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NOTE

Minimal Diameter of Certain Sets in the Plane

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We investigate the problem of finding the smallest diameter $D(n)$ of a set of n points such that all the mutual distances between them are at least 1. The asymptotic behaviour of $D(n)$ is known; the exact value of $D(n)$ can be easily found up to 6 points. Bateman and Erdős proved that $D(7) = 2$. In this paper we determine $D(8)$. © 1999 Academic Press

1. INTRODUCTION AND RESULTS

Points of the Euclidean plane $\mathbb{E}^2$ are usually denoted by capitals and sets of points by script capitals. The segment with endpoints P and Q is denoted by PQ. PQ will also denote the length of the segment PQ. $\triangle PQR$ denotes the triangle whose vertices are the points P, Q and R. By distance we mean the usual Euclidean distance of $\mathbb{E}^2$. The *diameter* of a pointset $\mathscr{K}$ is the least upper bound of the distances between its points. Note that the

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diameter of a polygon is equal to its longest diagonal. A line l is called a *supporting* line of $\mathscr{K}$ if l meets $\mathscr{K}$ and $\mathscr{K}$ is entirely contained in one of the closed halfplanes determined by l. We say that a convex region has a *constant width* w if the distance between any two of its parallel supporting lines is w.

We shall deal with the following question. Of all sets of n points in the plane such that the mutual distances are at least 1 what set has the minimum diameter $D(n)$? Bateman and Erdős mention in [3]without proof that " The asymptotic behaviour of $D(n)$ is well-known; in fact, for large n the hexagonal lattice gives about the best results, so that $D(n) \sim (12/\pi^2)^{1/4}\, n^{1/2}$ as $n \to \infty$." For completeness we give here our argument, which we think is short and straightforward: Let $\mathscr{C}_n$ be the set of n points with minimal diameter $D(n)$. Consider the packing of circles which are centered at the points of $\mathscr{C}_n$ and have radii $\frac{1}{2}$. These circles are contained in the $\frac{1}{2}$-paralleldomain of the convex hull of $\mathscr{C}_n$, i.e. in the region $\mathscr{C} = \text{conv}(\mathscr{C}_n) + B(\frac{1}{2})$, where conv(.) denotes the convex hull and $B(\frac{1}{2})$ denotes the circular disk of radius $\frac{1}{2}$. Then, according to a theorem of L. Fejes Tóth the area of $\mathscr{C}$ is at least n area$(B(\frac{1}{2}))(\sqrt{12}/\pi) = n(\sqrt{3}/2)$. Clearly the diameter of $\mathscr{C}$ is equal to $D(n)+1$. Thus, in view of Bieberbach theorem [4], which says that among convex regions of given area the circle has the smallest diameter, we have that $D(n)+1 \geqslant 2(n(\sqrt{3}/2\pi))^{1/2}$. Finally, the hexagonal lattice arrangement shows that asymptotically $D(n)$ is equal to this bound.

It is easy to determine $D(n)$ up to 6, [3]. In fact for $n = 2, 3, 4, 5$ the vertices of the regular n-gon of sidelength 1 give the optimal configuration and for $n = 6$ the minimal figure consists of the circumcenter and the vertices of the regular pentagon of circumradius 1. Thus $D(2) = D(3) = 1$, $D(4) = \sqrt{2}$, $D(5) = \frac{1}{2}(1+\sqrt{5})$ and $D(6) = 2 \sin 72°$. Bateman and Erdős [3] proved that $D(7) = 2$ and in the only optimal configuration 6 of the 7 points are the vertices of a regular hexagon of sidelength 1 and the 7th coincides with the circumcenter of this hexagon. Theorem 1.1 answers the question for $n = 8$.

THEOREM 1.1. *The minimal diameter* $D(8)$ *of a set of* 8 *points in the plane with mutual distances at least* 1 *is* $(2 \sin(\pi/14))^{-1} = 2.246...$. *The minimal diameter is attained only if the convex hull of the points is a regular heptagon of sidelength* 1.

Throughout this paper the critical value $(2 \sin(\pi/14))^{-1} = 2.246...$ will be denoted by d_0. Note that in Theorem 1.1 the optimal arrangement is not unique in the sense that the position of the eighth point is not uniquely determined. As for the next case $n = 9$ we know that the convex hull of the optimal arrangement is not the regular octagon. In fact the best known configuration is obtained by using a construction of Vincze. In [8] Vincze shows that there is an octagon with sides equal to 1 of diameter 2.58..., which is less than the diameter of the regular octagon (2.61...) of side length 1. It turns out that one can place an additional point inside this octagon maintaining that all the mutual distances are at least 1. Therefore $D(9) \leqslant 2.58\ldots$.

The same type of question can be considered in higher dimension too. For results and references see [1], [8]. Recently K. Bezdek and Connelly [2] found a rather general method (using rigidity theory) to determine minimal diameters. Their method works for those higher dimensional cases which were considered before.

In the course of our proof we shall use the following theorem which is a special case of a general theorem of Reinhardt [8] (essentially the same result later was rediscovered by Vincze [8] and then by Larman and Tamvakis [5]).

THEOREM 1.2. *If p is an odd prime, then the minimal diameter of a p-gon with sides at least* 1 *is* $(2\sin(\pi/2p))^{-1}$ *and equality holds only for the regular p-gon of sidelength* 1.

At the end of this introduction we describe the main ideas of the proof. We will consider cases similar to those of Bateman and Erdős [3]. Namely, we consider cases according to what the convex hull of the 8 points is. In each individual case we show that the diameter of the convex hull exceeds d_0 except when the convex hull is a regular heptagon of sidelength 1. This can be easily done once we argue that the perimeter of the convex hull is large enough. We simply recall the well-known formula $p = \frac{1}{2}\int_0^{2\pi} B(\alpha)\, d\alpha$, where p is the perimeter of a given convex curve, $B(\alpha)$ means the distance between two parallel lines of support, both belonging to the direction α. This implies that the diameter of the convex hull is at least p/π, in fact it can exceed d_0 depending on p. This fact is also known as the Cauchy lemma. We shall also have cases where the perimeter of the convex hull is not large enough to conclude that $p/\pi \geqslant d_0$. In these cases we will use the following facts.

It is known that if $\mathscr{P}$ is convex polygon of diameter d, then it can be extended to a convex figure $\mathscr{P}'$ of constant width d. Moreover, for any two points P, Q in $\mathscr{P}$ a circular arc of radius d through P and Q is fully contained in $\mathscr{P}'$.

We can now extend the convex hull of the points to a convex set, which has the same diameter and has sufficiently large perimeter. In fact, we draw circular arcs of radius d_0 over some of the sides of $\mathscr{P}$ such that we can still guarantee that the resulting figure is convex and has perimeter greater than πd_0.

2. PROOFS

We start the proof by stating the following two basic lemmas:

LEMMA 2.1. *If a convex polygon $\mathscr{P}$ has perimeter p, then its diameter d is at least p/π. In particular, if $p > 7.06$, then $d > d_0$.*

This is the direct consequence of the Cauchy lemma, which we have already discussed above.

LEMMA 2.2. *Let a, b and c be the sidelengths all greater than or equal to* 1 *of a given triangle and let P be a point of the triangle whose distance from each vertex is at least* 1. *Then*

(a) *the longest side of the triangle is at least $\sqrt{3}$ and the second longest side is at least $\sqrt{2}$.*

(b) *if $\sqrt{3} > a \geqslant b$ then $c \geqslant 2\sqrt{1-(1-(a^2/2))^2}$.*

(c) *in particular, if two sides of the triangle have lenghts less than* 1.55, *then the third side is not shorter than* 1.959.

Proof of Lemma 2.2. (a) One side of the triangle must subtend an angle of 120° or more at P. One of the remaining two sides must subtend an angle of 90° or more at P.

(b) Label the vertices of the triangle A, B and C such that A is across side a, B is across side b. Consider the angles $\angle APC$, $\angle BPC$. It is clear that these angles cannot exceed $\alpha = \arccos(1-(a^2/2))$. Hence $\angle APB \geqslant 360° - 2\alpha$. Therefore $c^2 \geqslant 2 - 2\cos(360° - 2\alpha) = 2 - 2\cos 2\alpha = 4(1-(1-(a^2/2))^2)$.

(c) This follows from (b).

Proof of Theorem 1.1. We shall consider 8 points in the plane with mutual distances at least 1. We distinguish six cases according as the convex hull of the 8 points is a triangle, quadrilateral, pentagon, ..., octagon. In each case we suppose that a circle of radius $\frac{1}{2}$ is drawn around each of the eight given points. Note that these circles do not overlap. Those circles whose center is different from the vertices of the convex hull of the eigth points will be called *inner circles*. We show throughout the proof that in all cases, except when the convex hull is a regular heptagon of sidelength 1, the diameter of the set exceeds $d_0 = (2\sin(\pi/14))^{-1} = 2.24\ldots$.

The triangular case. If one side of the triangle is intersected by two inner circles, then that side has length at least $3\sqrt{3}/2 = 2.59\cdots > d_0$ by the Pythagorean theorem. Otherwise it is easy to see that the total area of the circles belonging to the triangle is greater than the total area of 4 circles, i.e. it is π. It is easy to see that at least one side of the triangle is at least $\geqslant 2\pi^{1/2}3^{-1/4} = 2.69\ldots$, which greater than d_0.

The quadrilateral case. As we have seen in the triangular case, no side of the quadrangle can be intersected by two inner circles. If all sides of the quadrangle are intersected by exactly one inner circle, then all sides must be longer than $\sqrt{3}$. Therefore one of the diagonals must be at least $\sqrt{6} = 2.44 \cdots > d_0$. If there is a side which is disjoint from the inner circles, then the area of the quadrangle is at least the total area of 3.5 circles, namely $3.5\pi/4 = 2.74\ldots$. Since the product of the diagonals of a quadrilateral is at least twice the area of the quadrilateral we have that at least one of the diagonals of the quadrangle must be longer than $2.34 \cdots > d_0$.

The pentagonal case. Again each side of the pentagon is intersected by at most one inner circle. There are at most two sides which are intersected by inner circles. Otherwise the perimeter of the pentagon is at least $3\sqrt{3} + 2 = 7.19\ldots$ and we have by Lemma 2.1 that the diameter of the pentagon is greater than d_0. Thus there is an inner circle which is contained by the pentagon. Let P be the center of this circle and let Q, R be the centers of the other two inner circles. Let A, B be a pair of consecutive vertices of the pentagon such that Q belongs to $\triangle ABP$. We consider the cases when R belongs to this triangle and when it does not belong to it separately.

Case (i). *Assume that R belongs to $\triangle ABP$.* We claim that the if one of the segments AP and BP, say AP, is at least $\sqrt{3}$ then the diameter of the pentagon is greater than d_0. To see this let us extend AP until it meets a side EF at S. Since $PS < d_0 - \sqrt{3} = 0.51 \cdots < 1$ and both PE and PF are at least 1, we have that the projection of P onto the line EF belongs to the side EF and then $EF \geqslant 2\sqrt{1 - (d_0 - \sqrt{3})^2} = 1.71\ldots$. As P is at least $\frac{1}{2}$ from EF, AS must not deviate from the perpendicular more than 14° or AS is longer than d_0. So the component of AS perpendicular to EF is at least $\sqrt{3}\cos 14° + \frac{1}{2} = 2.18\ldots$. Thus one of AE and AF is at least $\sqrt{2.18^2 + (1.71/2)^2} = 2.34 \cdots > d_0$.

We may now assume that the sides PA and PB are not longer than $\sqrt{3}$. This means that none of the inner circles centered at Q and R can overlap the segments PA and PB. But we know that at least one of them, say the one centered at Q, is disjoint from the side AB, so it lies entirely in $\triangle ABP$. Thus the area of $\triangle ABP$ is at least $\pi/2$. It follows that $PAm_{PA}/2 \geqslant \pi/2$, so $m_{PA} \geqslant \pi/\sqrt{3} = 1.81\ldots$. Now both PB and AB must be at least as long as the altitude m_{PA}, hence $PB > 1.81 > \sqrt{3}$, a contradiction.

Case (ii). *Assume that R does not belong to $\triangle ABP$.* Let C, D be a pair of consecutive vertices of the pentagon such that R belongs to $\triangle CDP$. We can immediately assume that none of the sides PA, PB, PC, P through D

are longer than $\sqrt{3}$, otherwise the diameter would exceed d_0 as we have already proved it above. If none of these four sides are longer than 1.55, then we can extend our pentagon with drawing circular arcs of radius d_0 above the two sides of length 1.959 and another side such that the resulting figure is still convex. Then the perimeter of this figure is more than 7.06, so by Lemma 2.1 the diameter of the pentagon is greater than d_0. If some of the four sides are longer than 1.55 and PA is the longest among them, then there are two cases.

One is when the extension of PA hits CD in a point S. In this case notice the following. In $\triangle PCD$ the sides PC and PD are shorter than $\sqrt{3}$, so the one half of the circle around R is in $\triangle PCD$, so its area is at least $\pi/4$. Also $CDm_{CD}/2 \geqslant \pi/4$ and $CD \leqslant d_0$. Therefore $m_{CD} \geqslant \pi/2d_0 > 0.699$ and $AS = PA + PS \geqslant PA + m_{CD} = 2.249 > d_0$.

In the other case the extension of PA meets a side EF, different from CD. Suppose that PA has length a. Then $EF \geqslant 2\sqrt{1-(d_0-a)^2}$. By Lemma 2.2.b the lengths of AB and CD are at least $2\sqrt{(1-(1-(a^2/2))^2)}$. Therefore the perimeter of the pentagon is not shorter than $2\sqrt{1-(d_0-a)^2} + 4\sqrt{(1-(1-(a^2/2))^2)}$. Simple calculus shows that whenever a in between 1.55 and $\sqrt{3}$ the perimeter greater than 7.06, so the diameter exceeds d_0.

The hexagonal case. Again we may assume that no side of the hexagon is intersected by two inner circles. If there are at least two sides of the hexagon which are intersected by inner circles, then the perimeter is at least $2\sqrt{3} + 4 = 7.46...$ which, by Lemma 2.1, implies that the diameter of the hexagon is greater than d_0.

If at most one side of the hexagon has common point with inner circles, then there is an inner circle, which lies entirely in the hexagon. Let P be the center of this circle. Then the other inner point Q lies in some of the triangles determined by P and two consecutive vertices A, B of the hexagon. By Lemma 2.2.a $\triangle ABP$ has a side of length $> \sqrt{3}$. If this side is one of PA and PB, then the diameter of the hexagon exceeds d_0 as we have already seen in the pentagonal case. If both PA and PB are shorter than 1.55, then by Lemma 2.2.c, AB must be at least 1.959. Also, by Lemma 2.2.a, one of PA and PB is at least $\sqrt{2}$. Thus, extending this side until it meets the hexagon, we can estimate the length of the side it hits by the Pythagorean theorem. Namely, it is at least $2\sqrt{1-(d_0-\sqrt{2})^2} = 1.107...$. Therefore the perimeter of the hexagon exceeds 7.06 whis implies, by Lemma 2.1, that its diameter is greater than d_0.

If PA is longer than 1.55, then the side it hits is at least $2\sqrt{1-(d_0-1.55)^2} = 1.43...$. Thus the perimeter of the hexagon exceeds 7.06 and so its diameter is greater than d_0.

The heptagonal case. According to Theorem 1.2 the diameter of the convex hull of the points is at least d_0 and the minimum is attained only if the convex hull is a regular heptagon.

The octagonal case. The diameter of the octagon is greater than the diameter of the convex hull of seven of its vertices. But the later is at least d_0 according to the heptagonal case.

Thus we have finished the proof of Theorem 1.1.

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