Geometry of Spinors*

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Covariant differentiation of spinors from the point of view of a connexion on a principal fiber bundle of spinor frames has been studied by Guy [3] and Lichnerowicz [7]. The bundle studied has Spin (n) as structural group. In this paper, among other fiber bundles we introduce a principal fiber bundle \( P[M] \) of spinor frames which has a full linear group as structural group and contains the previously mentioned bundle as a subbundle. We then study connections in the rather general setting of \( P[M] \). In addition to the well-known structure equation for a connexion on a principal fiber bundle (see p. 77, [6]), we discuss a second structure equation for a connexion on \( P[M] \). These two equations furnish necessary and sufficient conditions that a form on \( P[M] \) be a connection form. (Note that although [3] and [7] deal with \( n = 4 \), the study generalizes rather easily.)

We then study the reduction of connections on \( P[M] \). To this end we will develop a theory for spinors similar to the theory of metric connections on the bundle of (vector) frames \( L[M] \). Actually our study will contain in a natural way this theory of metric connections. An application of this study in terms of the electromagnetic potential vector in general relativity will be given.

Latin letters will be used to index tensors and range from 1 to \( n \) (\( n = 2v \) or \( 2v + 1 \)). Greek letters will index spinors and range from 1 to \( 2^v \). The Einstein summation convention is used throughout.

1. Spinors and Related Fiber Bundles

Let \( M \) be an \( n \)-dimensional orientable Riemannian manifold. The following discussion applies to manifolds with an indefinite metric with only minor modifications (see [2]).

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We consider a set of \( 2v + 1, 2^v \times 2^v \) complex matrices (where \( n = 2^v \) or \( 2^v + 1 \)) called flat Dirac matrices

\[
\hat{\beta}^i = (\hat{\beta}^i_\alpha),
\]

where \( i = 1, 2, ..., 2v + 1; \alpha, \beta = 1, 2, ..., 2^v \). These matrices satisfy

\[
\hat{\beta}^i \cdot \hat{\beta}^i + \hat{\beta}^i \cdot \hat{\beta}^i = 2\delta^{ij}I,
\]

where \( \delta^{ij} \) is the Kronecker delta and \( I \) is the identity matrix. We define the group \( \hat{S}(n) \) as the set of all \( 2^v \times 2^v \) complex matrices \( a = (a_{\beta}^\alpha) \) such that there are real numbers \( h_j^i \) ( \( i, j = 1, 2, ..., n \) ) with the property

\[
\frac{\partial a_{\beta}^\alpha}{\partial x^p} = h_j^i a_{\beta}^\alpha \frac{\partial h_j^i}{\partial x^p},
\]

\( (i = 1, 2, ..., n; \alpha, \beta = 1, 2, ..., 2^v) \) and \( (a_{\beta}^{-1}) = a^{-1}. \) In this case we define a map \( \hat{\chi}_n \) on \( \hat{S}(n) \) by \( \hat{\chi}_n(a) = h - (h_j^i). \) It is known (see p. 5-16, [9]) that if \( n \) is even \( \hat{\chi}_n(\hat{S}(n)) = O(n) \) (the orthogonal group) and if \( n \) is odd \( \hat{\chi}_n(\hat{S}(n)) = O^+(n) \) (the proper orthogonal group). We define \( S(n) \) as all \( a \in \hat{S}(n) \) such that \( \det a = 1. \) The map \( \chi_n \) is the restriction of \( \hat{\chi}_n \) to \( S(n). \) Note that the classical group \( Spin(n) \) although defined in a manner very similar to \( S(n) \) is actually a proper subgroup of \( S(n). \)

We assume that it is possible to construct by extension, using \( \chi_n \) and the bundle of orthonormal frames \( 0[M], \) a principal fiber bundle \( \mathcal{F}[M] \) with base \( M \) and structure group \( S(n). \) For conditions concerning this possibility see [4].

Associated with \( 0[M] \) we have \( V = \{ U_A \} \) a covering of \( M \) by coordinate neighborhoods. On each \( U_A \) we have defined a vector field \( e_i \) \( i = 1, 2, ..., n \) such that \( (e_i | m) \) is an orthonormal frame at \( m \in U_A. \) Hereafter, we deal only with coordinate neighborhoods belonging to \( V. \) Also we have the projection map \( \pi : 0[M] \to M \) and a collection of diffeomorphisms

\[
\sigma_A : U_A \times 0(n) \to 0[M]
\]

given by

\[
\sigma (m, \ell) = (\ell^i e_i | m).
\]

Given the coordinate neighborhoods \( U_A, U_B \in V \) with \( U_A \cap U_B \neq \phi, \) then if

\[
e_i | m = h_{\ell}^i(m) e_i | m
\]

for \( m \in U_A \cap U_B, \) the transition function of \( (U_A, U_B) \) defined on \( U_A \cap U_B \) is given by

\[
\gamma_{AB}(m) = h(m) = (h_{\ell}^i(m)).
\]
Associated with the bundle $\mathcal{S}[M]$ we have corresponding quantities which we indicate with the symbol $\sim$. For example, if

$$y_{AB}(m) = h \in O(n),$$

then the transition function of $(U_A, U_B)$ for the bundle $\mathcal{S}[M]$ is denoted by $\tilde{y}_{AB}$ and $\tilde{y}_{AB}(m)$ is required to be one of the elements of the set $\chi^{-1}(h)$. Note that in order to be able to construct $\mathcal{S}[M]$ we must be able to choose the transition functions in such a way that if $U_A, U_B, U_C$ are in $V$, then

$$\tilde{y}_{AB}\tilde{y}_{BC}\tilde{y}_{CA}(m) = I,$$

for $m \in U_A \cap U_B \cap U_C$. If this can be done then there is a standard procedure for constructing $\mathcal{S}[M]$ (see p. 52, [6]).

**Definition.** A contravariant spinor $\psi$ at $m \in M$ is a map $p \mapsto \psi(p)$ of $\tilde{\Pi}^{-1}(m)$ (which is a subset of $\mathcal{S}[M]$) to the vector space of $2^r \times 1$ complex matrices, such that

$$\psi(p \cdot a^{-1}) = a \cdot \psi(p), \quad (1.1)$$

where $a \in S(n)$ and $p \cdot a^{-1}$ means right translation of $p$ by $a^{-1}$.

If $\tilde{\Pi}(p) = m \in U_A$ we can define a set of basis spinors at $m$ associated with $U_A$ as follows.

**Definition.** If $\tilde{\sigma}_A^{-1}(p_0) = (m, I) \in U_A \times S(n)$ then

$$e_{\alpha} |_m(p_0) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

for $\alpha = 1, 2, \ldots, 2^r$ where the 1 is in the $\alpha$th row. Note that the effect of $e_{\alpha} |_m$ on any $p \in \tilde{\Pi}^{-1}(m)$ can be seen using (1.1).

Now each contravariant spinor $\psi$ at $m$ can be expressed as a linear combination of the $e_{\alpha} |_m$. If $\tilde{\Pi}(p) = m \in U_A$ and $\psi(p_0) = (\psi^2)$ where $p_0 = \tilde{\sigma}_A(mI)$ then

$$\psi(p) = \psi^2 e_{\alpha} |_m(p)$$

Further the set $(e_{\alpha} |_m)$ is linearly independent and thus forms a basis for the vector space of contravariant spinors at $m \in U_A$. Such an ordered set is
called a contra-spinor frame at \( m \). The frame \((e_{\alpha} \mid m)\) will be referred to as the natural contra-spinor frame at \( m \) associated with \( U_A \).

We can consider the bundle \( \mathcal{F}[M] \) as a collection of contra-spinor frames as follows. Suppose \( \rho \in \mathcal{F}[M] \) and \( \tilde{\sigma}_A(m, a) = \rho \). Then we introduce the correspondence

\[
\rho = (a_{\beta}^\alpha e_{\alpha} \mid m) = (e_{\beta} \mid m).
\]

It can be shown that for \( m \in U_A \cap U_B \),

\[
(e_{\alpha} \mid m) = \tilde{\gamma}_{AB}(m) \cdot (e_{\alpha} \mid m).
\]

**DEFINITION.** A covariant spinor \( \phi \) at \( m \) is a map \( \rho \to \phi(\rho) \) of \( \mathcal{F}\Gamma^{-1}(m) \) to the vector space of \( 1 \times 2^r \) complex matrices such that if \( a \in S(n) \)

\[
\phi(\rho \cdot a^{-1}) = \phi(\rho) \cdot a^{-1}.
\]

We define the natural co-spinor frame associated with \( U_A \) at \( m \in U_A \), \((e^\alpha \mid m)\) by

**DEFINITION.** Let \( \tilde{\sigma}_A^{-1}(\rho_0) = (m, I) \), then

\[
e^\alpha \mid m(\rho_0) = (0, 0, \ldots, 0, 1, 0, \ldots, 0)
\]

for \( \alpha = 1, 2, \ldots, 2^r \) where the 1 is in the \( \alpha \)th place.

As in the contravariant case if \( \phi \) is a covariant spinor at \( m \in U_A \) and \( \tilde{\Pi}(\rho) = m \), and if \( \phi(\rho_0) = (\phi_\alpha) \) where \( \rho_0 = \tilde{\sigma}_A(m, I) \), then

\[
\phi(\rho) = \phi_\alpha e^\alpha \mid m(\rho).
\]

Note that it can easily be shown that the co-spinor frame \((e^\alpha \mid m)\) transforms contragradiently to the contra-spinor frame \((e_{\alpha} \mid m)\).

We pointed out that \( \mathcal{F}[M] \) can be thought of as a bundle of contra-spinor frames

\[
(e_{\alpha} \mid m) = (e_{\alpha}^\beta e_{\beta} \mid m),
\]

where \( a = (a_{\alpha}^\lambda) \in S(n) \). We can enlarge this bundle to obtain two additional bundles.

**DEFINITION.** The bundle \( \mathcal{F}\hat{\mathcal{F}}[M] \) consists of all contra-spinor frames \((e_{\alpha} \mid m) \ m \in M \) such that if \( m \in U_A \in V \), then

\[
e_{\alpha} \mid m = a_{\alpha}^\lambda e_{\lambda}^\alpha \mid m,
\]

where \( a = (a_{\alpha}^\lambda) \in S(n) \).
Definition. The bundle $\mathbf{P}[M]$ consists of all contra-spinor frames $(e_\alpha |_m) m \in M$ such that if $m \in U_\alpha \subset V$, then

$$e_\alpha |_m = a_\alpha \lambda e_\lambda |_m$$

where

$$a = (a_\alpha) \in GL(2^*, \mathbb{C}).$$

Thus corresponding to the groups

$$S(n) \subset \hat{S}(n) \subset GL(2^*, \mathbb{C})$$

we have the bundles

$$\mathcal{S}[M] \subset \hat{\mathcal{S}}[M] \subset \mathbf{P}[M]$$

with the above groups as their corresponding structure groups.

It is not difficult to show that $S(n)$ is closed in $\hat{S}(n)$ and $\hat{S}(n)$ is closed in $GL(2^*, \mathbb{C})$. (The manifold structure on $GL(2^*, \mathbb{C})$ arises from the fact that $GL(2^*, \mathbb{C})$ can be identified with an open set in $\mathbb{C}^{2^* \times 2^*} = \mathbb{R}^{2^* \times 2^*}$). We know therefore that there is a unique analytic structure for $\hat{S}(n)$ which makes it a submanifold of $GL(2^*, \mathbb{C})$ with the relative topology. Similarly, $S(n)$ is a submanifold of $\hat{S}(n)$ with the relative topology (see p. 105, [5]). Thus the same statements apply to the corresponding bundles $\mathcal{S}[M]$, $\hat{\mathcal{S}}[M], \mathbf{P}[M]$.

2. Coordinates and Complex Tangent Vectors to $\mathbf{P}[M]$

We assign coordinates on $\mathbf{P}[M]$ as follows. Suppose $p \in \mathbf{P}[M]$, $\tilde{\mathcal{N}}(p) = m \in U_\alpha$, the coordinates of $m$ in $U_\alpha$ are $(x^i)_i = 1, 2, ..., n$, and

$$p = (X_\alpha^\gamma |_m)$$

for some $X = (X_\alpha^\gamma) \in GL(2^*, \mathbb{C})$. If

$$X_\gamma^\alpha = V_\gamma^\alpha + \sqrt{-1} W_\gamma^\alpha$$

(where $\sqrt{-1}$ denotes the complex number $(0, 1)$) then the real coordinates of $p$ associated with the coordinate neighborhood $\tilde{\mathcal{N}}^{-1}(U_\alpha)$ are

$$(x_1^1, x_2^2, ..., x_n^n, V_1^1, V_2^2, ..., V_n^n, W_1^1, ..., W_2^n) = (x_i^\alpha, V_\gamma^\alpha, W_\gamma^\alpha).$$

The complex coordinates of $p$ in $\tilde{\mathcal{N}}^{-1}(U_\alpha)$ are $(x_i^\alpha, X_\gamma^\alpha)$. We will find these more convenient to work with.
Suppose \( \hat{\Pi}(p) = m \in U_A \cap U_B \) and
\[
p = (X_i^m e_i | m) = (X_i^m e_i | m).
\]
If
\[
\hat{y}_{BA}(m) = c = (c_\alpha) \in S(n)
\]
then clearly
\[
'X_\alpha = c_\alpha X_\alpha.
\]
(2.1)
If \((x^k)\) are the coordinates of \(m\) associated with \(U_B\), then the coordinates of \(\hat{p}\) associated with the coordinate neighborhood \(\hat{\pi}^{-1}(U_B)\) are \((x^k, 'X_\alpha)\).
Of course
\[
x_B^k = f^k(x^l),
\]
(2.2)
where the \(f^k\) are \(C^\infty\) functions of the \(x^l\). We will restrict our attention to coordinate systems on \(P[M]\) of the type described above. They are related to one another by (2.1) and (2.2).

A complex tangent vector to \(P[M]\) at \(p \in P[M]\) is defined in a manner similar to real tangent vectors, the difference being that if \((x^l, X_\alpha)\) is a coordinate system about \(p\), a complex tangent vector operates on complex-valued functions which are \(C^\infty\) in the \(x^l\) and holomorphic in the \(X_\alpha\) in a neighborhood about \(p\). Then in terms of the local coordinate system \((x^l, X_\alpha)\) a complex tangent vector at \(p\) is of the form
\[
v^l \frac{\partial}{\partial x^l} \bigg|_{\#(p)} + B_\alpha \frac{\partial}{\partial X_\alpha} \bigg|_{\#(p)},
\]
where \(v^l, B_\alpha \in \mathbb{C}\). We denote the vector space of all complex tangent vectors to \(P[M]\) at \(p\) by \(P_p\). A complex 1-form at \(p\) is an element of the space \(P_p^*\), i.e. the dual of \(P_p\). If we agree that
\[
\left\langle dX_\alpha | p, \frac{\partial}{\partial X_\alpha} \bigg|_{\#(p)} \right\rangle = \delta_\gamma^\alpha \delta_\alpha^\gamma, \quad \left\langle dx^j |_{\#(p)}, \frac{\partial}{\partial x^j} \bigg|_{\#(p)} \right\rangle = \delta^j_i
\]
\[
\left\langle dx^l |_{\#(p)}, \frac{\partial}{\partial x^l} \bigg|_{\#(p)} \right\rangle = 0, \quad \left\langle dX_\alpha | p, \frac{\partial}{\partial X_\alpha} \bigg|_{\#(p)} \right\rangle = 0,
\]
then in terms of local coordinates a complex 1-form looks like
\[
u_i dx^i |_{\#(p)} + E_\alpha dx_\alpha | p
\]
where \(u_i, E_\alpha \in \mathbb{C}\).

Many ideas involving real tangent vectors and forms have their analogues involving complex tangent vectors and complex forms. For instance, we may discuss higher order complex tensors, complex forms, exterior derivative, etc., on \(P[M]\). Hereafter when we use the terms "vector" and "form" we
mean "complex vector" and "complex form," then considering the exterior derivative of a form we assume that in a local coordinate system \((x^i, X_\beta^a)\) the components of the form are complex valued functions which are \(C^\infty\) in the \(x^i\) and holomorphic in the \(X_\beta^a\).

We can establish a 1-1 correspondence between the real tangent vectors to \(P[M]\) at \(p\) and a certain subset of complex tangent vectors as follows. Suppose \((x^i, X_\beta^a)\) is a local coordinate system about \(p \in P[M]\) and

\[
X_\beta^a = V_\beta^a + \sqrt{-1} W_\beta^a.
\]

A real tangent vector is of the form

\[
T = v^i \frac{\partial}{\partial x^i} \bigg|_{p} + C_\gamma^a \frac{\partial}{\partial V_\gamma^a} \bigg|_{p} + D_\nu^a \frac{\partial}{\partial W_\nu^a} \bigg|_{p},
\]

where \(v^i, C_\gamma^a, D_\nu^a \in R\). How make the correspondence

\[
\frac{\partial}{\partial V_\nu^a} \leftrightarrow \frac{\partial}{\partial X_\nu^a}, \quad \frac{\partial}{\partial W_\nu^a} \leftrightarrow \sqrt{-1} \frac{\partial}{\partial X_\nu^a},
\]

then

\[
T \rightarrow T' = v^i \frac{\partial}{\partial x^i} \bigg|_{p} + C_\gamma^a \frac{\partial}{\partial X_\gamma^a} + \sqrt{-1} D_\nu^a \frac{\partial}{\partial X_\nu^a}
\]

\[
= v^i \frac{\partial}{\partial x^i} \bigg|_{p} + \left(C_\gamma^a + \sqrt{-1} D_\nu^a\right) \frac{\partial}{\partial X_\gamma^a} \bigg|_{p}.
\]

Notice in the above equation the \(v^i\) are real. Given any such complex tangent vector as above where the \(v^i\) are real we can reverse the above process to obtain a real tangent vector. The above correspondence, although defined in terms of a local coordinate system, is independent of which of our local coordinate system we use. In fact the correspondence is preserved under the rather broad class of transformations

\[
'x^k = f^k(x^i), \quad 'X_\beta^a = g_\beta^a(x^i, X_\delta^a),
\]

where these functions are assumed to be \(C^\infty\) with respect to the \(x^i\) and holomorphic with respect to the \(X_\beta^a\). Similar statements apply to their inverses.

3. Connections on \(P[M]\)

On the basis of the above discussion concerning complex tangent vectors and their relation to real tangent vectors, it is not difficult to see that we can define what is meant by a connection on \(P[M]\) using complex tangent vectors.
**DEFINITION.** \( V_p \) is the subspace of \( P_p \) consisting of all vectors tangent to the fiber of \( P[M] \) at \( p \in P[M] \).

Thus if \((x^i, X^a)\) is a local coordinate system about \( p \), the space \( V_p \) is spanned by the vectors

\[
\frac{\partial}{\partial X^a} \bigg|_p.
\]

**DEFINITION.** A connexion \( \Gamma \) on \( P[M] \) is a selection of subspaces of \( P_p \) (called horizontal subspaces and denoted by \( H_p \)) for each \( p \in P[M] \) such that

1. \( P_p = V_p \oplus H_p \).
2. If \( R_a \) is the symbol for right translation by \( a \in GL(2^n, \mathbb{C}) \) and \( T \in H_p \), then \( (R_a)_* H_p = H_{p^a} \); i.e., \( H_p \) is invariant under right translation.
3. \( H_p \) depends differentiably on \( p \). That is if \((x^i, X^a)\) is a local coordinate system about \( p \) and \( X^a = V^a + \sqrt{-1} W^a \) and \( T \) is a complex tangent vector field in a neighborhood of \( p \) whose components in the local coordinate system \((x^i, X^a)\) are \( C^\infty \) functions of \((x^i, V^a, W^a)\), then if we write \( T = vT + hT \) where \( vT \in V \) and \( hT \in H \), the fields \( vT \) and \( hT \) are \( C^\infty \) functions in the above sense.

It can be shown from the general theory of connections or proved in a straightforward way on the basis of the above concepts that a connexion \( \Gamma \) on \( P[M] \) is equivalent to the assignment to each coordinate neighborhood \( U_A \) a set of \( n \cdot 2^n \) complex-valued \( C^\infty \) functions on \( U_A \) called connection components

\[
\Gamma^a_{\beta k} (x^\gamma, \gamma = 1, 2, ..., 2^n; k = 1, 2, ..., n).
\]

If \( U_B \) is another coordinate neighborhood and \((x^i)_B\), \((x^i)_A\) are the coordinate systems on \( U_A \) and \( U_B \), respectively, and

\[
\frac{\partial}{\partial x^i}_m = h^{-1}_i(m) \frac{\partial}{\partial x^i}_A |_m, \quad \tilde{\gamma}_{AB} = c(m) = (c^a(m)) \in S(n)
\]

for \( m \in U_A \cap U_B \), then

\[
\Gamma^a_{\beta k} = h^{-1}_i c^{-1}_a [c^a c^x_{\beta k} - \frac{\partial}{\partial x^i}_A (c^a)]
\]

on \( U_A \cap U_B \). In turn, if we define

\[
\omega^a_{\beta k} = \Gamma^a_{\beta k} dx^k_A
\]

then the assignment of connection components is equivalent to giving the forms \( \omega^a_{\beta k} \) with the transformation law

\[
\epsilon_{\beta k}^a = c^a_{\gamma \alpha} [c^\alpha_{\beta k} c^\gamma_{\alpha} - (d c^a_{\gamma}) c^{-1} c_{\beta k}].
\]
It can be shown that if \((x^i, \mathbf{X}^a)\) is a local coordinate system on \(\mathbf{P}[M]\) then the connexion form associated with a connexion \(\Gamma\) on \(\mathbf{P}[M]\) can be written locally in terms of \((x^i, \mathbf{X}^a)\) as the matrix valued form \(\Omega = (\Omega^a_{\beta})\),

\[
\Omega^a_{\beta} = Y_\alpha^a \, dX^\alpha_{\beta} + Y^a_{\lambda} \omega_{\gamma} X^\gamma_{\beta},
\]

where \(Y = X^{-1}\). Note that the above equation is independent of the coordinate system chosen, i.e., if \((x^i, \mathbf{X}^a)\) is another coordinate system and we let

\[
\Omega^a_{\beta} = Y^a_{\alpha} \, dX^\alpha_{\beta} + Y^a_{\lambda} \omega_{\gamma} X^\gamma_{\beta}
\]

then

\[
\Omega^a_{\beta} = \Omega_{\beta}^a.
\]

The relationship between the connection form and the horizontal subspace \(H_x\) is that these spaces are composed of precisely those vectors \(T\) for which \([\Omega^a_{\beta}, T] = 0\).

Suppose \(\gamma = \{x_t \mid x_t \in M, 0 \leq t \leq 1\}\) is a curve on \(M\) passing thru the coordinate neighborhood \(U\) and \((x^i)\) is the local coordinate system on \(U\). Then in \(U\)

\[
x^i(x_t) = y^i(t).
\]

Suppose \((x^i, \mathbf{X}^a)\) is the local coordinate system on \(\tilde{\pi}^{-1}(U)\). If \(\gamma^* = \{u_t\}\) is a lift of \(\gamma\), then we can write \(\gamma^*\) in coordinate form as

\[
x^i(u_t) = y^i(t) \quad \mathbf{X}^a(u_t) = a^a_b(t),
\]

where \(a(t) = (a^a_b(t)) \in GL(2, \mathbb{C}), 0 \leq t \leq 1\). Now if \(T\) is the complex tangent vector to \(\gamma^*\), that is

\[
T = \frac{dy^i}{dt} \frac{\partial}{\partial x^i} + \frac{da^a_b}{dt} \frac{\partial}{\partial \mathbf{X}^a_b},
\]

then since \(\gamma^*\) is horizontal \([\Omega, T] = 0\). This implies

\[
\frac{da^a_b}{dt} = - T^a_{\gamma} \omega_{\gamma} dy^k_{\gamma}.
\]

From the general theory of connections we know that a connection \(\Gamma\) on \(P[M]\) makes possible the definition of parallel translation of spinors and thus covariant differentiation of spinor fields in a given direction on \(M\) (see p. 114, [6]). Consider the covariant derivative in a local coordinate system. Let \(\psi\) be a contravariant spinor field on \(M\) and \(N\) a tangent vector field on \(M\). Further, let \(\gamma = \{x_t\}\) be the integral curve on \(N\) starting from \(x_0 = m \in U_A\). If

\[
x^i(x_t) = y^i(t) \quad \text{and} \quad \psi(x_t) = \psi^\alpha(t) \mathbf{e}_\alpha |_{x_t},
\]
then using the definition of the covariant derivative of \( \psi \) at \( m \) in the direction \( N \) and (3.2), it is not difficult to show

\[
(\nabla_N \psi)_m = \left[ \frac{d\psi^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma} \psi^\beta \frac{dy^\gamma}{dt} \right] e^\alpha_m.
\]

We will call the quantity in the above brackets \( \nabla_N \psi^\alpha \).

4. Structure Equations

Given a connection \( \Gamma \) on \( \mathbf{P}[M] \), we know from the general theory of connections that we can always write a structure equation namely the one involving the curvature form (see p. 77, [6]). Given the connection form of \( \Gamma \)

\[
\Omega^\sigma_\alpha = Y^\alpha_\nu dX^\sigma_\nu + Y^\alpha_\lambda \omega^\lambda_\nu X^\sigma_\nu, \tag{4.1}
\]

using the exterior derivative operator, one can show that

\[
d\Omega^\sigma_\alpha + \Omega^\sigma_\alpha \wedge \Omega^\alpha_\beta = R^\sigma_\beta, \tag{4.2}
\]

where

\[
R^\sigma_\beta = Y^\alpha_\nu (d\omega^\alpha_\nu + \omega^\alpha_\lambda \wedge \omega^\lambda_\nu) X^\beta_\nu.
\]

The matrix-valued form \( R = (R^\sigma_\beta) \) is the curvature form. In the remainder of this section we will make use of matrix notation. Thus if

\[
\Omega = (\Omega^\sigma_\alpha), \quad \omega = (\omega^\sigma_\alpha), \quad X = (X^\sigma_\alpha), \quad Y = X^{-1}, \quad dX = (dX^\alpha_\sigma)
\]

then (4.1) becomes in matrix form

\[
\Omega = Y \cdot dX + Y \cdot \omega \cdot X
\]

and (4.2) becomes

\[
d\Omega + \Omega \wedge \Omega = R.
\]

This last relation will be called the first structure equation.

We now derive a second structure equation and show that the first and second structure equations constitute necessary and sufficient conditions that a \( 2^r \times 2^r \) matrix of 1-forms on \( \mathbf{P}[M] \) be a connection form for some connection on \( \mathbf{P}[M] \). We use a procedure similar to that used by A. Seiken who obtained and discussed a second structure equation for the bundle \( B^r_s \) of tensor frames of type \((r, s)\) (see p. 39, [8]).

Suppose \( p \in \mathbf{P}[M] \) and

\[
p = (X^\sigma_\alpha e^\alpha_m),
\]

where \( m \in U_A \). At \( p \in \mathbf{P}[M] \) we define:
DEFINITION. The covariant frame-valued function \( \theta(\mathcal{P}) = (\theta^\alpha(\mathcal{P})) \), where

\[
\theta^\alpha(\mathcal{P}) = Y^\beta_{\alpha} e^\beta |_m
\]

\( \alpha = 1, 2, \ldots, 2^r \); \( Y = (Y^\beta) = X^{-1} \) and \((e^\beta |_m)\) is dual to \((e_\alpha |_m)\). In matrix form we write

\[
\theta = Y \cdot e.
\]

(4.3)

A different type of exterior derivative is defined by

DEFINITION.

\[
d\theta^\alpha = dY^\beta_{\alpha} \otimes e^\beta |_m.
\]

In matrix form we write this last relation as

\[
d\theta = dY \otimes e._A.
\]

Notice that \( \theta \) is defined invariantly, that is, independent of the local coordinate system on \( \mathcal{P}[M] \). However, \( d\theta \) is not defined invariantly. Its transformation law under a change of coordinates \((\mathcal{X}', \mathcal{X}') \rightarrow (\mathcal{X}, \mathcal{X})\) is

\[
(d\theta)' = d\theta - (Y \cdot e^{-1} \cdot dc \cdot X) \otimes \theta,
\]

where \( e = (e^\alpha_A) = \vec{Y}_{BA} \).

Now let a connection \( \Gamma \) on \( \mathcal{P}[M] \) be given with connection form

\[
\Omega = Y \cdot dX + Y \cdot \omega \cdot X.
\]

THEOREM. \( \theta \) and \( \Omega \) satisfy the equation

\[
d\theta + \Omega \otimes \theta = (Y \cdot \omega \cdot X) \otimes \theta.
\]

(4.5)

Further, the above equation is independent of the choice of coordinates on \( \mathcal{P}[M] \) and thus globally defined.

PROOF. Now \( X \cdot Y = I \) so \( dY = - Y \cdot dX \cdot Y \). Therefore

\[
d\theta = - Y \cdot dX \cdot Y \otimes e = - Y \cdot dX \otimes Y \cdot e_A,
\]

\[
d\theta = - Y \cdot dX \otimes \theta.
\]

Now substitute \( Y \cdot dX = \Omega - Y \cdot \omega \cdot X \) into the above equation to obtain (4.5). To show (4.5) is independent of coordinate system, suppose we write in another coordinate system \((\mathcal{X}', X_\beta')\)

\[
(d\theta)' + \Omega' \otimes \theta' = (Y' \cdot \omega' \cdot X') \otimes \theta'.
\]

(4.6)

Since the equations giving \( \Omega \) and \( \theta \) are independent of the local coordinate system involved, we have \( \Omega' = \Omega \) and \( \theta' = \theta \). We have seen in (2.1) that
'X = c \cdot X$ and so $Y = Y \cdot c^{-1}$. The transformation law for $(d\theta)'$ is given in (4.4). Substitution for the primed quantities in (4.6) we obtain
\[ d\theta - (Y \cdot c^{-1} \cdot dc \cdot X) \otimes \theta + \Omega \otimes \theta = [Y \cdot c^{-1} \cdot (c \cdot \omega \cdot c^{-1} - dc \cdot c^{-1}) \cdot c \cdot X] \otimes \theta. \]
Simplifying, we have
\[ d\theta + \Omega \otimes \theta = (Y \cdot \omega \cdot X) \otimes \theta. \tag{4.7} \]
Thus (4.6) is equivalent to (4.7) which is the desired result.

We call Eq. (4.5) the second structure equation.

We have seen that if $\Omega$ is a connection form on $\mathcal{P}[M]$ then it is necessary that it satisfy the first and second structure equations. We now show that these are also sufficient conditions.

**Theorem.** Let $\Omega$ be a $2r \times 2r$ matrix of independent 1-forms on $\mathcal{P}[M]$ and let $\theta$ be as given by (4.3). Assume $\Omega$ and $\theta$ satisfy
\[ d\Omega + \Omega \wedge \Omega = R \tag{4.8} \]
\[ d\theta + \Omega \otimes \theta = \psi \otimes \theta, \tag{4.9} \]
where $\psi$ is a $2r \times 2r$ matrix of local 1-forms in $V_p^0$ for each $p$ and $R$ is a matrix of 2-forms in $V_p^0 \wedge V_p^0$ for each $p$ ($V_p^0$ is the annihilator of $V_p$).

Under these conditions $\Omega$ is a connection form for some connection on $\mathcal{P}[M]$.

**Proof.** We know $d\theta = -Y \cdot dX \otimes \theta$. If we substitute this in (4.9), we obtain
\[ -Y \cdot dX \otimes \theta + \Omega \otimes \theta = \psi \otimes \theta, \quad (-Y \cdot dX + \Omega - \psi) \otimes \theta = 0. \]
This implies
\[ -Y \cdot dX + \Omega - \psi = 0. \]
Define $\omega$ to be $X \cdot \psi \cdot Y$. If we substitute $\psi = Y \cdot \omega \cdot X$ into above equation we obtain
\[ \Omega = Y \cdot dX + Y \cdot \omega \cdot X. \tag{4.10} \]
Now to show that $\Omega$ is a connection form it will suffice to show: (a) $\omega$ depends on $m \in M$ alone; and (b) $\omega$ satisfies the transformation law (3.1). To show statement (a) we multiply (4.10) on the left by $X$ and obtain
\[ -dX + X \cdot \Omega = \omega \cdot X. \tag{4.11} \]
Take the exterior derivative of both sides of above equation
\[ dX \wedge \Omega + X \cdot d\Omega = d\omega \cdot X - \omega \wedge dX. \tag{4.12} \]
Substitute $dX$ obtained from (4.11) into (4.12)

$$(X \cdot \Omega - \omega \cdot X) \wedge \Omega + X \cdot d\Omega = d\omega \cdot X - \omega \wedge (X \cdot \Omega - \omega \cdot X).$$

Simplifying, we obtain

$$X \cdot (d\Omega + \Omega \wedge \Omega) = (d\omega + \omega \wedge \omega) \cdot X$$
or

$$d\Omega + \Omega \wedge \Omega = Y \cdot (d\omega + \omega \wedge \omega) \cdot X.$$

Now using (4.8) we have

$$Y \cdot (d\omega + \omega \wedge \omega) \cdot X = R, \quad d\omega = X \cdot R \cdot Y - \omega \wedge \omega.$$

By our original assumption about $R$, $\psi$ and the definition of $\omega$, we see that $d\omega$ is in $V_p^0 \wedge V_p^0$ for each $p$, i.e., it is composed of terms involving $dx^i \wedge dx^j$. Therefore it cannot depend on $X^a$ for if it did it would involve the $dX^a$.

We now prove statement (b). The change of coordinates $(x^i) \rightarrow (\tilde{x}^i)$ on $M$ induces a change of coordinates $(x^i, X^a) \rightarrow (\tilde{x}^i, \tilde{X}^a)$ on $\mathcal{P}[M]$. We know from part (a) above that in terms of $(x^i, X^a)$

$$Y \cdot dX = \Omega - Y \cdot \omega \cdot X.$$  \hspace{1cm} (4.13)

With respect to the $(\tilde{x}^i, \tilde{X}^a)$ we write

$$Y \cdot d'X = \Omega - Y \cdot \omega \cdot \tilde{X}.$$  \hspace{1cm} (4.14)

Subtract (4.13) from (4.14) to obtain

$$Y \cdot d'X - Y \cdot dX = - Y \cdot \omega \cdot \tilde{X} + Y \cdot \omega \cdot X$$

and since $\tilde{X} = c \cdot X$, write

$$Y \cdot c^{-1}[dc \cdot X + c \cdot dX] - Y \cdot dX = - Y \cdot c^{-1} \cdot \omega \cdot c \cdot X + Y \cdot \omega \cdot X$$

$$Y \cdot c^{-1} \cdot dc \cdot X = Y \cdot [- c^{-1} \cdot \omega \cdot c + \omega] \cdot X.$$

Multiplying above on left by $X$ and on right by $Y$, we obtain

$$c^{-1} \cdot dc = - c^{-1} \cdot \tilde{\omega} \cdot c + \omega, \quad \tilde{\omega} = c \cdot \omega \cdot c^{-1} - dc \cdot c^{-1},$$

which is the transformation law (3.1).

5. The Fundamental Tensor-Spinor and Reduction of Connections

Let $m \in U_A$, let $(e_i \mid_m)$ be the natural orthonormal (vector) frame at $m$ associated with $U_A$ and $ct \ (e_a \mid_m)$ be the natural contra-spinor frame at $m$. 

associated with $U_A$. Let $(e^a | m)$ be the dual frame to $(e_a | m)$. The fundamental tensor-spinor $P$ at $m_A$ is

$$P = \tilde{P}_\beta^i e_i | m \otimes e_a | m \otimes e^\beta | m.$$ 

Let $(e_i | m)$ be an arbitrary (vector) frame and $(e_a | m)$ be an arbitrary contravariant frame

$$e_i | m = \ell^{-1}_i \bar{e}_j | m, \quad e_a | m = a^{-1}_a \bar{e}_\beta | m,$$

where $\ell = (\ell_i^j) \in GL(n, R)$, $a = (a^\alpha_\beta) \in GL(2^n, C)$. Then

$$P = 'P_\beta^i e_i | m \otimes e_a | m \otimes e^\beta | m,$$

where

$$'P_\beta^i = \ell_j^i a_\alpha^\beta \tilde{P}_n^\alpha \delta_{\beta}^{1-n}.$$ 

By definition of the map $\hat{\chi}_n$ it is clear that $'P_\beta^i = \tilde{P}_\beta^i$ if and only if $a \in \hat{S}(n)$, $\ell \in 0(n)$, and $\hat{\chi}_n(a) = \ell$. In this case we also say $(e_i | m) \in 0[M]$, $(e_a | m) \in \hat{S}(n)$ and $\hat{\chi}_n([e_a | m]) = (e_i | m)$. This correspondence is preserved under change of coordinates. The map $\hat{\chi}_n$ can also be extended to a map of connections. That is, if $\Gamma$ is a connection on $\hat{\mathcal{P}}[M]$ then the map $\hat{\chi}_n$ from $\hat{\mathcal{P}}[M]$ to $0[M]$ gives rise in a natural way to a connection $F$ on $0[M]$ which we call $\chi_n(\Gamma)$. Also if a connection $\Gamma$ on $\mathcal{P}[M]$ is reducible to a connection $\Gamma'$ on $\hat{\mathcal{P}}[M]$ and a connection $F$ on $L[M]$ is reducible to a connection $F'$ on $0[M]$ and $\chi_n(\Gamma') = F'$, then we also say $\chi_n(\Gamma) = F$.

We define the principal bundle $B[M]$ to be the set of tensor-spinor frames $(e^{\alpha}_a | m)$

$$e^{\beta}_a | m = e_i | m \otimes e_a | m \otimes e^\beta | m,$$

where $(e_i | m) \in L[M]$ the bundle of linear frames, $(e_a | m) \in \mathcal{P}[M]$ and $(e^\beta | m)$ is its dual frame. The fundamental tensor-spinor provides a means of selecting a subbundle of $B[M]$ similar to the way in which the metric tensor determines the set of orthonormal frames $0[M]$ as a subbundle of $L[M]$. We define $\bar{B}[M]$ to be the bundle of preferred frames of $B[M]$ where $(e^{\beta}_a | m)$ is a preferred frame provided the components of $P$ with respect to it are $\tilde{P}_\beta^i$.

The structure group of the bundles $B[M]$ and $B[M]$ can be constructed in a rather obvious way from the groups $GL(2^n, C)$, $GL(n, R)$, $\hat{S}(n)$ and $0(n)$ by observing the transformation law of the frames $(e^{\beta}_a)$. An element of the structure group $H$ of $B[M]$ is of the form

$$A = (h_i^j a_\beta^\gamma a_\gamma^\beta).$$
where \( h = (h_i^j) \in \text{GL}(n, R) \) and \( a = (a_\alpha^\beta) \in \text{GL}(2^\nu, C) \).

If

\[
A_1 = (h_i^j a_\alpha^\beta q_i^\gamma q_\beta^\gamma), \quad A_2 = (h_i^j a_\alpha^\beta q_i^\gamma q_\beta^\gamma)
\]

then

\[
A_1 \cdot A_2 = (h_i^j h_k^j a_\alpha^\beta q_n^\gamma q_i^\gamma q_\beta^\gamma q_\gamma^\delta q_\delta^\gamma).
\]

If \( A_1 = A_2 \) then \( h_1 = h_2 \) and \( a_1 = \lambda a_2 \). Thus \( H \) can be identified with \( \text{GL}(n, R) \times \text{GL}(2^\nu, C)/\{\lambda I\} \) where \( \text{GL}(2^\nu, C)/\{\lambda I\} \) has the usual manifold structure (see p. 43, [6]). So \( H \) is a Lie group. We define a continuous function

\[
g : A \rightarrow (P^i - h_j^i a \cdot \dot{P}^i \cdot a^{-1}).
\]

Now the structure group of \( \hat{B}[M] \) is

\[
\hat{H} = g^{-1}(0, 0, \ldots, 0),
\]

i.e., all \( A \) such that \( h \in O(n), a \in S(n) \), and \( \hat{q}_n(a) = h \). So \( \hat{H} \) is closed in \( H \) and thus has a unique analytic structure which makes it a subspace (i.e., has the relative topology) of \( H \). Therefore \( \hat{B}[M] \) has a unique manifold structure which makes it a subspace of \( B[M] \).

Suppose now we have a connection \( F \) on \( L[M] \) and a connection \( \Gamma' \) on \( P[M] \). These induce a connection \( L \) on \( B[M] \) (and thus enable us to find the covariant derivative of the fundamental tensor-spinor in a given direction \( N \) on \( M \)). This induced connection is the one whose lifts are the tensor products of the lifts of \( \Gamma \) and \( F \). Thus if \( \{v_i\} \) is a lift determined by \( F \) and \( \{\omega_i\} \) is a lift determined by \( \Gamma' \) where \( v_i = (e_i \mid t) \) and \( \omega_i = (e_\alpha \mid t) \) then a lift of the induced connexion will be \( u_i = (e_i \mid t \otimes e_\alpha \mid t \otimes \varepsilon_\beta \mid t) \) where \( (\varepsilon_\beta \mid t) \) is dual to \( (e_\alpha \mid t) \).

**Theorem.** \( L \) is reducible to a connexion \( L' \) on \( \hat{B}[M] \) if and only if the fundamental tensor-spinor \( P \) is parallel with respect to \( L \) (i.e., its covariant derivative vanishes).

**Proof.** Suppose \( P \) is parallel with respect to \( L \). Let \( \gamma = \{x_t\} \) be a curve on \( M \) with tangent vector \( N(t) \) at \( x_t \). Let \( \gamma^* = \{u_t\} \) be the lift of \( \gamma \) in \( B \) starting from a point \( u_0 \in \hat{B}[M] \). Let \( P^\beta_\mu(t) \) be the components of \( P \) in reference to the frame \( u_t \) at \( x_t \). Now if we calculate \( \nabla_{N(t)}P \) at \( x_t \) in terms of the components \( P^\beta_\mu \) we get, assuming \( P \) is parallel,

\[
\nabla_{N(t)}P^i_\mu = \frac{dP^i_\beta}{dt} \bigg|_{x_t} = 0.
\]
This implies \( \dot{P}^{\alpha}_\beta(t) = \text{constant along } \gamma \). But \( \dot{P}^{\alpha}_\beta(0) = \dot{P}^{\alpha}_\beta \) since \( u_0 \) is a preferred frame. Therefore

\[
\dot{P}^{\alpha}_\beta(t) = \dot{P}^{\alpha}_\beta \text{ (along } \gamma).$

So \( u_t \in \dot{B}[M] \) for each \( t \). Hence the tangent to \( \gamma^* \) is tangent to \( B[M] \) at \( u_0 \) (\( \dot{B}[M] \) has the relative topology) and so every horizontal vector at \( u_0 \) is tangent to \( \dot{B}[M] \).

Conversely suppose \( L \) is reducible to a connection \( L' \) on \( \dot{B}[M] \). Let \( \gamma, \gamma^* \) be as above. Then since \( L \) is reducible to \( L' \), \( u_t \in \dot{B}[M] \) for each \( t \) i.e. \( u_t \) is a preferred frame. So

\[
\dot{P}^{\alpha}_\beta(t) = \dot{P}^{\alpha}_\beta \text{ (along } \gamma).
\]

Thus

\[
\nabla_{\gamma(t)} \dot{P}^{\alpha}_\beta = \frac{d\dot{P}^{\alpha}_\beta}{dt} = 0,
\]

which means \( P \) is parallel with respect to \( L \).

DEFINITION. A connection \( \Gamma \) on \( P[M] \) is called fundamental provided it is reducible to a connection \( L \) on \( \dot{P}[M] \).

THEOREM. Let \( F \) be connection on \( L[M] \) and \( \Gamma \) a connection on \( P[M] \) which induce a connection \( L \) on \( B[M] \) then \( L \) is reducible to a connection on \( \dot{B}[M] \) if and only if \( F \) is metric, \( \Gamma \) is fundamental, and \( \hat{\xi}_n(\Gamma) = F \).

PROOF. Let \( \gamma = \{x_t\} \) be a curve on \( M \), \( \gamma^* = \{u_t\} \) be its lift in \( B[M] \) starting from \( u_0 \in \dot{B}[M] \). Now \( L \) is reducible to a connection on \( \dot{B}[M] \) provided \( u_t \in \dot{B}[M] \) for each \( t \). We can write

\[
u_t = (e_t | x_t) \otimes (e_\alpha | x_t) \otimes e^\beta | x_t),
\]

where \( \{e_t | x_t\} \) is the lift of \( \gamma \) in \( L[M] \) determined by the connection \( F \) and \( \{e_\alpha | x_t\} \) is the lift of \( \gamma \) in \( P[M] \) determined by \( \Gamma \). By definition of \( \dot{B}[M] \), we know that \( u_t \in \dot{B}[M] \) provided \( (e_\alpha | x_t) \in \mathcal{P}[M], (e_t | x_t) \in 0[M] \) and

\[
\hat{\xi}_n[(e_\alpha | x_t)] = (e_t | x_t)
\]

for each \( t \). But this is true if and only if \( \Gamma \) is reducible to a connection on \( \mathcal{P}[M] \) (i.e., \( \Gamma \) is fundamental), \( F \) is reducible to a connection on \( 0[M] \) (i.e. \( F \) is metric) and

\[
\hat{\xi}_n(\Gamma) = F.
\]

COROLLARY. If the connections \( F \) on \( L[M] \) and \( \Gamma \) on \( P[M] \) induce the connection \( L \) on \( B[M] \), then the fundamental tensor-spinor \( P \) is parallel with respect to \( L \) if and only if \( F \) is metric, \( \Gamma \) fundamental and \( \hat{\xi}_n(\Gamma) = F \).
It is natural to ask the following question: If \( \Gamma \) is a connection on \( \mathcal{P}[M] \), when is \( \Gamma \) reducible to a connection on \( \mathcal{S}[M] \)? The next theorem answers this question in terms of the connection components.

**Theorem.** If \( \Gamma \) is a connection on \( \mathcal{P}[M] \) and \( \Gamma_k = (\Gamma^a_{\beta k}) \) are the \( n \) matrices of the connection components (given on each \( U_A \)) then \( \Gamma \) is reducible to a connection on \( \mathcal{S}[M] \) if and only if \( \text{Trace } \Gamma_k = 0 \) \((k = 1, 2, \ldots, n)\).

**Proof.** Suppose \( \gamma = \{x_t\} \) \(0 \leq t \leq 1\) is a curve on \( M \). We may as well assume \( \gamma \) lies entirely in a coordinate neighborhood \( U_A \) with coordinate system \((x^i)\). We write \( \gamma \) in coordinate form

\[
x^i(x_t) = y^i(t).
\]

Let \( \gamma^* = \{u_t\} \) be a lift of \( \gamma \) (determined by \( \Gamma \)) such that \( u_0 \in \mathcal{S}[M] \). We can write

\[
u_t = (a_\alpha(t)e_\alpha, a_\alpha(t),)
\]

where \( a(t) = a_\alpha(t) \in \mathcal{S}(n) \) and \( a(0) \in \mathcal{S}(n) \) since \( u_0 \in \mathcal{S}[M] \). In coordinate form we write \( \gamma^* \) as

\[
x^i(u_t) = y^i(t), \quad X^i_\beta(u_t) = a_\beta(t),
\]

where we are using a coordinate system \((x^i, X^i_\beta)\) on \( \tilde{\mathcal{P}}^{-1}(U_A) \) in \( \mathcal{P}[M] \). Now since \( \gamma^* \) is a lift, its tangent vector is horizontal. Hence by (3.2)

\[
\frac{d a_\alpha^\gamma}{dt} a_\beta^\gamma = -\Gamma^a_{\alpha k} \frac{dy^k}{dt}.
\]

Setting \( \alpha = \beta \) and summing, we obtain

\[
\frac{d a_\alpha^\gamma}{dt} a_\alpha^\gamma = \frac{1}{\rho} \frac{d \rho}{dt} = -\Gamma^a_{\alpha k} \frac{dy^k}{dt}, \tag{5.1}
\]

where \( \rho = \det a \). Now suppose \( \Gamma \) is reducible to a connection on \( \mathcal{S}[M] \). Since \( u_0 \in \mathcal{S}[M] \) this implies that \( u_t \in \mathcal{S}[M] \) for each \( t \), thus \( a(t) \in \mathcal{S}(n) \) for each \( t \). Hence \( \rho(t) = 1 \). So (5.1) becomes

\[
0 = -\Gamma^a_{\alpha k} \frac{dy^k}{dt}.
\]

But \( \gamma \) and its initial point \( x_0 \) were arbitrary so \( \Gamma^a_{\alpha k} = \text{Trace } \Gamma_k = 0 \) \((k = 1, 2, \ldots, n)\).

Conversely, suppose \( \text{Trace } \Gamma_k = 0 \) on \( U_A \), then from (5.1) we obtain

\[
\frac{1}{\rho} \frac{d \rho}{dt} = -\Gamma^a_{\alpha k} \frac{dy^k}{dt} = 0.
\]

Hence \( \rho(t) \equiv \text{constant along } \gamma \).
But since \( u_0 \in \mathcal{F}[M] \), \( \rho(0) = 1 \). Therefore \( \rho(t) = 1 \) for each \( t \). This means \( a(t) \in S(n) \) for each \( t \). So \( u_t \in \mathcal{F}[M] \) for each \( t \). So \( \Gamma \) is reducible to a connection on \( \mathcal{F}[M] \).

Let \( U_A \) be a coordinate neighborhood with coordinates \((x^i)\). Let \( \Gamma \) be a connection \( \mathbf{P}[M] \) with connection components \( \Gamma_{ab}^c \) on \( U_A \) and \( F \) a connection on \( L[M] \) with components \( F_{jk}^l \) on \( U_A \).

On \( U_A \), suppose
\[
P = P_{ab}^{ia} \frac{\partial}{\partial x^a} \otimes e^i_A \otimes e^j_A.
\]

Now if we assume the covariant derivative of \( P \) is zero, this leads to the equation
\[
\partial_a P_{ab}^{ia} + F_{sk} P_{ab}^{ia} + \Gamma_{nk}^a P_{ab}^{ia} - \Gamma_{ab}^k P_{sk}^{ia} = 0
\]
on \( U_A \). It is well known ([9] and [1]) that this equation implies
\[
F_{sk}^l = \frac{1}{2^\nu} P_{ab}^{ia} \left[ \partial_a P_{ab}^{ia} - \Gamma_{nk}^a P_{ab}^{ia} - \Gamma_{ab}^k P_{sk}^{ia} \right] \tag{5.2}
\]
and if \( \Gamma_k = (\Gamma_{ab}^k), P_t = (P_{ab}^{ia}) \)
\[
\Gamma_k = \frac{1}{2} P^i \cdot (\partial_a P_{ab} - F_{sk} P_{sk}) + a_k l, \tag{5.3}
\]
where \( a_k \) is an arbitrary complex number and \( \text{Trace } \Gamma_k = 2^\nu a_k \). Equation (5.2) corresponds to the equation \( \chi_n(\Gamma) = F \) and (5.3) corresponds to \( \Gamma = \chi_n^{-1}(F) \). If \( \chi_n \) is the restriction of \( \chi \) to \( \mathcal{F}[M] \) we see from (5.3) and the last theorem that given a connection \( F \) on \( 0[M] \) there is only one connection \( \Gamma \) on \( \mathcal{F}[M] \) such that \( \chi_n(\Gamma) = F \). This is essentially the same connection studied by Lichnerowicz and Guy.

It is the underlying principle of general relativity that one should try to explain physical phenomena geometrically. Thus the presence of a mass in space has the effect of "distorting" the space-time manifold (from a flat manifold) in its vicinity and objects such as planets move along geodesics in the space-time manifold.

What is the situation in relativistic quantum mechanics? When considering a spinor connection \( \Gamma \), the physicists assume \( \nabla P = 0 \). This means \( \Gamma \) is a connection on \( \mathcal{F}[M] \) and its connection components are given by (5.3). The trace of the connection components (Trace \( \Gamma_k \)) is considered to be a multiple of the electromagnetic potential vector. From the above considerations we see that we can look for a geometric effect of the electromagnetic potential in the fiber bundles considered above. Thus when working with a free field i.e. where the electromagnetic potential is zero, the proper bundle setting is \( \mathcal{F}[M] \). However, the presence of a nonzero potential has the effect of "distorting" the lifts of curves out of \( \mathcal{F}[M] \) and into \( \mathcal{F}[M] \) so that in this case the proper bundle setting is \( \mathcal{F}[M] \).
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REFERENCES


