Journal of Differential Equations **164**, 92–109 (2000)
doi:10.1006/jdeq.1999.3743, available online at http://www.idealibrary.com on IDEAL®

Some Nonlinear Wave Equations with Acoustic Boundary Conditions

Cícero Lopes Frota[1]

Departamento de Matemática, Universidade Estadual de Maringá, Av. Colombo, 5790, CEP 87020-900, Maringá, PR, Brazil

and

Jerome A. Goldstein

We prove the existence and uniqueness of global solutions to the mixed problem for the Carrier equation

$$u_{tt} - M\left(\int_{\Omega} u^2\,dx\right)\Delta u + |u_t'|^{\alpha}\,u_t' = 0$$

with acoustic boundary conditions.

1. INTRODUCTION

Beale and Rosencrans [3] introduced acoustic boundary conditions into the rigorous wave propagation literature, and Beale carried out a detailed analysis of them for the wave equation in both bounded domains [1] and exterior domains [2]. See also [8] for a classical heuristic discussion. The idea is that each boundary point acts as a spring. The boundary is "locally reacting" in that these springs do not influence one another. Think of the solution of the wave equation $u_{tt} = c^2 \Delta u$ as the velocity potential of a fluid (in three dimensions) undergoing acoustic wave motion; the acoustic boundary condition says that each point on the boundary reacts to the

[1] This work was done while this author was a visiting scholar at The University of Memphis in a post-doctoral program supported by CAPES-Brasil Grant 0306/98-5.

0022-0396/00 $35.00

excess pressure of the wave like a resistive harmonic oscillator. The precise formulation is

$$\rho u_t + f\delta_{tt} + g\delta_t + h\delta = 0; \tag{1}$$

here $\delta(x, t)$ is the normal displacement to the boundary at time t with the boundary point x, ρ is the fluid density, and f, g, h are nonnegative functions on the boundary with f, h strictly positive. Condition (1) must be coupled with a condition expressing the impenetrability of the boundary,

$$\frac{\partial u}{\partial \nu} = \delta_t, \tag{2}$$

on the boundary for all time; here ν is the unit outer normal. The wave equation with (1), (2) is governed by a (C_0) contraction semigroup (or unitary group if $g \equiv 0$) on a "four-component" Hilbert space of vectors $(u, u_t, \delta, \delta_t)$ with a suitable energy norm (cf. [1–3]).

These acoustic boundary conditions have great intuitive appeal. It is easy to imagine a music hall designed with these conditions in mind, but with a portion of the boundary (e.g., the floor) absorbing.

We shall consider problems of this sort, with a homogeneous Dirichlet condition on a portion of the boundary and acoustic boundary conditions on the rest of the boundary. Our result will be new when specialized to $u_{tt} = c^2 \Delta u$. But we shall work in a much more general context, namely, that of nonlinear wave equations related to problems studied earlier by Kirchhoff, Carrier, and others (see [4, 5]). The equation we consider is

$$u_{tt} - M\left(\int_\Omega u^2\,dx\right)\Delta u + C\,|u_t|^\alpha\,u_t = 0; \tag{3}$$

here $x \in \Omega \subset \mathbf{R}^n$ and $0 \leqslant t \leqslant T$, where u is a position function on $\mathbf{R}^+ = [0, \infty)$ and Ω is a smooth bounded domain. The boundary $\partial\Omega = \Gamma$ is made up of two disjoint pieces, Γ_0, Γ_1, each having nonnempty interior. The Dirichlet condition $u(x, t) = 0$ is imposed for $(x, t) \in \Gamma_0 \times [0, T]$, while the acoustic boundary conditions (2), (3) are imposed for $(x, t) \in \Gamma_1 \times [0, T]$. Here T is any fixed but otherwise arbitrary positive number. Thus we are dealing with global existence.

2. EXISTENCE THEORY

Let $\Omega \subset \mathbf{R}^n$ be a bounded open connected set with a (sufficiently) smooth boundary $\Gamma = \partial\Omega$. Suppose $\Gamma = \Gamma_0 \cup \Gamma_1$ where Γ_0 is a measurable subset of

Γ such that meas$(\Gamma_0) > 0$, and $\Gamma_1 = \Gamma \backslash \Gamma_0$. We shall study the existence and uniqueness of solutions to the initial boundary value problem

$$u'' - M\left(\int_\Omega u^2\,dx\right)\Delta u + C\,|u'|^\alpha u' = 0 \qquad \text{in} \quad Q = \Omega \times (0, T); \tag{4}$$

$$u = 0 \qquad \text{on} \quad \Sigma_0 = \Gamma_0 \times (0, T); \tag{5}$$

$$\rho u' + f\delta'' + g\delta' + h\delta = 0 \qquad \text{on} \quad \Sigma_1 = \Gamma_1 \times (0, T); \tag{6}$$

$$\frac{\partial u}{\partial \nu} - \delta' = 0 \qquad \text{on} \quad \Sigma_1 = \Gamma_1 \times (0, T); \tag{7}$$

$$u(x, 0) = u_0(x), \qquad u'(x, 0) = u_1(x) \quad \text{in} \quad \Omega. \tag{8}$$

Here $' = \frac{\partial}{\partial t}$; ρ and T are positive constants; C is a nonnegative constant; $f(x)$, $g(x)$, and $h(x)$ are continuous real functions on $\overline{\Gamma}_1$ such that $f(x) > 0$, $h(x) > 0$ and $g(x) \geqslant 0$ for all $x \in \overline{\Gamma}_1$; and $M \in C^1([0, \infty); \mathbf{R})$ satisfies

$$0 < m_0 \leqslant M(\lambda), \qquad \frac{|M'(\lambda)\,\lambda^{1/2}|}{M(\lambda)} \leqslant k_0, \qquad \text{for all } \lambda \geqslant 0;$$

Where m_0 and k_0 are constants.

We employ the usual notation for the standard functional spaces (see [7]). The inner product and norm on $L^2(\Omega)$ and $L^2(\Gamma)$ are denoted by $(\cdot, \cdot)$, $\|\cdot\|$ and $(\cdot, \cdot)_\Gamma$, $\|\cdot\|_\Gamma$ respectively. We denote the Hilbert space $H(\Delta, \Omega) = \{u \in H^1(\Omega);\ \Delta u \in L^2(\Omega)\}$, provided with the norm

$$\|u\|_{H(\Delta, \Omega)} = (\|u\|^2_{H^1(\Omega)} + \|\Delta u\|^2)^{1/2},$$

where $H^1(\Omega)$ is the usual real Sobolev space of first order.

$\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ and $\gamma_1 : H(\Delta, \Omega) \to H^{-1/2}(\Gamma)$ are the trace map of order zero and the Neumann trace map on $H(\Delta, \Omega)$, respectively. Therefore

$$\gamma_0(u) = u_{|\Gamma},\ \gamma_1(u) = \left(\frac{\partial u}{\partial \nu}\right)_{|\Gamma} \qquad \text{for all} \quad u \in \mathscr{D}(\overline{\Omega}),$$

and the generalized Green's formula

$$\int_\Omega (-\Delta u).v\,dx = \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i}\,dx - \langle \gamma_1(u), \gamma_0(v) \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}$$

holds for all $u \in H(\Delta, \Omega)$ and $v \in H^1(\Omega)$.

We denote by V the closure in $H^1(\Omega)$ of $\{u \in C^1(\overline{\Omega});\ u = 0 \text{ on } \Gamma_0\}$. Since Ω is a regular connected domain we have that $V = \{u \in H^1(\Omega);$

$\gamma_0(u)=0$ a.e. on $\Gamma_0\}$, V is a closed subspace of $H^1(\Omega)$, the Poincaré inequality holds on V, and the norm

$$\|u\|_V=\left(\sum_{i=1}^{n}\int_{\Omega}\left(\frac{\partial u}{\partial x_i}\right)^2 dx\right)^{1/2}$$

is equivalent to the usual norm from $H^1(\Omega)$.

THEOREM 1. *Let* $\alpha>1$, $u_0\in V\cap H^2(\Omega)$, $u_1\in V\cap L^{2\alpha+2}(\Omega)$ *be given. Then there exists a pair of functions* $(u(x,t),\delta(x,t))$ *which comprise a solution to the problem* (4)–(8) *in the class*

$$u\in L^\infty(0,T;V);\quad u(t)\in H(\Delta,\Omega)\qquad a.e.\ in\ [0,T]; \tag{9}$$

$$u'\in L^\infty(0,T;V)\cap L^{\alpha+2}(0,T;L^{\alpha+2}(\Omega)); \tag{10}$$

$$u''\in L^\infty(0,T;L^2(\Omega)); \tag{11}$$

$$\delta,\delta',\delta''\in L^\infty(0,T;L^2(\Gamma)). \tag{12}$$

Proof. Let $\{\omega_j\}_{j\in\mathbf{N}}$, $\{z_j\}_{j\in\mathbf{N}}$ be orthonormal bases of V and $L^2(\Gamma)$, respectively. For each $m\in\mathbf{N}$ we consider

$$u_m(x,t)=\sum_{j=1}^{m}\xi_{jm}(t)\,\omega_j(x),\qquad x\in\Omega\quad\text{and}\quad t\in[0,T_m],$$

$$\delta_m(x,t)=\sum_{j=1}^{m}\beta_{jm}(t)\,z_j(x),\qquad x\in\Gamma\quad\text{and}\quad t\in[0,T_m],$$

which are solutions to the approximate problem

$$\begin{aligned}(u''_m(t),\omega_j)&+M(\|u_m(t)\|^2)[(\nabla u_m(t),\nabla\omega_j)-(\delta'_m(t),\gamma_0(\omega_j))_\Gamma]\\&+C(|u'_m(t)|^\alpha u'_m(t),\omega_j)=0,\quad 1\leqslant j\leqslant m;\end{aligned}$$

$$(\rho\gamma_0(u'_m(t))+f\delta''_m(t)+g\delta'_m(t)+h\delta_m(t),z_j)_\Gamma=0,\qquad 1\leqslant j\leqslant m;$$

$$u_m(0)=u_{0m},\qquad u'_m(0)=u_{1m},\qquad \delta_m(0)=\delta_0,\qquad \delta'_m(0)=\gamma_1(u_{0m}),$$

where $\delta_0\in L^2(\Gamma)$, $u_{0m}=\sum_{j=1}^m(u_0,\omega_j)\,\omega_j$, $u_{1m}=\sum_{j=1}^m(u_1,\omega_j)\,\omega_j$, and $0<T_m\leqslant T$.

Therefore we have the approximate equations

$$\frac{(u''_m(t),\omega)}{M(\|u_m(t)\|^2)}+(\nabla u_m(t),\nabla\omega)-(\delta'_m(t),\gamma_0(\omega))_\Gamma+C\frac{(|u'_m(t)|^\alpha u'_m(t),\omega)}{M(\|u_m(t)\|^2)}=0, \tag{13}$$

$$(\gamma_0(u'_m(t)),z)_\Gamma=-\frac{1}{\rho}(f\delta''_m(t)+g\delta'_m(t)+h\delta_m(t),z)_\Gamma, \tag{14}$$

for all $\omega \in [\omega_1, ..., \omega_m] = \mathrm{Span}\{\omega_1, ..., \omega_m\}$ and $z \in [z_1, ..., z_m]$. The local existence (for some $T_m > 0$) is standard.

Estimate 1. Taking $\omega = 2u_m'(t)$ in (13) and $z = 2\delta_m'(t)$ in (14) we have

$$\frac{d}{dt}\left[\frac{\rho\,\|u_m'(t)\|^2}{M(\|u_m(t)\|^2)} + \rho\,\|u_m(t)\|_V^2 + \|f^{1/2}\delta_m'(t)\|_\Gamma^2 + \|h^{1/2}\delta_m(t)\|_\Gamma^2\right]$$

$$+\frac{2\rho C\,\|u_m'(t)\|_{L^{\alpha+2}(\Omega)}^{\alpha+2}}{M(\|u_m(t)\|^2)}$$

$$= -\frac{2\rho M'(\|u_m(t)\|^2)}{(M(\|u_m(t)\|^2))^2}\,(u_m'(t), u_m(t))\,\|u_m'(t)\|^2 - \|g^{1/2}\delta_m'(t)\|_\Gamma^2$$

$$\leqslant \frac{C_1}{M(\|u_m(t)\|^2)}\,\|u_m'(t)\|_{L^{\alpha+2}(\Omega)}^3 + \|g^{1/2}\delta_m'(t)\|_\Gamma^2.$$

Since $\alpha > 1$, it is elementary to see that there exists a function $C_2\colon (0, \infty) \to (0, \infty)$ such that

$$C_1\,\|u_m'(t)\|_{L^{\alpha+2}(\Omega)}^3 \leqslant C_2(\varepsilon) + \varepsilon\,\|u_m'(t)\|_{L^{\alpha+2}(\Omega)}^{\alpha+2} \qquad \text{for all} \quad \varepsilon > 0.$$

After we have chosen $\varepsilon > 0$ sufficiently small, we get, for some constants $C_3, C_4 > 0$,

$$\frac{d}{dt}\left[\frac{\rho\,\|u_m'(t)\|^2}{M(\|u_m(t)\|^2)} + \rho\,\|u_m(t)\|_V^2 + \|f^{1/2}\delta_m'(t)\|_\Gamma^2 + \|h^{1/2}\delta_m(t)\|_\Gamma^2\right]$$

$$+ C_3\,\frac{\|u_m'(t)\|_{L^{\alpha+2}(\Omega)}^{\alpha+2}}{M(\|u_m(t)\|^2)} \leqslant C_4 + \|g^{1/2}\delta_m'(t)\|_\Gamma^2.$$

Integrating this from 0 to $t \leqslant T_m$ we find

$$\frac{\rho\,\|u_m'(t)\|^2}{M(\|u_m(t)\|^2)} + \rho\,\|u_m(t)\|_V^2 + \|f^{1/2}\delta_m'(t)\|_\Gamma^2 + \|h^{1/2}\delta_m(t)\|_\Gamma^2$$

$$+ C_3 \int_0^t \frac{\|u_m'(s)\|_{L^{\alpha+2}(\Omega)}^{\alpha+2}}{M(\|u_m(s)\|^2)}\,ds \leqslant C_5\left(1 + \int_0^t \|\delta_m'(s)\|_\Gamma^2\,ds\right), \qquad (15)$$

where C_5 is a positive constant which depends on u_0, u_1, M, T and $\|g\|_{L^\infty(\Gamma)}$.

From (15), Gronwall's inequality gives

$$\|\delta_m'(t)\|_\Gamma^2 \leqslant C_6.$$

This and (15) imply that there exists a constant C_7 independent of m and $t \in [0, T_m]$ such that

$$\frac{\rho\,\|u'_m(t)\|^2}{M(\|u_m(t)\|^2)} + \rho\,\|u_m(t)\|_V^2 + \|f^{1/2}\delta'_m(t)\|_\Gamma^2 + \|h^{1/2}\delta_m(t)\|_\Gamma^2 + C_3 \int_0^t \frac{\|u'_m(s)\|_{L^{\alpha+2}(\Omega)}^{\alpha+2}}{M(\|u_m(s)\|^2)}\,ds \leqslant C_7.$$

Using this and the Poincaré inequality we can see that there exists a constant M_0, independent of m and t, such that

$$m_0 \leqslant M(\|u_m(t)\|^2) \leqslant M_0. \tag{16}$$

Thus there exists a constant C_8 such that

$$\|u'_m(t)\|^2 + \|u_m(t)\|_V^2 + \|\delta'_m(t)\|_\Gamma^2 + |\delta_m(t)\|_\Gamma^2 + \int_0^t \|u'_m(s)\|_{L^{\alpha+2}(\Omega)}^{\alpha+2}\,ds \leqslant C_8, \tag{17}$$

which completes the first estimate. Taking into account (17) we can extend the approximate solutions u_m and δ_m to the whole interval $[0, T]$.

Estimate 2. Taking $t=0$ in (13) and (14) we get

$$(u''_m(0), \omega) - M(\|u_{0m}\|^2)(\Delta u_{0m}, \omega) + C(|u_{1m}|^\alpha u_{1m}, \omega) = 0,$$

$$(f\delta''_m(0), z)_\Gamma + (g\gamma_1(u_{0m}), z)_\Gamma + (h\delta_0, z)_\Gamma + \rho(\gamma_0(u_{1m}), z)_\Gamma = 0.$$

Putting $\omega = u''_m(0)$ and $z = \delta''_m(0)$ we obtain

$$\|u''_m(0)\|^2 \leqslant (M(\|u_{0m}\|^2)\,\|\Delta u_{0m}\| + C\,\|u_{1m}\|_{L^{2\alpha+2}(\Omega)}^{\alpha+1})\,\|u''_m(0)\|,$$

$$\|\delta''_m(0)\|_\Gamma^2 \leqslant C_9(1 + \|u_{0m}\|_{H^2(\Omega)} + \|u_{1m}\|_{H^1(\Omega)})\,\|\delta''_m(0)\|_\Gamma.$$

Therefore

$$\|u''_m(0)\| + \|\delta''_m(0)\| \leqslant C_{10}. \tag{18}$$

Differentiating (13) and (14) with respect to t and taking $\omega = 2u''_m(t)$, $z = 2\delta''_m(t)$ we obtain

$$\frac{d}{dt}\left[\frac{\rho\,\|u''_m(t)\|^2}{M(\|u_m(t)\|^2)}+\rho\,\|u'_m(t)\|^2_V+\|f^{1/2}\delta''_m(t)\|^2_\Gamma+\|h^{1/2}\delta'_m(t)\|^2_\Gamma\right]$$
$$+\frac{2\rho(\alpha+1)\,C(|u'_m(t)|^\alpha,(u''_m(t))^2)}{M(\|u_m(t)\|^2)}$$
$$=\frac{2\rho M'(\|u_m(t)\|^2)}{(M(\|u_m(t)\|^2))^2}(u'_m(t),u_m(t))\,\|u''_m(t)\|^2$$
$$+\frac{4\rho CM'(\|u_m(t)\|^2)}{M(\|u_m(t)\|^2)}(u'_m(t),u_m(t))(|u'_m(t)|^\alpha\,u'_m(t),u''_m(t))$$
$$-2\,\|g^{1/2}\delta''_m(t)\|^2_\Gamma. \tag{19}$$

We observe that

$$\frac{2\rho(\alpha+1)\,C(|u'_m(t)|^\alpha,(u''_m(t))^2)}{M_0}\leqslant\frac{2\rho(\alpha+1)\,C(|u'_m(t)|^\alpha,(u''_m(t))^2)}{M(\|u_m(t)\|^2)};$$

$$\frac{2\rho M'(\|u_m(t)\|^2)}{(M(\|u_m(t)\|^2))^2}(u'_m(t),u_m(t))\,\|u''_m(t)\|^2\leqslant C_{11}\,\|u''_m(t)\|^2;$$

$$\frac{4\rho CM'(\|u_m(t)\|^2)}{M(\|u_m(t)\|^2)}(u'_m(t),u_m(t))(|u'_m(t)|^\alpha\,u'_m(t),u''_m(t))$$
$$\leqslant C_{12}\varepsilon(|u'_m(t)|^\alpha,(u''_m(t))^2)+\frac{C_{12}}{\varepsilon}\,\|u'_m(t)\|^{\alpha+2}_{L^{\alpha+2}(\Omega)},\qquad\text{for all}\quad\varepsilon>0;$$

$$2\,\|g^{1/2}\delta''_m(t)\|^2_\Gamma\leqslant C_{13}\,\|\delta''_m(t)\|^2_\Gamma.$$

Choosing $\varepsilon>0$ sufficiently small and applying this to (19) we get

$$\frac{d}{dt}\left[\frac{\rho\,\|u''_m(t)\|^2}{M(\|u_m(t)\|^2)}+\rho\,\|u'_m(t)\|^2_V+\|f^{1/2}\delta''_m(t)\|^2_\Gamma+\|h^{1/2}\delta'_m(t)\|^2_\Gamma\right]$$
$$\leqslant C_{14}\,\|u'_m(t)\|^{\alpha+2}_{L^{\alpha+2}(\Omega)}+C_{15}(\|u''_m(t)\|^2+\|\delta''_m(t)\|^2_\Gamma).$$

Integrating over $(0,T)$, using (17), (18), and applying Gronwall's inequality we have

$$\|u''_m(t)\|^2+\|u'_m(t)\|^2_V+\|\delta''_m(t)\|^2_\Gamma+\|\delta'_m(t)\|^2_\Gamma\leqslant C_{16}, \tag{20}$$

which is the second estimate.

From (17) and (20) there exist a subsequence of $(u_m)_{m\in\mathbf{N}}$ and a subsequence of $(\delta_m)_{m\in\mathbf{N}}$, which we denote by the same notations, and functions u,δ such that

$$
\begin{aligned}
&u_m \overset{\star}{\rightharpoonup} u && \text{in } L^\infty(0, T; V),\\
&u'_m \overset{\star}{\rightharpoonup} u' && \text{in } L^\infty(0, T; V),\\
&u'_m \rightharpoonup u' && \text{in } L^{\alpha+2}(0, T; L^{\alpha+2}(\Omega)),\\
&u''_m \overset{\star}{\rightharpoonup} u'' && \text{in } L^\infty(0, T; L^2(\Omega)),\\
&\delta_m \overset{\star}{\rightharpoonup} \delta && \text{in } L^\infty(0, T; L^2(\Gamma)),\\
&\delta'_m \overset{\star}{\rightharpoonup} \delta' && \text{in } L^\infty(0, T; L^2(\Gamma)),\\
&\delta''_m \overset{\star}{\rightharpoonup} \delta'' && \text{in } L^\infty(0, T; L^2(\Gamma)).
\end{aligned}
$$

Since $V \overset{c}{\hookrightarrow} L^2(\Omega)$, using the compactness theorem of Aubin and Lions [7], we obtain

$$
\begin{aligned}
&u_m \rightarrow u && \text{in} \quad L^2(0, T; L^2(\Omega)),\\
&u'_m \rightarrow u' && \text{in} \quad L^2(0, T; L^2(\Omega)).
\end{aligned}
$$

Taking into account the above convergences and passing to the limit in the approximate equations we have

$$
\begin{aligned}
|(u''(t), \omega) + M(\|u(t)\|^2)[(\nabla u(t), \nabla \omega) - (\delta'(t), \gamma_0(\omega))_\Gamma]&\\
+ C(|u'(t)|^\alpha u'(t), \omega) = 0,& \qquad (21)
\end{aligned}
$$

$$
(\gamma_0(u'(t)), z)_\Gamma = -\frac{1}{\rho}(f\delta''(t) + g\delta'(t) + h\delta(t), z)_\Gamma, \qquad (22)
$$

for all $\omega \in V$, $z \in L^2(\Gamma)$ a.e. in $[0, T]$.

From (21) we obtain

$$
\int_\Omega u''(x, t)\, \varphi(x)\, dx - \langle M(\|u(t)\|^2)\, \Delta u(t), \varphi \rangle_{\mathscr{D}'(\Omega) \times \mathscr{D}(\Omega)}
$$

$$
+ \int_\Omega |u'(x, t)|^\alpha u'(x, t)\, \varphi(x)\, dx = 0, \qquad \text{for all } \varphi \in \mathscr{D}(\Omega), \text{ a.e. in } [0, T].
$$

Therefore $\Delta u(t) \in L^2(\Omega)$ a.e. in $[0, T]$ and

$$
u'' - M(\|u\|^2)\, \Delta u + C\, |u'|^\alpha u' = 0 \qquad \text{a.e. in } Q = \Omega \times (0, T), \qquad (23)
$$

which shows that u satisfies (4). Since $u(t) \in V$ a.e. in $[0, T]$ we have that (5) is proved. From (22) we can see that u and δ satisfy the boundary condition (6).

Now we shall interpret the sense in which u and δ satisfy (7). Multiplying (23) by $\omega \in V$ and integrating over Ω we find

$$(u''(t), \omega) - M(\|u(t)\|^2)(\Delta u(t), \omega) + C(|u'(t)|^\alpha u'(t), \omega) = 0.$$

Since $u(t) \in H(\Delta, \Omega)$ a.e. in $[0, T]$, using the generalized Green's formula we have

$$(u''(t), \omega) + M(\|u(t)\|^2)\,[(\nabla u(t), \nabla \omega) - \langle \gamma_1(u(t)), \gamma_0(\omega)\rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}] + C(|u'(t)|^\alpha u'(t), \omega) = 0.$$

This and (21) give

$$\langle \gamma_1(u(t)), \gamma_0(\omega)\rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} = (\delta'(t), \gamma_0(\omega))_\Gamma, \tag{24}$$

for all $\omega \in V$ and a.e. in $[0, T]$, which proves (7).

The initial conditions (8) can be proved in a standard way and the proof of Theorem 1 is complete. ∎

THEOREM 2 (Uniqueness, regularity, and continuous dependence on the parameters). *Let $n \leqslant 3$. Under the hypotheses of Theorem 1, if*

$$\alpha > 1 \qquad \textit{for} \qquad n = 1, 2 \tag{25}$$

or

$$1 < \alpha \leqslant 2 \qquad \textit{for} \qquad n = 3, \tag{26}$$

then for each $u_0 \in V \cap H^2(\Omega)$ and $u_1 \in V$ there exists a unique pair of functions $(u(x, t), \delta(x, t))$, which is a solution to the problem (4)–(8) in the class

$$u \in C([0, T]; V) \cap C^1([0, T]; L^2(\Omega)); \qquad u(t) \in H(\Delta, \Omega) \quad \textit{a.e. in } [0, T]; \tag{27}$$

$$u' \in L^\infty(0, T; V); \qquad u'' \in L^\infty(0, T; L^2(\Omega)); \tag{28}$$

$$\delta \in C^1(0, T; L^2(\Gamma)); \qquad \delta'' \in L^\infty(0, T; L^2(\Gamma)). \tag{29}$$

Moreover the solution u, δ depends continuously on the parameters f, g, h, u_0 and u_1.

Proof. Taking into account (25) and (26), the Sobolev imbedding theorem gives $V \hookrightarrow L^{2\alpha+2}(\Omega)$. This allows us to get one more estimate for the approximate solutions. Let $(u_m)_{m \in \mathbf{N}}$ and $(\delta_m)_{m \in \mathbf{N}}$ be the approximate solutions as in the proof of Theorem 1. Thus we have the following extra estimate.

Estimate 3. Let μ and m be arbitrary fixed natural numbers, $\mu \geqslant m$. We define $\xi_{jm}(t) = \beta_{jm}(t) = 0$ for all $j = m+1, ..., \mu$, and we put

$$v_m = u_\mu - u_m, \qquad \theta_m = \delta_\mu - \delta_m.$$

From (13) and (14) we have

$$\frac{M(\|u_m(t)\|^2)(u''_\mu(t), \omega) - M(\|u_\mu(t)\|^2)(u''_m(t), \omega)}{M(\|u_\mu(t)\|^2)\, M(\|u_m(t)\|^2)} + (\nabla v_m(t), \nabla \omega)$$

$$- (\theta'_m(t), \gamma_0(\omega))_\Gamma + C\left[\frac{(|u'_\mu(t)|^\alpha\, u'_\mu(t), \omega)}{M(\|u_\mu(t)\|^2)} - \frac{(|u'_m(t)|^\alpha\, u'_m(t), \omega)}{M(\|u_m(t)\|^2)}\right] = 0,$$

$$(\gamma_0(v'_m(t)), z) = -\frac{1}{\rho}\,(f\theta''_m(t) + g\theta'_m(t) + h\theta_m(t), z)_\Gamma,$$

for all $\omega \in [\omega_1, ..., \omega_\mu]$ and $z \in [z_1, ..., z_\mu]$.

Adding and subtracting appropriate terms, taking $w = 2v'_m(t)$ and $z = \theta'_m(t)$ we find

$$\frac{d}{dt}\left[\frac{\rho\, \|v'_m(t)\|^2}{M(\|u_\mu(t)\|^2)} + \rho\, \|v_m(t)\|_V^2 + \|f^{1/2}\theta'_m(t)\|_\Gamma^2 + \|h^{1/2}\theta_m(t)\|_\Gamma^2\right]$$

$$+ \frac{2\rho C(|u'_\mu(t)|^\alpha\, u'_\mu(t) - |u'_m(t)|^\alpha\, u'_m(t), v'_m(t))}{M(\|u_\mu(t)\|^2)}$$

$$= -2\, \|g^{1/2}\theta'_m(t)\|_\Gamma^2 + 2\rho\left[\frac{M(\|u_\mu(t)\|^2) - M(\|u_m(t)\|^2)}{M(\|u_\mu(t)\|^2)\, M(\|u_m(t)\|^2)}\right](u''_m(t), v'_m(t))$$

$$+ 2\rho C\left[\frac{M(\|u_\mu(t)\|^2) - M(\|u_m(t)\|^2)}{M(\|u_\mu(t)\|^2)\, M(\|u_m(t)\|^2)}\right](|u'_m(t)|^\alpha\, u'_m(t), v'_m(t))$$

$$- 2\rho\, \frac{M'(\|u_\mu(t)\|^2)}{M(\|u_\mu(t)\|^2)}\,(u'_\mu(t), u_\mu(t))\, \|v'_m(t)\|^2. \tag{30}$$

Now we observe that

$$\frac{2\rho C}{M(\|u_\mu(t)\|^2)}\,(|u'_\mu(t)|^\alpha\, u'_\mu(t) - |u'_m(t)|^\alpha\, u'_m(t), v'_m(t)) \geqslant 0;$$

$$2\, \|g^{1/2}\theta'_m(t)\|_\Gamma^2 \leqslant C_1\, \|\theta'_m(t)\|_\Gamma^2;$$

$$2\rho\left[\frac{M(\|u_\mu(t)\|^2) - M(\|u_m(t)\|^2)}{M(\|u_\mu(t)\|^2)\, M(\|u_m(t)\|^2)}\right](u''_m(t), v'_m(t))$$

$$\leqslant C_2(\|v_m(t)\|_V^2 + \|v'_m(t)\|^2).$$

Here we have used

$$
\begin{aligned}
&|M(\|u_\mu(t)\|^2)-M(\|u_m(t)\|^2)| \\
&\quad =\left|\int_{\|u_m(t)\|^2}^{\|u_\mu(t)\|^2} M'(s)\,ds\right| \\
&\quad \leqslant C(\|u_\mu(t)\|+\|u_m(t)\|)\,\|v_m(t)\| \leqslant C\,\|v_m(t)\|_V; \\
&2\rho C\left[\frac{M(\|u_\mu(t)\|^2)-M(\|u_m(t)\|^2)}{M(\|u_\mu(t)\|^2)\,M(\|u_m(t)\|^2)}\right](|u'_m(t)|^\alpha u'_m(t), v'_m(t)) \\
&\quad \leqslant C_3\,\|v_m(t)\|_V\,\|\,|u'_m(t)|^\alpha u'_m(t)\|\,\|v'_m(t)\| \\
&\quad \leqslant \|u'_m(t)\|^{2\alpha+2}_{L^{2\alpha+2}(\Omega)}\,\|v_m(t)\|^2_V + C_3^2\,\|v'_m(t)\|^2 \\
&\quad \leqslant C_4\,\|u'_m(t)\|^{2\alpha+2}_V\,\|v_m(t)\|^2_V + C_3^2\,\|v'_m(t)\|^2 \\
&\quad \leqslant C_5(\|v_m(t)\|^2_V+\|v'_m(t)\|^2).
\end{aligned}
$$

Here we have used $V \hookrightarrow L^{2\alpha+2}(\Omega)$. Next,

$$
2\rho\,\frac{M'(\|u_\mu(t)\|^2)}{M(\|u_\mu(t)\|^2)}\,(u'_\mu(t), u_\mu(t))\,\|v'_m(t)\|^2 \leqslant C_6\,\|v'_m(t)\|^2.
$$

Applying this to (30) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{\rho\,\|v'_m(t)\|^2}{M(\|u_\mu(t)\|^2)}+\rho\,\|v_m(t)\|^2_V+\|f^{1/2}\theta'_m(t)\|^2_\Gamma+\|h^{1/2}\theta_m(t)\|^2_\Gamma\right] \\
&\qquad \leqslant C_7(\|v'_m(t)\|^2+\|v_m(t)\|^2_V+\|\theta'_m(t)\|^2_\Gamma+\|\theta_m(t)\|^2_\Gamma).
\end{aligned}
$$

Integrating from 0 to t and using Gronwall's inequality we have

$$
\begin{aligned}
&\|v'_m(t)\|^2+\|v_m(t)\|^2_V+\|\theta'_m(t)\|^2_\Gamma+\|\theta_m(t)\|^2_\Gamma \\
&\qquad \leqslant C_8(\|v'_m(0)\|^2+\|v_m(0)\|^2_V+\|\theta'_m(0)\|^2_\Gamma+\|\theta_m(0)\|^2_\Gamma);
\end{aligned}
$$

then

$$
\|v'_m(t)\|^2+\|v_m(t)\|^2_V+\|\theta'_m(t)\|^2_\Gamma+\|\theta_m(t)\|^2_\Gamma \leqslant C_9(\|v'_m(0)\|^2+\|v_m(0)\|_{H^2(\Omega)}).
$$

This estimate shows that $(u_m(t))_{m\in\mathbf{N}}$ is a Cauchy sequence in $C([0,T];V)$ and in $C^1([0,T];L^2(\Omega))$. Moreover, it shows also that $(\delta_m(t))_{m\in\mathbf{N}}$ is a Cauchy sequence in $C^1([0,T];L^2(\Gamma))$. Therefore (27)–(29) is proved.

To prove that the solution u, δ depends continuously on the parameters f, g, h, u_0, and u_1 let us consider two sets of parameters, $\{f, g, h, u_0, u_1\}$

and $\{\tilde{f}, \tilde{g}, \tilde{h}, \widetilde{u_0}, \widetilde{u_1}\}$, with associated solutions u, δ and $\tilde{u}, \tilde{\delta}$, respectively. Putting

$$v = u - \tilde{u} \qquad \text{and} \qquad \theta = \delta - \tilde{\delta}$$

and proceeding as in Estimate 3 we have

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{\rho\,\|v'(t)\|^2}{M(\|u(t)\|^2)} + \rho\,\|v(t)\|_V^2 + \|f^{1/2}\theta'(t)\|_\Gamma^2 + \|h^{1/2}\theta(t)\|_\Gamma^2\right]\\
&\quad + \frac{2\rho C(|u'(t)|^\alpha u'(t) - |\tilde{u}'(t)|^\alpha \tilde{u}'(t), v'(t))}{M(\|u(t)\|^2)}\\
&= -2\,\|g^{1/2}\theta'(t)\|_\Gamma^2 + \frac{2\rho[M(\|u(t)\|^2) - M(\|\tilde{u}(t)\|^2)]}{M(\|u(t)\|^2)\,M(\|\tilde{u}(t)\|^2)}(\tilde{u}''(t), v'(t))\\
&\quad + \frac{2\rho C[M(\|u(t)\|^2) - M(\|\tilde{u}(t)\|^2)]}{M(\|u(t)\|^2)\,M(\|\tilde{u}(t)\|^2)}(|\tilde{u}'(t)|^\alpha \tilde{u}'(t), v'(t))\\
&\quad - \frac{2\rho M'(\|u(t)\|^2)}{M(\|u(t)\|^2)}(u'(t), u(t))\,\|v'(t)\|^2\\
&\quad + 2((\tilde{f} - f)\,\tilde{\delta}''(t), \theta'(t))_\Gamma + 2((\tilde{g} - g)\,\tilde{\delta}'(t), \theta'(t))_\Gamma\\
&\quad + 2((\tilde{h} - h)\,\tilde{\delta}(t), \theta'(t))_\Gamma\\
&\leqslant C_{10}(\|v'(t)\|^2 + \|v(t)\|_V^2 + \|\theta'(t)\|_\Gamma^2 + \|\theta(t)\|_\Gamma^2)\\
&\quad + C_{11}(\|f - \tilde{f}\|_{C(\bar{\Gamma}_1)}^2 + \|g - \tilde{g}\|_{C(\bar{\Gamma}_1)}^2 + \|h - \tilde{h}\|_{C(\bar{\Gamma}_1)}^2).
\end{aligned}$$

Integrating this from 0 to t and using Gronwall's inequality we have

$$\begin{aligned}
&\|v'(t)\|^2 + \|v(t)\|_V^2 + \|\theta'(t)\|_\Gamma^2 + |\theta(t)\|_\Gamma^2\\
&\quad \leqslant C_{12}(\|v'(0)\|^2 + \|v(0)\|_{H^2(\Omega)}^2 + \|f - \tilde{f}\|_{C(\bar{\Gamma}_1)}^2 + \|g - \tilde{g}\|_{C(\bar{\Gamma}_1)}^2 + \|h - \tilde{h}\|_{C(\bar{\Gamma}_1)}^2).
\end{aligned}$$

From this we can see that

$$\begin{aligned}
&\|u - \tilde{u}\|_{C^1([0,T];L^2(\Omega))}^2 + \|\delta - \tilde{\delta}\|_{C^1([0,T];L^2(\Gamma))}^2\\
&\quad \leqslant C_{12}(\|f - \tilde{f}\|_{C(\bar{\Gamma}_1)}^2 + \|g - \tilde{g}\|_{C(\bar{\Gamma}_1)}^2 + \|h - \tilde{h}\|_{C(\bar{\Gamma}_1)}^2\\
&\qquad + \|u_0 - \widetilde{u_0}\|_{H^2(\Omega)}^2 + \|u_1 - \widetilde{u_1}\|^2),
\end{aligned}$$

which completes the proof of Theorem 2. ∎

3. THE ACOUSTIC BOUNDARY CONDITION

Thus far we have divided the boundary Γ into two parts, one of which (Γ_0) has positive measure and has the homogeneous Dirichlet boundary condition imposed upon it. Now we let that positive measure shrink to zero. For each $m \in \mathbf{N}$, let Γ_{0m} be a subset of Γ such that

$$\operatorname{meas}(\Gamma_{0m}) > 0, \qquad \text{for all} \quad m \in \mathbf{N};$$

$$\Gamma_{0(m+1)} \subset \Gamma_{0m}, \qquad \text{for all} \quad m \in \mathbf{N};$$

$$\lim_{m \to \infty} \operatorname{meas}(\Gamma_{0m}) = 0.$$

We denote $V_m = \{u \in H^1(\Omega);\ \gamma_0(u) = 0 \text{ a.e. on } \Gamma_{0m}\}$, and $V_\infty = \bigcup_{m=1}^{\infty} V_m$. Therefore, putting $W = \overline{V_\infty}^{H^1(\Omega)}$ we have that W is a closed subspace of $H^1(\Omega)$, $H_0^1(\Omega) \subset V_1 \subset V_2 \subset \cdots \subset V_\infty \subset W \subset H^1(\Omega)$, and the Poincaré inequality is satisfied in W. Moreover,

$$\|u\|_W = \left(\sum_{i=1}^{n} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^2 dx \right)^{1/2}$$

is a norm in W equivalent to the usual norm of $H^1(\Omega)$.

THEOREM 3. *Under the assumptions of Theorem 2 with $n = 2$ or $n = 3$, for each $u_0 \in W \cap H^2(\Omega)$ and $u_1 \in W$ there exists a unique pair of functions $(u(x, t), \delta(x, t))$ in the class*

$$\begin{aligned} &u \in C([0, T]; W) \cap C^1([0, T]; L^2(\Omega)); \\ &u(t) \in H(\Delta, \Omega) \qquad \textit{a.e. in} \quad [0, T]; \end{aligned} \tag{31}$$

$$u' \in L^\infty(0, T; W); \qquad u'' \in L^\infty(0, T; L^2(\Omega)); \tag{32}$$

$$\delta \in C^1(0, T; L^2(\Gamma)); \qquad \delta'' \in L^\infty(0, T; L^2(\Gamma)); \tag{33}$$

such that

$$u'' - M\left(\int_\Omega u^2\, dx \right) \Delta u + C\, |u'|^\alpha\, u' = 0 \qquad \textit{in} \quad Q = \Omega \times (0, T); \tag{34}$$

$$\rho u' + f\delta'' + g\delta' + h\delta = 0 \qquad \textit{on} \quad \Sigma = \Gamma \times (0, T); \tag{35}$$

$$\frac{\partial u}{\partial \nu} - \delta' = 0 \qquad \textit{on} \quad \Sigma = \Gamma \times (0, T); \tag{36}$$

$$u(x, 0) = u_0(x), \qquad u'(x, 0) = u_1(x) \qquad \textit{in } \Omega. \tag{37}$$

Moreover the solution u, δ depends continuously on the parameters f, g, h, u_0 and u_1.

Proof. Let $(u_{0m})_{m\in\mathbf{N}}$ and $(u_{1m})_{m\in\mathbf{N}}$ be two sequences such that

$$u_{0m} \in V_m \cap H^2(\Omega) \quad \text{and} \quad u_{0m} \to u_0 \quad \text{in} \quad W \cap H^2(\Omega);$$
$$u_{1m} \in V_m \quad \text{and} \quad u_{1m} \to u_1 \quad \text{in } W.$$

Thus, by Theorem 2, for each $m \in \mathbf{N}$ there exist a unique pair of functions u_m, δ_m in the class

$$u_m \in C([0, T]; V_m) \cap C^1([0, T]; L^2(\Omega));$$
$$u_m(t) \in H(\Delta, \Omega) \qquad \text{a.e. in } [0, T];$$

$$u'_m \in L^\infty(0, T; V_m); \qquad u''_m \in L^\infty(0, T; L^2(\Omega));$$
$$\delta_m \in C^1(0, T; L^2(\Gamma)); \qquad \delta''_m \in L^\infty(0, T; L^2(\Gamma)),$$

such that

$$u''_m - M(\|u_m\|^2)\, \Delta u_m + C\, |u'_m|^\alpha\, u'_m = 0 \qquad \text{in } Q;$$
$$\rho\gamma_0(u'_m) + f\delta''_m + g\delta'_m + h\delta_m = 0 \qquad \text{on } \Sigma;$$
$$\langle \gamma_1(u_m(t)), \gamma_0(\omega) \rangle = (\delta'_m(t), \omega) \qquad \text{for all} \quad \omega \in V_m, \text{ a.e. in } [0, T];$$
$$u_m(0) = u_{0m}, \qquad u'_m(0) = u_{1m}.$$

Since the constant of the Poincaré inequality on W does not depend on the measure of Γ_{0m} (see the Appendix), we can see that all estimates that we have done before still true for (u_m) and (δ_m). Therefore by taking the limit when m goes to infinity, Theorem 3 is proved. ∎

Remark 1. The boundary condition (36) is satisfied in a weak sense. This means that

$$u(t) \in H(\Delta, \Omega) \qquad \text{a.e. in} \quad [0, T]$$

and

$$\langle \gamma_1(u(t)), \gamma_0(\omega) \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} = (\delta'(t), \gamma_0(\omega))_\Gamma, \qquad \text{for all} \quad \omega \in W.$$

Remark 2. When $M(\lambda) = 1$, $C = 0$ and $n = 3$ we can consider the Hilbert space $H_D(\Omega) = H^1(\Omega)$ modulo constant functions, instead of W.

Our result in this case was obtained in Beale [1] using semigroup methods. He considered an equivalent initial value problem

$$v'(t) = Av(t) \qquad t > 0,$$
$$v(0) = v_0$$

in the Hilbert space $\mathscr{H} = H_D(\Omega) \times L^2(\Omega) \times L^2(\Gamma) \times L^2(\Gamma)$ with norm

$$\|v\|_{\mathscr{H}} = \rho\, \|v_1\|^2_{H_D(\Omega)} + \rho\, \|v_2\|^2 + \|h^{1/2} v_3\|^2_\Gamma + \|f^{1/2} v_4\|^2_\Gamma$$

and A: $D(A) \subset \mathscr{H} \to \mathscr{H}$ the operator defined by

$$D(A) = \{v \in \mathscr{H};\, \Delta v_1 \in L^2(\Omega),\, v_2 \in H^1(\Omega) \text{ and } \gamma_1(v_1) = v_4\}$$

$$Av = \left(v_2,\, \Delta v_1,\, v_4,\, -\frac{1}{f}\,(\rho \gamma_0(v_2) + h v_3 + g v_4) \right).$$

Then he proved that A generates a (C_0) contraction semigroup (a unitary group if $g \equiv 0$), which solves the initial value problem. He also showed that A has a noncompact resolvent and a nonempty essential spectrum. Moreover a description of the spectrum of the semigroup generator A was obtained. The remarkable feature here is that A does not have a compact resolvent; rather $\lambda \to (\lambda I - A)^{-1}$ has essential singularities in **C**. This is a fascinating result for which we have no nonlinear analogue.

Remark 3. The damping term $C\,|u_t|^\alpha\, u_t$ in Eq. (3) is not sufficient for the global solvability when $\alpha = 1$, $C > 0$. However, one can consider this case with additional assumptions. When the measure of Ω is sufficiently small or the size of the initial data (u_0, u_1) is sufficiently small, then global solvability holds. The details will appear elsewhere. Finally we remark that in our basic equation (3) the right-hand side (zero) may be replaced by a given forcing function $F(x, t)$, where $F \in H^1(Q)$. The proof goes through with only inessential changes; we omit the details.

APPENDIX

Let Ω be a bounded connected open set in $\mathbf{R}^n$ with a sufficiently smooth boundary. Then the canonical injection of $H^1(\Omega)$ into $L^2(\Omega)$ is compact, and $H^1(\Omega) \subset C(\bar{\Omega})$ holds by a Sobolev inequality if $n \leqslant 3$. Let $x_0 \in \partial\Omega$. Then there is a constant $k = k(\Omega)$, depending only on Ω, such that

$$\|u\|^2 \leqslant k(\Omega)\, \|\nabla u\|^2 \tag{38}$$

holds for all $u \in H^1(\Omega)$ such that $u(x_0)=0$. (Recall $u \in H^1(\Omega)$ implies $u \in C(\bar{\Omega})$, so that $u(x_0)$ is well defined. More precisely, the equivalence class of u contains an everywhere continuous function on $\bar{\Omega}$, which we identify with u).

The proof of the Poincaré inequality (38) is similar to that in [6, pp. 127–129]. We proceed by contradiction. Assume (38) fails to hold. Then there is a sequence (u_m) in $H^1(\Omega)$ such that

$$\|u_m\|_{H^1(\Omega)}=1, \qquad u_m(x_0)=0, \qquad \text{and} \qquad \alpha_m \|\nabla u_m\|^2 \leqslant \|u_m\|^2 \leqslant 1$$

for all m, where $\alpha_m \to \infty$. (Recall $\|u\|^2_{H^1(\Omega)} = \|u\|^2 + \|\nabla u\|^2$, and $\|\cdot\|$ is the $L^2(\Omega)$ norm). By compactness, a subsequence of (u_m) (which we also denote by (u_m)) converges weakly in $H^1(\Omega)$ to $u \in H^1(\Omega)$. Then u_m converges to u strongly in $L^2(\Omega)$ and uniformly since $C(\bar{\Omega}) \hookrightarrow H^1(\Omega)$ is compact. Hence $u \in C(\bar{\Omega})$ and $u(x_0)=0$. Since $\alpha_m \to \infty$ we have $\|\nabla u_m\| \to 0$, and $\nabla u_m \to \nabla u$ in the sense of distributions. It follows that $\nabla u = 0$, and since $u \in H^1(\Omega)$, $u(x) \equiv C$ is a constant since Ω is connected. But $u \in C(\bar{\Omega})$ and $u(x_0)=0$, whence u is the zero function. Thus $\|u_m\| \to 0$, which coupled with $\|\nabla u_m\| \to 0$ implies $\|u_m\|_{H^1(\Omega)} \to 0$. This contradicts $\|u_m\|_{H^1(\Omega)}=1$, and so (38) follows. ∎

We present an alternative version of (38) in a special context which admits a simple proof and a concrete bound for $k(\Omega)$ with $\dim(\Omega)$ arbitrary. Now let Ω be a bounded convex set in $\mathbf{R}^n$ (for any n). Let $x_0 \in \partial\Omega$. Let D be the diameter of Ω and let $V = \{u \in C^1(\bar{\Omega});\, u(x_0)=0\}$. Then

$$\|u\|^2 \leqslant D^2 \|\nabla u\|^2 \tag{39}$$

holds for all $u \in V$. Thus (38) holds (on the $H^1(\Omega)$ closure of V) with $k(\Omega)=D^2$.

We prove (39). Let $x \in \Omega$. Let e_1 be a unit vector in $\mathbf{R}^n$ pointing from x_0 to x, and let $l = \|x - x_0\|_{\mathbf{R}^n}$. Then by the fundamental theorem of calculus

$$u(x) = \int_0^l \frac{\partial u}{\partial y_1}(x_0 + s e_1)\, ds; \tag{40}$$

here we extend e_1 to $e_1, e_2, ..., e_n$, an orthonormal basis of $\mathbf{R}^n$, and we let $y_1, ..., y_n$ be the corresponding Cartesian coordinate system. By the Cauchy–Schwarz inequality applied to (40),

$$(u(x))^2 \leqslant l \int_0^l \left| \frac{\partial u}{\partial y_1}(x_0 + s e_1) \right|^2 ds.$$

Let L be the portion of the line $\{x_0 + se_1, s \in \mathbf{R}\}$ which intersects Ω. Then

$$|u(x)|^2 \leqslant D \int_L \left(\frac{\partial u}{\partial y_1}\right)^2 dy_1 = D \int_0^\lambda \left|\frac{\partial u}{\partial y_1}(x_0 + se_1)\right|^2 ds, \tag{41}$$

where λ ($\leqslant D$) is the length of L. Integrating both sides of (41) over L gives

$$\int_L |u|^2 \, dy_1 \leqslant D^2 \int_L \left(\frac{\partial u}{\partial y_1}\right)^2 dy_1 \leqslant D^2 \int_L |\nabla u|^2 \, dy_1. \tag{42}$$

To review, we pick $x \in \Omega$ and then choose the coordinate system $y_1, ..., y_n$. Next, for any $f \in L^2(\Omega)$,

$$\int_\Omega |f(y)|^2 \, dy = \int_{E_n} \cdots \int_{E_2} \int_{E_1} |f(y)|^2 \, dy_1 \, dy_2 \, \cdots dy_n \tag{43}$$

where $y_1 \in E_1$ means $F_{11}(y_2, ..., y_n) \leqslant y_1 \leqslant F_{12}(y_2, ..., y_n)$, $y_2 \in E_2$ means $F_{21}(y_3, ..., y_n) \leqslant y_2 \leqslant F_{22}(y_3, ..., y_n)$, and so on until $y_n \in E_n$ means $F_{n1} \leqslant y_n \leqslant F_{n2}$, where F_{n1}, F_{n2} are constants. This is valid since Ω is convex. And our choice of the coordinate system $\{y_1, ..., y_n\}$ implies

$$\int_{E_1} |f(y)|^2 \, dy_1 = \int_L |f|^2 \, dy_1.$$

Now integrate over $y_2 \in E_2, ..., y_n \in E_n$ as in (43), taking for f the choices u and ∇u. By (42),

$$\int_\Omega |u(y)|^2 \, dy \leqslant D^2 \int_\Omega |\nabla u(y)|^2 \, dy,$$

which is (39). ∎

This argument can be extended to more general domains, but rather than taking L the line through x_0 and x, to be parallel to the y_1 axis, we take the path from x_0 to x to be a continuous broken line with sides parallel to the coordinate axes in the $y_1, ..., y_n$ coordinate system. The details are very messy and we omit them.

REFERENCES

1. J. T. Beale, Spectral properties of an acoustic boundary condition, *Indiana Univ. Math. J.* **25** (1976), 895–917.
2. J. T. Beale, Acoustic scattering from locally reacting surfaces, *Indiana Univ. Math. J.* **26** (1977), 199–222.

3. J. T. Beale and S. I. Rosencrans, Acoustic boundary conditions, *Bull. Amer. Math. Soc.* **80** (1974), 1276–1278.
4. G. F. Carrier, On the non-linear vibration problem of the elastic string, *Quart. Appl. Math.* **3** (1945), 157–165.
5. A. T. Cousin, C. L. Frota, N. A. Lar′kin, and L. A. Medeiros, On the abstract model of the Kirchhoff–Carrier equation, *Comm. Appl. Anal.* **1** (1997), 389–404.
6. R. Dautray and J. L. Lions, "Mathematical Analysis and Numerical Methods for Science and Technology," Springer-Verlag, New York, 1988.
7. J. L. Lions and E. Magenes, "Non-homogeneous Boundary Value Problems and Applications," Springer-Verlag, New York, 1972.
8. P. M. Morse and K. U. Ingard, "Theoretical Acoustics," McGraw–Hill, New York, 1968.