaystyle\int_0^t\int_0^1 c^\beta Q(m)^{\beta+1}u_x^2\,dx\,ds + C\max\limits_{[0,1]\times[0,t]}\left(cQ(m)\right)^{\gamma-\beta-1}\int_0^t \max\limits_{[0,1]}\left(cQ(m)\right)^\gamma ds \\ \quad \leqslant C. \end{cases} \tag{3.25}$$

Now we estimate J_6, which is divided into two cases:

Case 1. When the initial masses connect to vacuum continuously (corresponding to the boundary condition (2.10)), we have $J_6 = 0$.

Case 2. When the initial masses connect to vacuum discontinuously (corresponding to the boundary condition (2.8)), we have by Young inequality and Lemma 3.5

$$\begin{aligned} J_6 &= -(\beta\rho_l)^2\int_0^t \left(cQ(m)\right)^{\gamma-\beta}(1,s)\left(\left(cQ(m)\right)^\beta(1,s)u(1,s)\right)ds \\ &\quad + \left(\beta\rho_l\right)^2\int_0^t \left(\left(cQ(m)\right)^{\gamma-\beta}(0,s)\left(cQ(m)\right)^\beta(0,s)u(0,s)\right)ds \\ &\leqslant C\int_0^t \left\{\left|\left(cQ(m)\right)^{n\beta}(1,s)u^n(1,s)\right| + \left|\left(cQ(m)\right)^{n\beta}(0,s)u^n(0,s)\right|\right\}ds \\ &\quad + C\int_0^t \left\{\left(cQ(m)\right)^{(\gamma-\beta)\frac{n}{n-1}}(1,s) + \left(cQ(m)\right)^{(\gamma-\beta)\frac{n}{n-1}}(0,s)\right\}ds \\ &\leqslant C + C\int_0^t \left\|\left(cQ(m)\right)^{n\beta}(\cdot,s)u^n(\cdot,s)\right\|_{L^\infty([0,1])}ds. \end{aligned} \tag{3.26}$$

Substituting (3.25) and (3.26) into (3.24), we have for Case 1 and Case 2

$$\frac{1}{2}\int_0^1 \left(\left(cQ(m)\right)^\beta\right)_x^2 dx + \frac{4\gamma\beta^2\rho_l}{(\beta+\gamma)^2}\int_0^1\int_0^t \left(\left(cQ(m)\right)^{\frac{\beta+\gamma}{2}}\right)_x^2 ds\,dx \leqslant C + J_8, \tag{3.27}$$

where

$$J_8 = C\int_0^t \left\| \big(cQ(m)\big)^{n\beta}(\cdot,s)u^n(\cdot,s)\right\|_{L^\infty([0,1])}\,ds.$$

By the embedding theorem $W^{1,1}([0,1]) \hookrightarrow L^\infty([0,1])$, we have

$$\begin{aligned}
J_8 &\leqslant C\int_0^t\int_0^1 \big(cQ(m)\big)^{n\beta}u^n(x,s)\,dx\,ds + C\int_0^t\int_0^1 \Big(\big(cQ(m)\big)^{n\beta}u^n(x,s)\Big)_x\,dx\,ds \\
&\leqslant C\int_0^t\int_0^1 \big(cQ(m)\big)^{\gamma}u^{2n}(x,s)\,dx\,ds + C\int_0^t\int_0^1 \big(cQ(m)\big)^{2n\beta-\gamma}\,dx\,ds \\
&\quad + \int_0^t\int_0^1 n\beta\big(cQ(m)\big)^{n\beta-1}\big(cQ(m)\big)_x u^n(x,s)\,dx\,ds + \int_0^t\int_0^1 n\big(cQ(m)\big)^{n\beta}u^{n-1}u_x\,dx\,ds \\
&\leqslant C\int_0^t \max_{[0,1]}\big(cQ(m)\big)^{\gamma}\left(\int_0^1 u^{2n}\,dx\right)ds + C\int_0^t \max_{[0,1]}\big(cQ(m)\big)^{\gamma}\int_0^1 \big(cQ(m)\big)^{2n\beta-2\gamma}\,dx\,ds \\
&\quad + C\int_0^t\int_0^1 \big(cQ(m)\big)^{2n\beta-\gamma-\beta}u^{2n}\,dx\,ds + \frac{2\gamma\beta^2\rho_l}{(\beta+\gamma)^2}\int_0^1\int_0^t \Big(\big(cQ(m)\big)^{\frac{\beta+\gamma}{2}}\Big)_x^2\,ds\,dx \\
&\quad + C\int_0^t\int_0^1 \big(cQ(m)\big)^{\beta+1}u^{2n-2}u_x^2\,dx\,ds + C\int_0^t\int_0^1 \big(cQ(m)\big)^{2n\beta-\beta-1}\,dx\,ds \\
&\leqslant C + \max\big(cQ(m)\big)^{2n\beta-2\gamma-\beta}\int_0^t \max_{[0,1]}\big(cQ(m)\big)^{\gamma}\left(\int_0^1 u^{2n}\,dx\right)ds \\
&\quad + \frac{2\gamma\beta^2\rho_l}{(\beta+\gamma)^2}\int_0^1\int_0^t \Big(\big(cQ(m)\big)^{\frac{\beta+\gamma}{2}}\Big)_x^2\,ds\,dx + C\max\big(c_0(x)\big)\int_0^t\int_0^1 c^{\beta}Q(m)^{\beta+1}u^{2n-2}u_x^2\,dx\,ds \\
&\quad + C\max\big(cQ(m)\big)^{2n\beta-\beta-1-\gamma}\int_0^t \max_{[0,1]}\big(cQ(m)\big)^{\gamma}\,ds \\
&\leqslant C + \frac{2\gamma\beta^2\rho_l}{(\beta+\gamma)^2}\int_0^1\int_0^t \Big(\big(cQ(m)\big)^{\frac{\beta+\gamma}{2}}\Big)_x^2\,ds\,dx. \qquad (3.28)
\end{aligned}$$

Here we have used Lemma 3.3, Corollary 3.4, Lemma 3.7, and $n \geqslant \frac{2\gamma+\beta}{2\beta}$.

Substituting (3.28) into (3.27), we get (3.21). This proves Lemma 3.8. □

3.2. Asymptotic behavior of $cQ(m)$

To apply the uniform estimates obtained above to study the asymptotic behavior of the mass functions $m(x,t)$, $n(x,t)$ with respect to the time t, we introduce the following lemma. The proof is quite simple and the detail is omitted.

Lemma 3.9. *Suppose that $g(t) \geqslant 0$ for $t \geqslant 0$, $g(t) \in L^1(0,\infty)$ and $g'(t) \in L^1(0,\infty)$. Then $\lim_{t\to\infty} g(t) = 0$.*

Now we prove Theorem 2.2. Let

$$g(t) = \int_0^1 \big(cQ(m)\big)^{\gamma}(x,t)\,dx. \tag{3.29}$$

Integrating (3.9) with respect to x over $[0,1]$, we have

$$\int_0^t \int_0^1 \big(cQ(m)\big)^{\gamma}(x,s)\,dx\,ds \leqslant C, \tag{3.30}$$

which implies $g(t) \in L^1(0,\infty)$.

Now we prove $g'(t) \in L^1(0,\infty)$. By the second equation of (2.7) and using Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\int_0^\infty \big|g'(t)\big|\,dt &= \gamma \int_0^\infty \left| \int_0^1 \big(cQ(m)\big)^{\gamma-1} c(x) Q(m)_t\,dy \right| dt \\
&= \int_0^\infty \left| \int_0^1 \gamma\rho_l c^{\gamma} Q(m)^{\gamma+1} u_x\,dx \right| dt \\
&\leqslant C \int_0^\infty \int_0^1 c^{\beta} Q(m)^{1+\beta} u_x^2\,dx\,dt + C \int_0^\infty \int_0^1 c^{2\gamma-\beta} Q(m)^{2\gamma+1-\beta}\,dx\,dt.
\end{aligned} \tag{3.31}$$

By (3.8), (3.9), and the assumptions (A_1) or $(A_1)'$, we can estimate the last term on the right-hand side in (3.31) as follows:

$$\begin{aligned}
\int_0^t \int_0^1 c^{2\gamma-\beta} Q(m)^{2\gamma+1-\beta}\,dx\,ds &\leqslant \int_0^t \max_{[0,1]}\big(cQ(m)\big)^{\gamma} \int_0^1 c^{\gamma-\beta} Q(m)^{\gamma+1-\beta}\,dx\,ds \\
&\leqslant \max_{[0,1]\times[0,t]}\big(cQ(m)\big)^{\gamma+1-\beta} \left(\int_0^1 \frac{1}{c_0(x)}\,dx \right) \int_0^t \max_{[0,1]}\big(cQ(m)\big)^{\gamma}\,ds \\
&\leqslant C.
\end{aligned}$$

Substituting the above inequality into (3.31) and using Lemma 3.2, we deduce $g'(t) \in L^1(0,\infty)$.

Consequently,

$$\lim_{t\to\infty} g(t) = 0. \tag{3.32}$$

By (3.32), Lemma 3.3 or Hölder inequality, we have

$$\lim_{t\to\infty} \int_0^1 \big(cQ(m)\big)^{\lambda}(x,t)\,dx = 0, \tag{3.33}$$

for any $0 < \lambda < \infty$.

Now we prove Theorem 2.2, which is divided into two cases:

Case 1. When the initial masses connect to vacuum continuously (corresponding to the boundary condition (2.10)), choosing $k > \beta > 0$ and applying (3.33), Lemma 3.8 and Hölder inequality, we have

$$\begin{aligned}
0 \leqslant \big(cQ(m)\big)^k &= \int_0^x \big(\big(cQ(m)\big)^k\big)_y\,dy \\
&= \int_0^x k\big(cQ(m)\big)^{k-\beta}\big(cQ(m)\big)^{\beta-1}\big(cQ(m)\big)_y\,dy \\
&= \frac{k}{\beta}\int_0^x \big(cQ(m)\big)^{k-\beta}\big(\big(cQ(m)\big)^{\beta}\big)_y\,dy \\
&\leqslant C\left(\int_0^1 \big(cQ(m)\big)^{2k-2\beta}\,dx\right)^{\frac12}\left(\int_0^1 \big(\big(cQ(m)\big)^{\beta}\big)_x^2\,dx\right)^{\frac12} \\
&\leqslant C\left(\int_0^1 \big(cQ(m)\big)^{2k-2\beta}\,dx\right)^{\frac12} \to 0, \quad \text{as } t\to\infty.
\end{aligned}$$

Case 2. When the initial masses connect to vacuum discontinuously (corresponding to the boundary condition (2.8)), choosing $k > \beta > 0$ and applying (3.33), Corollary 3.6, Lemma 3.8 and Hölder inequality, we have

$$\begin{aligned}
0 \leqslant \big(cQ(m)\big)^k &= \big(cQ(m)\big)^k(0,t) + \int_0^x \big(\big(cQ(m)\big)^k\big)_y\,dy \\
&\leqslant C(1+t)^{-\frac{k}{\gamma-\beta}} + \left(\int_0^1 \big(cQ(m)\big)^{2k-2\beta}\,dx\right)^{\frac12}\left(\int_0^1 \big(\big(cQ(m)\big)^{\beta}\big)_x^2\,dx\right)^{\frac12} \\
&\leqslant C(1+t)^{-\frac{k}{\gamma-\beta}} + C\left(\int_0^1 \big(cQ(m)\big)^{2k-2\beta}\,dx\right)^{\frac12} \to 0, \quad \text{as } t\to\infty.
\end{aligned}$$

Combining the above two cases, we have

$$\big(cQ\,(m)\big)(x,t) \to 0, \quad \text{as } t \to \infty,$$

which implies

$$\lim_{t\to\infty} \frac{n}{m} \cdot \frac{m}{\rho_l - m} = \lim_{t\to\infty} \frac{n}{\rho_l - m} = 0.$$

Thus

$$\lim_{t\to\infty} n(x,t) = \lim_{t\to\infty} \frac{n}{\rho_l - m} \cdot (\rho_l - m) = 0,$$

and

$$\lim_{t\to\infty} m(x,t) = 0,$$

for any $x \in [0,1]$.

This completes the proof of Theorem 2.2.

4. Decay rates of the mass functions

Now we are in the position to estimate the stabilization rates of the mass functions $m(x,t)$, $n(x,t)$ as $t \to \infty$.

Firstly, since only the decay rates of the mass functions will be discussed, we assume without loss of generality that $\int_0^1 u_0(y)\,dy = 0$. Otherwise, we set

$$v = u - \int_0^1 u_0(y)\,dy,$$

so that the form of (2.7) remains unchanged but $\int_0^1 v_0(y)\,dy = 0$.

Secondly, introduce a new function $w(x,t)$ defined as follows (cf. [13]):

$$w(x,t) = \rho_l u(x,t) - \frac{1}{1+t}\int_0^x \frac{1}{Q\,(m)}\,dy + \frac{1}{1+t}\int_0^1\int_0^x \frac{1}{Q\,(m)}\,dy\,dx. \tag{4.1}$$

By direct calculation, we have

$$w_x = \rho_l u_x - \frac{1}{(1+t)Q\,(m)}, \tag{4.2}$$

and

$$w_t + \frac{w}{1+t} = \rho_l u_t. \tag{4.3}$$

Here we have used the fact (see (3.1)) that

$$\int_0^1 u(x,t)\,dx = \int_0^1 u_0(x)\,dx = 0.$$

Thus the auxiliary functions w and $Q(m)$ satisfy the following

$$\begin{cases} \partial_t c = 0, \\ \partial_t Q(m) + Q(m)^2 \partial_x w + \dfrac{Q(m)}{1+t} = 0, \\ \partial_t w + \dfrac{w}{1+t} = \partial_x \left(c^\beta Q(m)^{\beta+1} \partial_x w + \dfrac{(cQ(m))^\beta}{1+t} - \rho_l \big(cQ(m)\big)^\gamma \right). \end{cases} \tag{4.4}$$

Then we have

Lemma 4.1. *Let* $(c(x), u(x,t), Q(m)(x,t))$ *be a global weak solution to the fixed boundary value problem* (2.7), (2.8), (2.9), *or* (2.7), (2.10), (2.11). *Then for any* $\beta > 0$, $\gamma \geqslant 1+\beta$, *the following estimates hold for any* $t > 0$,

***Case* I: $0 < \beta < 1$.**

$$\begin{aligned} &\frac{1}{2}(1+t)^\theta \int_0^1 w^2\,dx + \frac{(1+t)^{\theta-1}}{1-\beta} \int_0^1 c^\beta Q(m)^{\beta-1}\,dx + \frac{\rho_l (1+t)^\theta}{\gamma-1} \int_0^1 c^\gamma Q(m)^{\gamma-1}\,dx \\ &\quad + \left(1-\frac{\theta}{2}\right) \int_0^t (1+s)^{\theta-1} \int_0^1 w^2\,dx\,ds + \int_0^t (1+s)^\theta \int_0^1 c^\beta Q(m)^{1+\beta} w_x^2\,dx\,ds \\ &\quad + \frac{\beta-\theta}{1-\beta} \int_0^t (1+s)^{\theta-2} \int_0^1 c^\beta Q(m)^{\beta-1}\,dx\,ds + \rho_l \frac{\gamma-1-\theta}{\gamma-1} \int_0^t (1+s)^{\theta-1} \int_0^1 c^\gamma Q(m)^{\gamma-1}\,dx\,ds \\ &\leqslant C, \end{aligned} \tag{4.5}$$

where

$$\theta = \min\{\gamma-1, \beta\} = \beta. \tag{4.6}$$

***Case* II: $\beta = 1$.**

$$\begin{aligned} &\frac{1}{2}(1+t)^\theta \int_0^1 w^2\,dx + \frac{\rho_l (1+t)^\theta}{\gamma-1} \int_0^1 c^\gamma Q(m)^{\gamma-1}\,dx \\ &\quad + \left(1-\frac{\theta}{2}\right) \int_0^t (1+s)^{\theta-1} \int_0^1 w^2\,dx\,ds + \int_0^t (1+s)^\theta \int_0^1 cQ(m)^2 w_x^2\,dx\,ds \end{aligned}$$

$$
\begin{aligned}
&+ \rho_l \frac{\gamma - 1 - \theta}{\gamma - 1} \int_0^t (1+s)^{\theta-1} \int_0^1 c^\gamma Q(m)^{\gamma-1}\,dx\,ds \\
&\leqslant C + C\ln(1+t), \qquad (4.7)
\end{aligned}
$$

where $\theta = 1$.

***Case* III: $\beta > 1$.**

$$
\begin{aligned}
&\frac{1}{2}(1+t)^\theta \int_0^1 w^2\,dx + \frac{\rho_l(1+t)^\theta}{2(\gamma-1)} \int_0^1 c^\gamma Q(m)^{\gamma-1}\,dx + \left(1 - \frac{\theta}{2}\right) \int_0^t (1+s)^{\theta-1} \int_0^1 w^2\,dx\,ds \\
&\quad + \int_0^t (1+s)^\theta \int_0^1 c^\beta Q(m)^{1+\beta} w_x^2\,dx\,ds + \frac{\rho_l(\gamma-1-\theta)}{2(\gamma-1)} \int_0^t (1+s)^{\theta-1} \int_0^1 c^\gamma Q(m)^{\gamma-1}\,dx\,ds \\
&\leqslant C + C\big(\ln(1+t)\big)^l, \qquad (4.8)
\end{aligned}
$$

where

$$
\theta = \begin{cases} 2, & \textit{for } \frac{\gamma-1}{\gamma-\beta} > 2, \\ \frac{\gamma-1}{\gamma-\beta}, & \textit{for } \frac{\gamma-1}{\gamma-\beta} \leqslant 2, \end{cases} \qquad (4.9)
$$

and $l = 0$, *when* $\frac{\gamma-1}{\gamma-\beta} > 2$, *whereas* $l = 1$, *when* $\frac{\gamma-1}{\gamma-\beta} \leqslant 2$.

Proof. Multiplying $(4.4)_3$ by w, integrating the resulting equation with respect to x over $[0,1]$, using integration by parts, we obtain by the boundary conditions (2.8) or (2.10)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_0^1 w^2\,dx + \frac{1}{1+t} \int_0^1 w^2\,dx \\
&= \int_0^1 \big(c^\beta Q(m)^{\beta+1} w_x\big)_x w\,dx + \frac{1}{1+t} \int_0^1 \big(c^\beta Q(m)^\beta\big)_x w\,dx - \rho_l \int_0^1 \big(c^\gamma Q(m)^\gamma\big)_x w\,dx \\
&= c^\beta Q(m)^{\beta+1} w_x w\Big|_0^1 + \frac{1}{1+t} c^\beta Q(m)^\beta w\Big|_0^1 - \rho_l c^\gamma Q(m)^\gamma w\Big|_0^1 \\
&\quad - \int_0^1 c^\beta Q(m)^{\beta+1} w_x^2\,dx - \frac{1}{1+t} \int_0^1 c^\beta Q(m)^\beta w_x\,dx + \rho_l \int_0^1 c^\gamma Q(m)^\gamma w_x\,dx \\
&= -\int_0^1 c^\beta Q(m)^{\beta+1} w_x^2\,dx - \frac{1}{1+t} \int_0^1 c^\beta Q(m)^\beta w_x\,dx + \rho_l \int_0^1 c^\gamma Q(m)^\gamma w_x\,dx, \qquad (4.10)
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 w^2\,dx+\frac{1}{1+t}\int_0^1 w^2\,dx+\int_0^1 c^\beta Q(m)^{\beta+1}w_x^2\,dx\\
&\quad=-\frac{1}{1+t}\int_0^1 c^\beta Q(m)^\beta w_x\,dx+\rho_l\int_0^1 c^\gamma Q(m)^\gamma w_x\,dx. \qquad (4.11)
\end{aligned}
$$

Now we will prove (4.5), (4.7) and (4.8).

Case I: $0<\beta<1$. (The proof of (4.5).)

Notes that

$$
w_x=\rho_l u_x-\frac{1}{(1+t)Q(m)}=\left(\frac{1}{Q(m)}\right)_t-\frac{1}{(1+t)Q(m)}.
$$

Thus we can estimate the first and second terms on the right-hand side in (4.11) as following:

$$
\begin{aligned}
&-\frac{1}{1+t}\int_0^1 c^\beta Q(m)^\beta w_x\,dx\\
&\quad=-\frac{1}{1+t}\int_0^1 c^\beta Q(m)^\beta\left\{\left(\frac{1}{Q(m)}\right)_t-\frac{1}{(1+t)Q(m)}\right\}dx\\
&\quad=-\frac{1}{(1-\beta)(1+t)}\int_0^1 c^\beta\left(Q(m)^{\beta-1}\right)_t dx+\frac{1}{(1+t)^2}\int_0^1 c^\beta Q(m)^{\beta-1}\,dx, \qquad (4.12)
\end{aligned}
$$

and

$$
\begin{aligned}
\rho_l\int_0^1 c^\gamma Q(m)^\gamma w_x\,dx&=\rho_l\int_0^1 c^\gamma Q(m)^\gamma\left\{\left(\frac{1}{Q(m)}\right)_t-\frac{1}{(1+t)Q(m)}\right\}dx\\
&=\frac{\rho_l}{1-\gamma}\int_0^1 c^\gamma\left(Q(m)^{\gamma-1}\right)_t dx-\frac{\rho_l}{1+t}\int_0^1 c^\gamma Q(m)^{\gamma-1}\,dx. \qquad (4.13)
\end{aligned}
$$

Substituting (4.12) and (4.13) into (4.11), we get

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1\left(\frac{w^2}{2}+\frac{\rho_l}{\gamma-1}c^\gamma Q(m)^{\gamma-1}\right)dx+\frac{1}{1+t}\int_0^1 w^2\,dx\\
&\quad+\int_0^1 c^\beta Q(m)^{\beta+1}w_x^2\,dx+\frac{\rho_l}{1+t}\int_0^1 c^\gamma Q(m)^{\gamma-1}\,dx\\
&\quad=\frac{1}{(\beta-1)(1+t)}\int_0^1 c^\beta\left(Q(m)^{\beta-1}\right)_t dx+\frac{1}{(1+t)^2}\int_0^1 c^\beta Q(m)^{\beta-1}\,dx. \qquad (4.14)
\end{aligned}
$$

Multiplying (4.14) by $(1+t)^{\theta}$ for some θ to be determined later, we deduce for any $0<\beta<1$

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}(1+t)^{\theta}\int_0^1 w^2\,dx+\frac{(1+t)^{\theta-1}}{1-\beta}\int_0^1 c^{\beta}Q(m)^{\beta-1}\,dx+\frac{\rho_l(1+t)^{\theta}}{\gamma-1}\int_0^1 c^{\gamma}Q(m)^{\gamma-1}\,dx\right\}\\
&\quad+\left(1-\frac{\theta}{2}\right)(1+t)^{\theta-1}\int_0^1 w^2\,dx+(1+t)^{\theta}\int_0^1 c^{\beta}Q(m)^{1+\beta}w_x^2\,dx\\
&\quad+\frac{\beta-\theta}{1-\beta}(1+t)^{\theta-2}\int_0^1 c^{\beta}Q(m)^{\beta-1}\,dx+\rho_l\frac{\gamma-1-\theta}{\gamma-1}(1+t)^{\theta-1}\int_0^1 c^{\gamma}Q(m)^{\gamma-1}\,dx\\
&=0.
\end{aligned}\tag{4.15}$$

Taking $\theta=\min\{\beta,\gamma-1\}=\beta$ in (4.15) and integrating (4.15) with respect to t over $[0,t]$, we deduce (4.5).

Consequently,

$$\int_0^1 c^{\gamma}Q(m)^{\gamma-1}\,dx\leqslant C(1+t)^{-\theta}.\tag{4.16}$$

Case II: $\beta=1$. (The proof of (4.7).) Under this case, the first term on the right-hand side in (4.11) can be rewritten as

$$\begin{aligned}
-\frac{1}{1+t}\int_0^1 cQ(m)w_x\,dx&=-\frac{1}{1+t}\int_0^1 cQ(m)\left\{\left(\frac{1}{Q(m)}\right)_t-\frac{1}{(1+t)Q(m)}\right\}dx\\
&=\frac{1}{(1+t)}\int_0^1 c\big(\ln Q(m)\big)_t\,dx+\frac{1}{(1+t)^2}\int_0^1 c_0(x)\,dx.
\end{aligned}\tag{4.17}$$

Similar to (4.15), we have:

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}(1+t)^{\theta}\int_0^1 w^2\,dx+\frac{\rho_l(1+t)^{\theta}}{\gamma-1}\int_0^1 c^{\gamma}Q(m)^{\gamma-1}\,dx\right\}+\left(1-\frac{\theta}{2}\right)(1+t)^{\theta-1}\int_0^1 w^2\,dx\\
&\quad+(1+t)^{\theta}\int_0^1 cQ(m)^2w_x^2\,dx+\rho_l\frac{\gamma-1-\theta}{\gamma-1}(1+t)^{\theta-1}\int_0^1 c^{\gamma}Q(m)^{\gamma-1}\,dx\\
&=\frac{d}{dt}\left\{(1+t)^{\theta-1}\int_0^1 c\ln\big(Q(m)\big)\,dx\right\}+(1+t)^{\theta-2}\int_0^1 c_0(x)\,dx\\
&\quad+(1-\theta)(1+t)^{\theta-2}\int_0^1 c\ln\big(Q(m)\big)\,dx.
\end{aligned}\tag{4.18}$$

By using $\ln x \leqslant x - 1$ for any $x > 0$ and Lemma 3.3, we have

$$\int_0^1 c \ln Q(m)\, dx \leqslant \int_0^1 c Q(m)\, dx \leqslant C$$

and the assumption (A_1) or $(A_1)'$ implies that

$$\int_0^1 c_0(x)\, dx \leqslant C.$$

Taking $\theta = 1$ in (4.18) and integrating (4.18) with respect to t over $[0, t]$, we deduce (4.7).

Consequently,

$$\int_0^1 c^\gamma Q(m)^{\gamma-1}\, dx \leqslant C(1+t)^{-\theta} \ln(1+t). \tag{4.19}$$

Case III: $\beta > 1$. (The proof of (4.8).)

Rewrite (4.15) as

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}(1+t)^\theta \int_0^1 w^2\, dx + \frac{\rho_l (1+t)^\theta}{\gamma - 1} \int_0^1 c^\gamma Q(m)^{\gamma-1}\, dx\right\} \\
&\quad + \left(1 - \frac{\theta}{2}\right)(1+t)^{\theta-1} \int_0^1 w^2\, dx + (1+t)^\theta \int_0^1 c^\beta Q(m)^{1+\beta} w_x^2\, dx \\
&\quad + \rho_l \frac{\gamma - 1 - \theta}{\gamma - 1}(1+t)^{\theta-1} \int_0^1 c^\gamma Q(m)^{\gamma-1}\, dx \\
&= \frac{d}{dt}\left\{\frac{(1+t)^{\theta-1}}{\beta - 1} \int_0^1 c^\beta Q(m)^{\beta-1}\, dx\right\} + \frac{\beta - \theta}{\beta - 1}(1+t)^{\theta-2} \int_0^1 c^\beta Q(m)^{\beta-1}\, dx.
\end{aligned} \tag{4.20}$$

Integrating (4.20) with respect to t over $[0, t]$, we have

$$\begin{aligned}
&\frac{1}{2}(1+t)^\theta \int_0^1 w^2\, dx + \frac{\rho_l (1+t)^\theta}{\gamma - 1} \int_0^1 c^\gamma Q(m)^{\gamma-1}\, dx \\
&\quad + \left(1 - \frac{\theta}{2}\right) \int_0^t (1+s)^{\theta-1} \int_0^1 w^2\, dx\, ds + \int_0^t (1+s)^\theta \int_0^1 c^\beta Q(m)^{1+\beta} w_x^2\, dx\, ds \\
&\quad + \rho_l \frac{\gamma - 1 - \theta}{\gamma - 1} \int_0^t (1+s)^{\theta-1} \int_0^1 c^\gamma Q(m)^{\gamma-1}\, dx\, ds
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^1 w_0^2\,dx + \frac{\rho_l}{\gamma-1}\int_0^1 c_0^\gamma Q(m_0)^{\gamma-1}\,dx - \frac{1}{\beta-1}\int_0^1 c_0^\beta Q(m_0)^{\beta-1}\,dx \\
&\quad + \frac{(1+t)^{\theta-1}}{\beta-1}\int_0^1 c^\beta Q(m)^{\beta-1}\,dx + \frac{\beta-\theta}{\beta-1}\int_0^t (1+s)^{\theta-2}\int_0^1 c^\beta Q(m)^{\beta-1}\,dx\,ds \\
&= -\frac{1}{\beta-1}\int_0^1 c_0^\beta Q(m_0)^{\beta-1}\,dx + \frac{1}{2}\int_0^1 w_0^2\,dx + \frac{\rho_l}{\gamma-1}\int_0^1 c_0^\gamma Q(m_0)^{\gamma-1}\,dx + I_1 + I_2. \qquad (4.21)
\end{aligned}
$$

By Young inequality, we have

$$
\begin{aligned}
I_1 &= \frac{(1+t)^{\theta-1}}{\beta-1}\int_0^1 c^\beta Q(m)^{\beta-1}\,dx \\
&= \frac{1}{\beta-1}\int_0^1 \big(cQ(m)\big)^{\beta-1}(1+t)^{\frac{(\beta-1)\theta}{\gamma-1}} c(1+t)^{\theta-1-\frac{(\beta-1)\theta}{\gamma-1}}\,dx \\
&\leqslant \frac{\rho_l(1+t)^\theta}{2(\gamma-1)}\int_0^1 c^\gamma Q(m)^{\gamma-1}\,dx + C(1+t)^{(\theta-1-\frac{(\beta-1)\theta}{\gamma-1})\frac{\gamma-1}{\gamma-\beta}}\int_0^1 c_0(x)\,dx \\
&\leqslant \frac{\rho_l(1+t)^\theta}{2(\gamma-1)}\int_0^1 c^\gamma Q(m)^{\gamma-1}\,dx + C(1+t)^{(\theta-1-\frac{(\beta-1)\theta}{\gamma-1})\frac{\gamma-1}{\gamma-\beta}}, \qquad (4.22)
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \frac{\beta-\theta}{\beta-1}\int_0^t (1+s)^{\theta-2}\int_0^1 c^\beta Q(m)^{\beta-1}\,dx\,ds \\
&\leqslant \frac{\rho_l(\gamma-1-\theta)}{2(\gamma-1)}\int_0^t (1+s)^{\theta-1}\int_0^1 c^\gamma Q(m)^{\gamma-1}\,dx + C\int_0^t (1+s)^{(\theta-2-\frac{(\beta-1)(\theta-1)}{\gamma-1})\frac{\gamma-1}{\gamma-\beta}}\,ds. \qquad (4.23)
\end{aligned}
$$

Substituting (4.22) and (4.23) into (4.21), we have

$$
\begin{aligned}
&\frac{1}{2}(1+t)^\theta\int_0^1 w^2\,dx + \frac{\rho_l(1+t)^\theta}{2(\gamma-1)}\int_0^1 c^\gamma Q(m)^{\gamma-1}\,dx \\
&\quad + \left(1-\frac{\theta}{2}\right)\int_0^t (1+s)^{\theta-1}\int_0^1 w^2\,dx\,ds + \int_0^t (1+s)^\theta\int_0^1 c^\beta Q(m)^{1+\beta} w_x^2\,dx\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\rho_l(\gamma-1-\theta)}{2(\gamma-1)}\int_0^t(1+s)^{\theta-1}\int_0^1 c^\gamma Q(m)^{\gamma-1}\,dx\,ds\\
&\leqslant C+C(1+t)^{\theta-\frac{\gamma-1}{\gamma-\beta}}+C\int_0^t(1+s)^{\theta-1-\frac{\gamma-1}{\gamma-\beta}}\,ds.
\end{aligned}
\tag{4.24}
$$

Taking $\theta=2$, when $\frac{\gamma-1}{\gamma-\beta}>2$ in (4.24), we have

$$
\int_0^1 c^\gamma Q(m)^{\gamma-1}\,dx\leqslant C(1+t)^{-\theta}.
\tag{4.25}
$$

Taking $\theta=\frac{\gamma-1}{\gamma-\beta}$, when $\frac{\gamma-1}{\gamma-\beta}\leqslant 2$ in (4.24), we have

$$
\int_0^1 c^\gamma Q(m)^{\gamma-1}\,dx\leqslant C(1+t)^{-\theta}\ln(1+t).
\tag{4.26}
$$

This completes the proof of Lemma 4.1. $\square$

Proof of Theorem 2.3. Under the boundary condition (2.8), for $0<\beta<1$ or $\beta>1$, $\frac{\gamma-1}{\gamma-\beta}>2$, choosing some constant $2k=\gamma-1+2\beta$ and using the assumption (A_1), Corollary 3.6, Lemma 3.8, (4.16) and (4.25), we have

$$
\begin{aligned}
\big(cQ(m)\big)^k(x,t)&=\big(cQ(m)\big)^k(0,t)+\int_0^x\Big(\big(cQ(m)\big)^k\Big)_y(y,t)\,dy\\
&\leqslant C(1+t)^{-\frac{k}{\gamma-\beta}}+C\left(\int_0^1\Big(\big(cQ(m)\big)^\beta\Big)_x^2\,dx\right)^{\frac12}\left(\int_0^1\big(cQ(m)\big)^{2k-2\beta}\,dx\right)^{\frac12}\\
&\leqslant C(1+t)^{-\frac{k}{\gamma-\beta}}+C\left(\int_0^1\frac{1}{c_0}c^\gamma Q(m)^{\gamma-1}\,dx\right)^{\frac12}\\
&\leqslant C(1+t)^{-\frac{k}{\gamma-\beta}}+C(1+t)^{-\frac{\theta}{2}}\\
&\leqslant C(1+t)^{-\frac{\theta}{2}},
\end{aligned}
\tag{4.27}
$$

which implies

$$
\big(cQ(m)\big)(x,t)\leqslant C(1+t)^{-\frac{\theta}{2k}}=C(1+t)^{-\frac{\theta}{\gamma-1+2\beta}},
\tag{4.28}
$$

i.e.,

$$
\frac{n(x,t)}{\rho_l-m(x,t)}\leqslant C(1+t)^{-\frac{\theta}{\gamma-1+2\beta}}.
\tag{4.29}
$$

Thus

$$n(x,t) = \frac{n(x,t)}{\rho_l - m(x,t)} \cdot \big(\rho_l - m(x,t)\big) \leqslant C(1+t)^{-\frac{\theta}{\gamma-1+2\beta}}, \tag{4.30}$$

and

$$m(x,t) = n(x,t) \cdot c(x)^{-1} \leqslant C(1+t)^{-\frac{\theta}{\gamma-1+2\beta}}, \tag{4.31}$$

for any $x \in [0,1]$.

Similarly, if $\beta = 1$ or $\beta > 1$, $\frac{\gamma-1}{\gamma-\beta} \leqslant 2$, we have

$$\big(cQ(m)\big)^k(x,t) \leqslant C(1+t)^{-\frac{\theta}{2}}\sqrt{\ln(1+t)},$$

which implies

$$n(x,t) \leqslant C(1+t)^{-\frac{\theta}{\gamma-1+2\beta}}\big(\ln(1+t)\big)^{\frac{1}{\gamma-1+2\beta}}, \tag{4.32}$$

and

$$m(x,t) \leqslant C(1+t)^{-\frac{\theta}{\gamma-1+2\beta}}\big(\ln(1+t)\big)^{\frac{1}{\gamma-1+2\beta}}, \tag{4.33}$$

for any $x \in [0,1]$. Here we have used (4.19) and (4.26).

Under the boundary condition (2.10), for $0 < \beta < 1$ or $\beta > 1$, $\frac{\gamma-1}{\gamma-\beta} > 2$, choosing some constant $2k_1 = \frac{\gamma-1}{2} + 2\beta$ and using the assumptions $(A_1)'$, Lemma 3.8, (4.16), (4.25) and Hölder's inequality, we have

$$\begin{aligned}
\big(cQ(m)\big)^{k_1}(x,t) &= \int_0^x \Big(\big(cQ(m)\big)^{k_1}\Big)_y(y,t)\,dy \\
&\leqslant C\left(\int_0^1 \Big(\big(cQ(m)\big)^{\beta}\Big)_x^2\,dx\right)^{\frac{1}{2}}\left(\int_0^1 \big(cQ(m)\big)^{2k_1-2\beta}\,dx\right)^{\frac{1}{2}} \\
&\leqslant C\left(\int_0^1 \frac{1}{c^{\frac{1}{2}}} \cdot c^{\frac{\gamma}{2}}Q(m)^{\frac{\gamma-1}{2}}\,dx\right)^{\frac{1}{2}} \\
&\leqslant C\left(\int_0^1 c^{\gamma}Q(m)^{\gamma-1}\,dx\right)^{\frac{1}{4}}\left(\int_0^1 \frac{1}{c_0(x)}\,dx\right)^{\frac{1}{4}} \\
&\leqslant C(1+t)^{-\frac{\theta}{4}},
\end{aligned} \tag{4.34}$$

which implies

$$\big(cQ(m)\big)(x,t) \leqslant C(1+t)^{-\frac{\theta}{4k_1}} = C(1+t)^{-\frac{\theta}{\gamma-1+4\beta}}, \tag{4.35}$$

i.e.,

$$\frac{n(x,t)}{\rho_l - m(x,t)} \leqslant C(1+t)^{-\frac{\theta}{\gamma-1+4\beta}}. \tag{4.36}$$

Thus

$$n(x,t) = \frac{n(x,t)}{\rho_l - m(x,t)} \cdot \big(\rho_l - m(x,t)\big) \leqslant C(1+t)^{-\frac{\theta}{\gamma-1+4\beta}}, \tag{4.37}$$

and

$$m(x,t) \leqslant C(1+t)^{-\frac{\theta}{\gamma-1+4\beta}}, \tag{4.38}$$

for any $x \in [0,1]$.

Similarly, if $\beta = 1$ or $\beta > 1$, $\frac{\gamma-1}{\gamma-\beta} \leqslant 2$, we have

$$\big(cQ(m)\big)^{k_1}(x,t) \leqslant C(1+t)^{-\frac{\theta}{4}}\big(\ln(1+t)\big)^{\frac{1}{4}},$$

then

$$n(x,t) \leqslant C(1+t)^{-\frac{\theta}{\gamma-1+4\beta}}\big(\ln(1+t)\big)^{\frac{1}{\gamma-1+4\beta}}, \tag{4.39}$$

and

$$m(x,t) \leqslant C(1+t)^{-\frac{\theta}{\gamma-1+4\beta}}\big(\ln(1+t)\big)^{\frac{1}{\gamma-1+4\beta}}, \tag{4.40}$$

for any $x \in [0,1]$. Here we have used (4.19) and (4.26).

The proof of Theorem 2.3 is completed. $\square$

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