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Uniqueness of unconditional bases in nonlocally convex ℓ_1-products ☆

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Dedicated to the memory of Professor Nigel Kalton, our mentor and friend

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ABSTRACT

We show that the ℓ_1-product $(X \oplus X \oplus \cdots \oplus X \oplus \cdots)_1$ has a unique unconditional basis up to permutation for a wide class of nonlocally convex quasi-Banach spaces X, even without knowing whether X has a unique unconditional basis or not.

1. Introduction

Suppose that X is a quasi-Banach space (in particular a Banach space) with a quasi-norm $\|\cdot\|$ and a normalized unconditional basis $(x_n)_{n=1}^{\infty}$. The space X is said to have a *unique unconditional basis* (*up to a permutation*) if whenever $(y_n)_{n=1}^{\infty}$ is another normalized unconditional basis of X, then $(y_n)_{n=1}^{\infty}$ is equivalent to (a permutation of) $(x_n)_{n=1}^{\infty}$, i.e., there exists an automorphism of X which takes one basis to (a permutation of) the other.

For a Banach space it is rather unusual to have a unique unconditional basis; in fact only the spaces c_0, ℓ_1, and ℓ_2 do [14]. If an unconditional basis is unique, in particular it must be equivalent to all its permutations and hence must be symmetric. Thus, the obvious modification for spaces whose canonical basis is unconditional but not symmetric is to require uniqueness of unconditional basis via a permutation, which in many ways is a more natural concept for unconditional bases, whose order is irrelevant. Classifying those Banach spaces with unique unconditional basis up to permutation, however, has turned out to be a much more difficult task. What is known of this topic can be found in [5] and for a modern overview of the subject see the survey article [15].

On the other hand, in the context of quasi-Banach spaces that are not Banach spaces, the uniqueness of unconditional basis seems to be the norm rather than an exception. For instance, it was shown in [8] that a wide class of nonlocally convex Orlicz sequence spaces, including the ℓ_p spaces for $0 < p < 1$, have a unique unconditional basis. The same is true in nonlocally convex Lorentz sequence spaces [12,3] and (up to a permutation) in the Hardy spaces $H_p(\mathbb{T})$ for $0 < p < 1$ [17].

The before mentioned *Memoir* by Bourgain et al. [5] left many open problems, most of which remain unsolved as of today. One of this questions was: do the spaces $\ell_1(X) = (X \oplus X \oplus \cdots \oplus X \oplus \cdots)_1$ and $c_0(X) = (X \oplus X \oplus \cdots \oplus X \oplus \cdots)_0$

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have a unique unconditional basis up to permutation whenever X does? [5, Problem 11.1]. We translated this question into the nonlocally convex setting and tackled the case of $c_0(X)$ in [4].

In this paper we deal with the corresponding problem in the $\ell_1(X)$ case. We show that, at least for an ample class of quasi-Banach spaces X, any complemented unconditional basic sequence of $\ell_1(X)$ must be equivalent to a subset of the canonical basis of the space. As a consequence we obtain that $\ell_1(X)$ has a unique unconditional basis up to permutation (even without knowing whether X has a unique unconditional basis or not!), extending thus a result from [2].

Throughout this article we use standard Banach space theory terminology and notation, as may be found in [1,11]. Other more specific references will be provided in context.

2. Preliminaries

Suppose X is an infinite-dimensional quasi-Banach space. A basis $(x_n)_{n=1}^\infty$ of X is said to be *strongly absolute* if given $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ so that

$$\sum_{n=1}^\infty |\alpha_n| \leqslant C_\varepsilon \sup_n |\alpha_n| + \varepsilon \left\| \sum_{n=1}^\infty \alpha_n x_n \right\|_X, \tag{2.1}$$

for any $(\alpha_n)_{n=1}^\infty \in c_{00}$. This definition was introduced in [12]. Its intuitive meaning is that if the space X has a strongly-absolute basis, then it is far from being a Banach space.

Let X be a quasi-Banach space, and let B_X denote the unit ball of X, i.e., $B_X = \{x \in X\colon \|x\| \leqslant 1\}$. Let $0 < q \leqslant 1$, the *q-convex hull* of B_X, denoted by $q\text{-co}\, B_X$, is the set

$$\left\{ \sum_{i=1}^n \alpha_i x_i\colon \sum_{i=1}^n \alpha_i^q \leqslant 1,\ \alpha_i \geqslant 0,\ \{x_i\}_{i=1}^n \subset B_X,\ n \in \mathbb{N} \right\},$$

i.e., the smallest q-convex set containing B_X, also known as the q-convex hull of B_X. If X has a separating dual, then the gauge functional of $q\text{-co}B_X$ is a q-norm on X that will be denoted by $\|\cdot\|_{(q)}$. The q-Banach space $\hat{X}_q$ resulting from the completion of $(X, \|\cdot\|_{(q)})$ is called the *q-Banach envelope* of X. $\hat{X}_q$ has the property that every continuous linear operator from X into a q-Banach space extends to $\hat{X}_q$ with preservation of norm. In particular, the dual of $\hat{X}_q$ is X^*. For $q = 1$, the space $(\hat{X}_1, \|\cdot\|_{(1)})$ is a Banach space that will be called the *Banach envelope* of X and will be simply denoted by $(\hat{X}, \|\cdot\|_c)$ (see [11,10]).

It follows readily from (2.1) that if an infinite-dimensional quasi-Banach space X with normalized unconditional basis $(x_n)_{n=1}^\infty$ has a q-Banach envelope which is isomorphic to ℓ_q for some $0 < q < 1$, then $(x_n)_{n=1}^\infty$ is strongly absolute, and the Banach envelope of X is isomorphic to ℓ_1.

Quasi-Banach spaces whose q-Banach envelope is isomorphic to ℓ_q for some $q < 1$ are abundant and amongst them we find most of the nonlocally convex classical spaces, like the sequence spaces ℓ_p and the spaces of analytic functions $H_p(\mathbb{T})$ for $0 < p < 1$. All these spaces have a 1-unconditional basis which induces a p-convex lattice structure for some $0 < p < 1$.

Recall that a quasi-Banach lattice X is said to be *p-convex*, where $0 < p < \infty$, if there is a constant M_p such that for any $\{y_i\}_{i=1}^n$ in X and $n \in \mathbb{N}$ we have

$$\left\| \left(\sum_{i=1}^n |y_i|^p \right)^{1/p} \right\| \leqslant M_p \left(\sum_{i=1}^n \|y_i\|^p \right)^{1/p}.$$

The procedure to define the element $(\sum_{i=1}^n |y_i|^p)^{1/p}$ is exactly the same as for Banach lattices [13]. If a quasi-Banach space X is isomorphic to a closed subspace of a p-convex quasi-Banach lattice, then X is also p-convex and it is called *natural* (see [9]).

The other notion that we will need is that of a lattice anti-Euclidean quasi-Banach lattice. A quasi-Banach lattice is said to be *sufficiently Euclidean* if there is a constant M so that for any $n \in \mathbb{N}$ there are operators $S_n : X \to \ell_2^n$ and $T_n : \ell_2^n \to X$ so that $S_n \circ T_n = I_{\ell_2^n}$, $\|S_n\|\|T_n\| \leqslant M$, and S_n is a lattice homomorphism, i.e., if ℓ_2 is finitely representable as a complemented sublattice of X. The space X is called *lattice anti-Euclidean* if it is not sufficiently Euclidean. We refer the reader to [6] for more details on this definition and the role it has played in the classification of complemented basic sequences in Banach lattices which are lattice anti-Euclidean.

Our proofs depend critically on the fact that ℓ_1, the Banach envelope of our spaces (up to isomorphism), is lattice anti-Euclidean. This will be made more explicit below as an application of the next lemma, which summarizes several results and ideas contained in [2].

Lemma 2.1. *Let Z be a quasi-Banach space such that*:

(i) *Z has a 1-unconditional basis $(e_k)_{k=1}^\infty$ that induces a p-convex lattice structure in Z for some $p > 0$,*
(ii) *the Banach envelope of Z is lattice anti-Euclidean, and*
(iii) *Z is lattice isomorphic to $Z \oplus Z$.*

If $(u_n)_{n=1}^\infty$ is a normalized unconditional complemented basic sequence in Z, then $(u_n)_{n=1}^\infty$ is equivalent to a normalized, unconditional, and complemented basic sequence $(v_n)_{n=1}^\infty$ in Z such that the sets $S_n := \{k \in \mathbb{N}\colon e_k^*(v_n) \neq 0\}$ are disjoint and finite, S_n coincides with $\{k \in \mathbb{N}\colon v_n^*(e_k) \neq 0\}$, and there exists a constant $\nu > 0$ such that $e_k^*(v_n) > 0$ and $v_n^*(e_k) > \nu$ for all $k \in S_n$ and all $n \in \mathbb{N}$.

The following technique is known as the "large coefficient technique" and has become crucial to determine the uniqueness of unconditional basis in quasi-Banach spaces. It was introduced by Kalton in [8] to prove the uniqueness of unconditional basis in nonlocally convex Orlicz sequence spaces, and was extended to the framework of quasi-Banach lattices in [12].

Lemma 2.2. *(See [12, Theorem 2.3].) Let Z be a p-convex quasi-Banach lattice $(0 < p < 1)$ with normalized unconditional basis $(e_n)_{n=1}^\infty$ and let Y be a complemented subspace of Z with a normalized unconditional basis $(u_n)_{n\in S}$ $(S \subseteq \mathbb{N})$. Let $(e_n^*)_{n=1}^\infty$ and $(u_n^*)_{n\in S}$ be the sequences of biorthogonal linear functionals associated to $(e_n)_{n=1}^\infty$ and $(u_n)_{n\in S}$ respectively. Suppose that there is a constant $\nu > 0$ and an injective map $\sigma : S \to \mathbb{N}$ so that*

$$\left|e_{\sigma(n)}^*(u_n)u_n^*(e_{\sigma(n)})\right| > \nu$$

for all $n \in S$. Then, the basic sequences $(u_n)_{n\in S}$ and $(e_{\sigma(n)})_{n\in S}$ are equivalent. That is, there exists a positive constant ρ so that

$$\rho^{-1}\left\|\sum_{n\in S}\alpha_n e_{\sigma(n)}\right\| \leqslant \left\|\sum_{n\in S}\alpha_n x_n\right\| \leqslant \rho\left\|\sum_{n\in S}\alpha_n e_{\sigma(n)}\right\|,$$

for any scalars $(\alpha_n) \in c_{00}$.

The following generalization of Lemma 2.2 will be also used. The proofs are similar.

Lemma 2.3. *Let Z be a p-convex quasi-Banach lattice $(0 < p < 1)$ with normalized unconditional basis $(e_n)_{n=1}^\infty$ and let Y be a complemented subspace of Z with a normalized unconditional basis $(u_n)_{n\in S}$ $(S \subseteq \mathbb{N})$ so that $\operatorname{supp}(u_n^*) \subseteq \operatorname{supp}(u_n)$ and the sets $\operatorname{supp}(u_n)$ are disjoint for all $n \in S$. Suppose that there is a constant $\nu > 0$ (independent of n) such that to each $n \in S$ corresponds a subset $T_n \subseteq \operatorname{supp}(u_n)$ for which*

$$\left|\sum_{k\in T_n} e_k^*(u_n)u_n^*(e_k)\right| > \nu.$$

Then, the basic sequence

$$v_n = \sum_{k\in T_n} e_k^*(u_n)e_k \quad (n \in S)$$

is equivalent to $(u_n)_{n\in S}$ and the subspace $[v_n]_{n\in S}$ is complemented in Z. Furthermore, $\operatorname{supp}(v_n^*) = \operatorname{supp}(v_n)$, and $v_n^* \geqslant 0$ if $u_n^* \geqslant 0$ $(n \in S)$.

Finally we remark that there is a Cantor–Bernstein type principle which helps determine whether two unconditional bases are permutatively equivalent. We will use this principle in the form in which it was reinterpreted by Wojtaszczyk in [17, Proposition 2.11].

Proposition 2.4. *Suppose $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty$ are two unconditional basic sequences of a quasi-Banach space X. Then (u_n) and (v_n) are permutatively equivalent if and only if (u_n) is equivalent to a permutation of a subbasis of (v_n) and (v_n) is equivalent up to permutation to a subbasis of (u_n).*

3. Uniqueness of unconditional basis in $\ell_1(X)$

Throughout this section $(X, \|\cdot\|_X)$ will be a quasi-Banach space with a normalized 1-unconditional basis $(x_k)_{k=1}^\infty$ that induces in X a p-convex lattice structure for some $0 < p < 1$, and such that for some $p \leqslant q < 1$, the q-Banach envelope of X is isomorphic to ℓ_q.

Let

$$\ell_1(X) = \left\{z = (z_l)_{l=1}^\infty\colon z_l \in X \text{ for each } l \text{ and } \left(\|z_l\|_X\right)_{l=1}^\infty \in \ell_1\right\}.$$

This set endowed with the quasi-norm

$$\|z\| = \sum_{l=1}^\infty \|z_l\|_X$$

is a (locally p-convex) quasi-Banach space.

For each $l \in \mathbb{N}$, we can write $z_l = \sum_{k=1}^{\infty} \alpha_{l,k} x_k$, and then identify $\ell_1(X)$ with the space of infinite real matrices $A = (\alpha_{l,k})_{l,k=1}^{\infty}$ such that

$$\|A\| = \sum_{l=1}^{\infty} \left\| \sum_{k=1}^{\infty} \alpha_{l,k} x_k \right\|_X < \infty.$$

The space $\ell_1(X)$ has a canonical 1-unconditional basis that will be denoted by $(e_{l,k})_{l,k=1}^{\infty}$. The lattice structure induced by $(e_{l,k})_{l,k=1}^{\infty}$ in $\ell_1(X)$ is p-convex.

The dual space of $\ell_1(X)$ is isomorphic to ℓ_∞ and the Banach envelope of $\ell_1(X)$ is isomorphic to ℓ_1. Both the quasi-norm in $\ell_1(X)$ and the norm in the dual $\ell_\infty(X^*)$ will be denoted without confusion by $\|\cdot\|$; in turn $\|\cdot\|_{(q)}$ will stand for the quasi-norms in the q-Banach envelopes, $\hat{X}_q$ and $\ell_1(\hat{X}_q)$, of X and $\ell_1(X)$ respectively, whereas $\|\cdot\|_c$ will denote both norms in the Banach envelopes $\hat{X}$ and $\ell_1(\hat{X})$.

Suppose Q is a bounded linear projection from $\ell_1(X)$ onto a subspace Y with normalized unconditional basis $(u_n)_{n\in S}$; the cardinality of S can be finite or infinite. We will denote by $(e^*_{l,k})_{l,k\in\mathbb{N}}$ and $(u^*_n)_{n\in S}$ the sequences in $\ell_\infty(X^*)$ of the biorthogonal linear functionals associated to $(e_{l,k})_{l,k\in\mathbb{N}}$ and $(u_n)_{n\in S}$, for which

$$z = \sum_{l,k=1}^{\infty} e^*_{l,k}(z) e_{l,k} \quad \text{and} \quad Q(z) = \sum_{n\in S} u^*_n(z) u_n,$$

for all $z \in \ell_1(X)$. Also, for each $n \in S$ we can write

$$u_n = \sum_{l,k=1}^{\infty} e^*_{l,k}(u_n) e_{l,k},$$

and

$$u^*_n = \sum_{l,k=1}^{\infty} u^*_n(e_{l,k}) e^*_{l,k},$$

where the convergence of this last series is understood in the weak*-sense. Then, we have

$$\left\|u^*_n\right\| \leqslant K\|Q\| \quad (n \in S). \tag{3.1}$$

We also recall that $(u_n)_{n\in S}$ is a K-unconditional basis of $\hat{Y}$, the Banach envelope of Y, which is complemented in $\ell_1(\hat{X})$, and from (3.1) we easily obtain

$$\left(\|Q\|K\right)^{-1} \leqslant \|u_n\|_c \leqslant 1 \quad (n \in S).$$

This is our main theorem.

Theorem 3.1. *Let Q be a bounded linear projection from $\ell_1(X)$ onto a subspace Y with a normalized K-unconditional basis $(u_n)_{n\in S}$. Then, $(u_n)_{n\in S}$ is equivalent to a permutation of a subbasis of the canonical basis $(e_{l,k})_{l,k=1}^{\infty}$ of $\ell_1(X)$.*

Before we see the proof, let us establish a reduction lemma that will allow us to unravel the form in which any complemented unconditional basic sequence in $\ell_1(X)$ can be written in terms of the canonical basis of the space. This is an application of Lemma 2.1, and exemplifies the virtues of the lattice anti-Euclidean ingredient.

For each u_n $(n \in S)$ we single out the sets

$$S_n = \operatorname{supp}(u_n) = \left\{(l,k) \in \mathbb{N}\times\mathbb{N}\colon\ e^*_{l,k}(u_n) \neq 0\right\},$$

and

$$F_n = \left\{l \in \mathbb{N}\colon\ (l,k) \in S_n \text{ for some } k\right\}.$$

Lemma 3.2. *Under the hypotheses of Theorem* 3.1, *we can assume* (*by taking a sequence equivalent to* $(u_n)_{n\in S}$) *that for all $n \in S$,*

(i) $\{(l,k) \in \mathbb{N}\times\mathbb{N}\colon\ u^*_n(e_{l,k}) \neq 0\} = S_n$, *and* $S_n \cap S_m = \emptyset$ *if* $n \neq m$;
(ii) *there exists a constant* $\nu > 0$ *such that* $e^*_{l,k}(u_n) > 0$ *and* $u^*_n(e_{l,k}) > \nu$ *for all* $(l,k) \in S_n$;
(iii) $S_n = \{(l,\sigma_n(l))\colon\ l \in F_n\}$, *and* $\sigma_n(l) \neq \sigma_m(l)$ *if* $n \neq m$.

Proof. The space $\ell_1(X)$ satisfies all the conditions of Lemma 2.1 since $\ell_1(X)$ is lattice isomorphic to $\ell_1(X) \oplus \ell_1(X)$ and the Banach envelope of $\ell_1(X)$ is isomorphic to ℓ_1, which is lattice anti-Euclidean. Consequently we can assume that the support sets

$$S_n = \{(l,k) \in \mathbb{N} \times \mathbb{N}\colon e^*_{l,k}(u_n) \neq 0\} = \{(l,k) \in \mathbb{N} \times \mathbb{N}\colon u^*_n(e_{l,k}) \neq 0\}$$

are disjoint and finite, and that there exists a constant $\nu > 0$ such that $e^*_{l,k}(u_n) > 0$ and $u^*_n(e_{l,k}) > \nu$ for all $(l,k) \in S_n$, and all $n \in S$.

Let us see that we can further simplify the supports.

From now on, for abbreviation we will put

$$b^n_{lk} = e^*_{l,k}(u_n) \quad \text{and} \quad a^n_{lk} = u^*_n(e_{l,k}).$$

Fix $n \in S$. Given $\varepsilon = 1/2K\|Q\|$, the strong-absoluteness of $(x_k)_{k=1}^\infty$ yields a constant C such that for every $l \in F_n$,

$$\sum_{k=1}^\infty a^n_{lk} b^n_{lk} \leqslant C \sup_k a^n_{lk} b^n_{lk} + \frac{1}{2K\|Q\|} \left\| \sum_{k=1}^\infty a^n_{lk} b^n_{lk} x_k \right\|_X$$
$$\leqslant C \sup_k a^n_{lk} b^n_{lk} + \frac{1}{2K\|Q\|} \sup_{l,k} a^n_{lk} \left\| \sum_{k=1}^\infty b^n_{l,k} x_k \right\|_X .$$

Summing in l on both sides of the previous inequality and using (3.1),

$$1 = u^*_n(u_n) = \sum_{l \in F_n} \sum_{k=1}^\infty a^n_{lk} b^n_{lk}$$
$$\leqslant C \sum_{l \in F_n} \sup_k a^n_{lk} b^n_{lk} + \frac{1}{2K\|Q\|} \|u^*_n\| \|u_n\|$$
$$\leqslant C \sum_{l \in F_n} \sup_k a^n_{lk} b^n_{lk} + \frac{1}{2}.$$

Thus,

$$\sum_{l \in F_n} \sup_k a^n_{lk} b^n_{lk} \geqslant \frac{1}{2C}.$$

For each $n \in S$ we pick a map $\sigma_n : F_n \to \mathbb{N}$, $l \to \sigma_n(l)$ such that

$$a^n_{l\sigma_n(l)} b^n_{l\sigma_n(l)} = \sup_k a^n_{lk} b^n_{lk}.$$

This way, we have

$$\sum_{l \in F_n} a^n_{l\sigma_n(l)} b^n_{l\sigma_n(l)} \geqslant \frac{1}{2C} \quad (n \in S),$$

and so by Lemma 2.3, the sequence defined as

$$v_n = \sum_{l \in F_n} b^n_{l\sigma_n(l)} e_{l\sigma_n(l)} \quad (n \in S),$$

is equivalent to $(u_n)_{n \in S}$ and complemented in $\ell_1(X)$. □

We are now ready to show Theorem 3.1.

Proof of Theorem 3.1. The basic sequence $(u_n)_{n \in S}$ is (semi-normalized) and complemented in the Banach envelope $\ell_1(\hat{X})$. Since ℓ_1 is prime and has a unique unconditional basis, we have an estimate

$$\left\| \sum_{n \in \mathcal{N}} \alpha_n u_n \right\|_c \geqslant c \sum_{n \in \mathcal{N}} |\alpha_n|, \tag{3.2}$$

for any scalars $(\alpha_n)_{n \in \mathcal{N}}$ finitely nonzero.

Notice also that $(x_n)_{n=1}^\infty$ is a (semi-normalized) unconditional basis of both ℓ_1 and ℓ_q, and so, by the uniqueness of unconditional basis in these spaces, there are constants C_1 and C_q so that

$$C_1^{-1}\sum_{n\in\mathcal{N}}|\alpha_n|\leqslant\Big\|\sum_{n\in\mathcal{N}}\alpha_n x_n\Big\|_c\leqslant C_1\sum_{n\in\mathcal{N}}|\alpha_n|, \tag{3.3}$$

and

$$C_q^{-1}\Big(\sum_{n\in\mathcal{N}}\big|\alpha_n^q\big|\Big)^{1/q}\leqslant\Big\|\sum_{n\in\mathcal{N}}\alpha_n x_n\Big\|_{(q)}\leqslant C_q\Big(\sum_{n\in\mathcal{N}}\big|\alpha_n^q\big|\Big)^{1/q}, \tag{3.4}$$

for any scalars $(\alpha_n)_{n\in\mathcal{N}}$ finitely nonzero.

Then, on the one hand we have

$$\begin{aligned}\Big\|\sum_{n\in\mathcal{N}}\alpha_n u_n\Big\|_c&=\sum_{l\in\bigcup_{n\in\mathcal{N}}F_n}\Big\|\sum_{n\in\mathcal{N}}\alpha_n b^n_{l\sigma_n(l)}x_{\sigma(n)}\Big\|_c\\&\leqslant C_1\sum_{l\in\bigcup_{n\in\mathcal{N}}F_n}\sum_{n\in\mathcal{N}}\big|\alpha_n b^n_{l\sigma_n(l)}\big|,\end{aligned} \tag{3.5}$$

while, on the other

$$\begin{aligned}C_q^{-1}\sum_{l\in\bigcup_{n\in\mathcal{N}}F_n}\Big(\sum_{n\in\mathcal{N}}\big|\alpha_n b^n_{l\sigma_n(l)}\big|^q\Big)^{1/q}&\leqslant\sum_{l\in\bigcup_{n\in\mathcal{N}}F_n}\Big\|\sum_{n\in\mathcal{N}}\alpha_n b^n_{l\sigma_n(l)}x_{\sigma(n)}\Big\|_{(q)}\\&=\Big\|\sum_{n\in\mathcal{N}}\alpha_n u_n\Big\|_{(q)},\end{aligned} \tag{3.6}$$

for any scalars $(\alpha_n)_{n\in\mathcal{N}}$ finitely nonzero.

We will split the set S in two disjoint subsets according to the size of the coefficients of $(u_n)_{n\in S}$. Let us fix

$$\delta=\frac{1}{2}\left(\frac{c}{C_1C_q^q\rho^q}\right)^{\frac{1}{1-q}}$$

and put

$$\mathcal{A}=\Big\{n\in S\colon\ \sup_{l\in F_n}b^n_{l\sigma_n(l)}>\delta\Big\},$$

and $\mathcal{B}=S\setminus\mathcal{A}$.

By appealing to Lemma 2.2, the sequence $(u_n)_{n\in\mathcal{A}}$ is permutatively equivalent to a subbasis of $(e_{l,k})_{l,k=1}^\infty$. Therefore the proof will be over once we prove the following statement.

Claim. *$(u_n)_{n\in\mathcal{B}}$ is equivalent (in $\ell_1(X)$) to $(e_n)_{n\in\mathcal{B}}$, where $(e_n)_{n=1}^\infty$ denotes the canonical basis of ℓ_1.*

Given any finite set of vectors $\{u_{n_j}\}_{j=1}^N$ with $n_j\in\mathcal{B}$, we consider a bipartite graph whose sets of vertices are, on the one hand, the set of indexes $\{n_1,\dots,n_N\}$, and on the other hand $\bigcup_{j=1}^N F_{n_j}$. A vertex n_i is joined with a vertex l in this graph iff $l\in F_{n_i}$.

Our goal is to prove that there exists a matching of $\{n_1,\dots,n_N\}$, which is equivalent to showing (by the Hall–König lemma) that, for every subset $\mathcal{M}\subset\{n_1,\dots,n_N\}$ we have

$$\Big|\bigcup_{n_j\in\mathcal{M}}F_{n_j}\Big|\geqslant|\mathcal{M}|.$$

Suppose the contrary. Then, there would exist a minimal $\mathcal{M}\subset\{n_1,\dots,n_N\}$ for which

$$\Big|\bigcup_{n_j\in\mathcal{M}}F_{n_j}\Big|<|\mathcal{M}|.$$

By the minimality of $\mathcal{M}$, if we remove one element from it, then the resultant set $\mathcal{M}^*$ verifies sharp the identity

$$\Big|\bigcup_{n_j\in\mathcal{M}^*}F_{n_j}\Big|=\big|\mathcal{M}^*\big|, \tag{3.7}$$

and the Hall–König lemma applies to $\mathcal{M}^*$. Hence we will have an injective map

$$\psi : \mathcal{M}^* \to \bigcup_{n_j \in \mathcal{M}^*} F_{n_j}, \quad n_j \to \psi(n_j) = l_j.$$

As a consequence, by Lemma 2.2 there will exist a constant $\rho > 0$ such that

$$\Bigl\| \sum_{n_j \in \mathcal{M}^*} \alpha_j u_{n_j} \Bigr\|_{(q)} \leqslant \rho \Bigl\| \sum_{n_j \in \mathcal{M}^*} \alpha_j e_{l_j \sigma_{n_j}(l_j)} \Bigr\|_{(q)} = \rho \sum_{n_j \in \mathcal{M}^*} |\alpha_j|, \tag{3.8}$$

for any scalars (α_j). Combining (3.2) and (3.5),

$$c\bigl|\mathcal{M}^*\bigr| \leqslant \Bigl\| \sum_{n_j \in \mathcal{M}^*} u_{n_j} \Bigr\|_c \leqslant C_1 \sum_{l \in \bigcup_{n_j \in \mathcal{M}^*} F_{n_j}} \sum_{n_j \in \mathcal{M}^*} b^{n_j}_{l\sigma_{n_j}(l)}. \tag{3.9}$$

By definition, given $n \in \mathcal{B}$ we have $|b^n_{l\sigma_n(l)}| < \delta$ for all $l \in F_n$. Since $0 < q < 1$, it follows that

$$\bigl|b^n_{l\sigma_n(l)}\bigr| < \delta^{1-q} \bigl|b^n_{l\sigma_n(l)}\bigr|^q, \quad l \in F_n.$$

Hence,

$$\sum_{l \in \bigcup_{n_j \in \mathcal{M}^*} F_{n_j}} \sum_{n_j \in \mathcal{M}^*} b^{n_j}_{l\sigma_{n_j}(l)} \leqslant \delta^{1-q} \sum_{l \in \bigcup_{n_j \in \mathcal{M}^*} F_{n_j}} \sum_{n_j \in \mathcal{M}^*} \bigl|b^{n_j}_{l\sigma_{n_j}(l)}\bigr|^q.$$

Using Hölder's inequality in combination with (3.6), (3.7), and (3.8) gives

$$\begin{aligned}
\sum_{l \in \bigcup_{n_j \in \mathcal{M}^*} F_{n_j}} \sum_{n_j \in \mathcal{M}^*} \bigl|b^{n_j}_{l\sigma_{n_j}(l)}\bigr|^q &\leqslant \biggl(\sum_{l \in \bigcup_{n_j \in \mathcal{M}^*} F_{n_j}} \biggl(\sum_{n_j \in \mathcal{M}^*} \bigl|b^{n_j}_{l\sigma_{n_j}(l)}\bigr|^q \biggr)^{1/q} \biggr)^q \biggl(\sum_{l \in \bigcup_{n_j \in \mathcal{M}^*} F_{n_j}} 1 \biggr)^{1-q} \\
&\leqslant C_q^q \Bigl\| \sum_{n \in \mathcal{M}^*} u_n \Bigr\|_{(q)}^q \Bigl| \bigcup_{n_j \in \mathcal{M}^*} F_{n_j} \Bigr|^{1-q} \\
&\leqslant C_q^q \rho^q \bigl|\mathcal{M}^*\bigr|^q \bigl|\mathcal{M}^*\bigr|^{1-q},
\end{aligned}$$

which jointly with (3.9) yields

$$c\bigl|\mathcal{M}^*\bigr| \leqslant \delta^{1-q} C_1 C_q^q \rho^q \bigl|\mathcal{M}^*\bigr|.$$

This implies that

$$\delta \geqslant \biggl(\frac{c}{C_1 C_q^q \rho^q} \biggr)^{\frac{1}{1-q}},$$

contradicting our choice of δ. This means we must have

$$\Bigl| \bigcup_{n_j \in \mathcal{M}} F_{n_j} \Bigr| \geqslant |\mathcal{M}|,$$

for every subset $\mathcal{M} \subset \{n_1, \ldots, n_N\}$, and so there is an injective map

$$\{n_1, \ldots, n_N\} \to \bigcup_{j=1}^N F_{n_j}, \quad n_j \mapsto l_j.$$

Since by Lemma 3.2, the coefficients $u^*_{n_j}(e_{l_j,\sigma_{n_j}(l_j)})$ are uniformly bigger than some $\nu > 0$, Lemma 2.2 comes into play again to produce a constant $\rho' > 0$ so that for any scalars $(\alpha_j)_{j=1}^N$,

$$\Bigl\| \sum_{j=1}^N \alpha_j u_{n_j} \Bigr\| \leqslant \rho' \Bigl\| \sum_{j=1}^N \alpha_j e_{l_j,\sigma_{n_j}(l_j)} \Bigr\| = \rho' \sum_{j=1}^N |\alpha_j|. \tag{3.10}$$

Now the claim follows since

$$\Bigl\| \sum_{j=1}^N \alpha_j u_{n_j} \Bigr\| \geqslant \Bigl\| \sum_{j=1}^N \alpha_j u_{n_j} \Bigr\|_c \geqslant c \sum_{j=1}^N |\alpha_j|. \qquad \square$$

Corollary 3.3. *Let X be a quasi-Banach space with unconditional basis $(x_k)_{k=1}^{\infty}$ and whose q-Banach envelope is isomorphic to ℓ_q for some $0 < q < 1$. For each $l \in \mathbb{N}$, let $X_l = [x_k]_{k \in \mathcal{N}_l}$ with $|\mathcal{N}_l| \leqslant \infty$. Then any unconditional basis $(u_n)_{n=1}^{\infty}$ of a complemented subspace of $\ell_1(X_l)_{l=1}^{\infty} = (X_1 \oplus X_2 \oplus \cdots \oplus X_l \oplus \cdots)_1$ is equivalent to a permutation of a subbasis of the canonical basis of $\ell_1(X_l)_{l=1}^{\infty}$.*

Proof. We just note that $\ell_1(X_l)_{l=1}^{\infty}$ is a complemented subspace of $\ell_1(X)$, which makes it lattice anti-Euclidean. Now the proof would follow the same steps as the proof of Theorem 3.1. We omit the details. □

Corollary 3.4. *Suppose that X is a quasi-Banach space with unconditional basis and whose q-Banach envelope is isomorphic to ℓ_q for some $0 < q < 1$. Then, the space $\ell_1(X)$ has a unique unconditional basis up to permutation.*

Proof. Suppose that $(u_n)_{n=1}^{\infty}$ is a normalized unconditional basis of $\ell_1(X)$. By Theorem 3.1, $(u_n)_{n=1}^{\infty}$ is equivalent to a permutation of a subbasis $(e_{l,k})_{(l,k)\in\mathcal{M}}$ of $(e_{l,k})_{l,k=1}^{\infty}$, the canonical basis of $\ell_1(X)$. In order to obtain the equivalence up to permutation of $(u_n)_{n=1}^{\infty}$ and $(e_{l,k})_{l,k=1}^{\infty}$, appealing to Proposition 2.4 it suffices to show the converse, i.e., that $(e_{l,k})_{l,k=1}^{\infty}$ is permutatively equivalent to a subbasis of $(e_{l,k})_{(l,k)\in\mathcal{M}}$.

Clearly, $(e_{l,k})_{(l,k)\in\mathcal{M}}$ is the canonical basis of a space $\ell_1(X_l)_{l=1}^{\infty}$, where each $X_l = [x_k]_{k\in\mathcal{N}_l}$ and $\mathcal{N}_l$ is a subset of integers of cardinality $|\mathcal{N}_l| \leqslant \infty$. Since $\ell_1(X_l)_{l=1}^{\infty}$ is isomorphic to $\ell_1(X)$, there exists a basis $(v_n)_{n=1}^{\infty}$ in $\ell_1(X_l)_{l=1}^{\infty}$ equivalent to $(e_{l,k})_{l,k=1}^{\infty}$. Now, Corollary 3.3 yields that $(v_n)_{n=1}^{\infty}$ must be equivalent to a permutation of a subbasis of $(e_{l,k})_{(l,k)\in\mathcal{M}}$ and the proof is over. □

Corollary 3.5. *The following spaces have a unique unconditional basis up to permutation*:

(i) $\ell_1(\ell_p)$ *for* $0 < p < 1$;
(ii) $\ell_1(\ell_p^{k_n})$, *where* $(k_n)_{n=1}^{\infty}$ *is any increasing sequence of positive integers and* $0 < p < 1$;
(iii) $\ell_1(\ell_p(\ell_q))$ *for* $0 < p, q < 1$;
(iv) $\ell_1(\ell_p(\ell_q^{k_n}))$ *where* $(k_n)_{n=1}^{\infty}$ *is any increasing sequence of positive integers and* $0 < p < 1$;
(v) $\ell_1(H_p(\mathbb{T}))$ *for* $0 < p < 1$.

Proof. If $p < q \leqslant 1$, the q-Banach envelope of the spaces whose infinite ℓ_1-product are considered in (i)–(iv) is isomorphic to ℓ_q, as the reader can easily check. For the case (v), see [7] or [16]. □

Remark 3.6. It is very likely that the hypothesis that $\hat{X}_q \approx \ell_q$ for some $q < 1$ can be replaced for the weaker condition that X have a strongly absolute basis. This would extend the uniqueness of unconditional basis to ℓ_1-products of other nonlocally convex quasi-Banach spaces like some Lorentz sequence spaces or Orlicz sequence spaces.

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