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# Iterating lowering operators

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Dedicated to Eric M. Friedlander on his 60th birthday

**Abstract**

For an algebraically closed base field of positive characteristic, an algorithm to construct some non-zero $GL(n-1)$-high weight vectors of irreducible rational $GL(n)$-modules is suggested. It is based on the criterion proved in this paper for the existence of a set $A$ such that $S_{i,n}(A)f_{\mu,\lambda}$ is a non-zero $GL(n-1)$-high weight vector, where $S_{i,n}(A)$ is Kleshchev's lowering operator and $f_{\mu,\lambda}$ is a non-zero $GL(n-1)$-high weight vector of weight $\mu$ of the costandard $GL(n)$-module $\nabla_n(\lambda)$ with highest weight $\lambda$.



*MSC:* 20G05

## 1. Introduction

Classical lowering operators were introduced by Carter in [2]. Kleshchev used them in [4] to define generalized lowering operators. Following [1] and [5], we denote these operators by $S_{i,j}(A)$. Kleshchev's lowering operators are useful in constructing $GL(n-1)$-high weight vectors from the first level of irreducible rational $GL(n)$-modules. In fact, [4, Theorem 4.2] shows that every such vector has the form $S_{i,n}(A)v_+$, where $v_+$ is the $GL(n)$-high weight vector. A natural idea is to continue to apply lowering operators $S_{i,n}(A)$ to the $GL(n-1)$-high weight vectors already obtained in order to construct new $GL(n-1)$-high weight vectors belonging to higher levels. For example, this method was used in [5] to construct all $GL(n-1)$-high weight vectors of irreducible modules $L_n(\lambda)$, where $\lambda$ is a

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generalized Jantzen–Seitz weight. The main aim of this paper is to find all $\mathrm{GL}(n-1)$-high weight vectors that can be constructed in this way (Theorem 13 and Remark 2).

Let $K$ be an algebraically closed field of characteristic $p>0$ and $\mathrm{GL}(m)$ denote the group of invertible $m\times m$-matrices over $K$. We generally follow the notations of [5] and [1] and actually work with hyperalgebras rather than algebraic groups. For the connection between representations of the latter two, we refer the reader to [3]. Let $U(m,\mathbb{Z})$ denote the $\mathbb{Z}$-subalgebra of the universal enveloping algebra $U(m,\mathbb{C})$ of the Lie algebra $\mathfrak{gl}(m,\mathbb{C})$ that is generated by the identity element and

$$X_{i,j}^{(r)} := \frac{(X_{i,j})^r}{r!} \quad \text{for } 1\leqslant i, \quad j\leqslant m, i\neq j \quad \text{and} \quad r\geqslant 1,$$

$$\binom{X_{i,i}}{r} := \frac{X_{i,i}(X_{i,i}-1)\cdots(X_{i,i}-r+1)}{r!} \quad \text{for } 1\leqslant i\leqslant m \quad \text{and} \quad r\geqslant 1,$$

where $X_{i,j}$ denotes the $m\times m$-matrix with 1 in the *ij*-entry and zeros elsewhere. We define the *hyperalgebra* $U(m)$ to be $U(m,\mathbb{Z})\otimes_{\mathbb{Z}}K$. For $1\leqslant i<j\leqslant m$ we denote by $E_{i,j}^{(r)}$ and $F_{i,j}^{(r)}$ the images of $X_{i,j}^{(r)}$ and $X_{j,i}^{(r)}$, respectively and for $1\leqslant i\leqslant m$ denote by $\binom{H_i}{r}$ the image of $\binom{X_{i,i}}{r}$ under the above base change. If $r=1$ then we omit the superscripts in the above definitions and write $H_i$ for $\binom{H_i}{1}$. We also put $E_i^{(r)} := E_{i,i+1}^{(r)}$ and $F_{i,i}^{(r)} := 1$.

Let $U^0(m)$ denote the subalgebra of $U(m)$ generated by 1 and $\binom{H_i}{r}$ for $1\leqslant i\leqslant m$ and $r\geqslant 1$ and $X^+(m)$ denote the set of integer sequences $(\lambda_1,\ldots,\lambda_m)$ such that $\lambda_1\geqslant\cdots\geqslant\lambda_m$. We say that a vector $v$ of a $U(m)$-module *has weight* $\lambda\in X^+(m)$ if $\binom{H_i}{r}v=\binom{\lambda_i}{r}v$ for any $1\leqslant i\leqslant m$ and $r\geqslant 1$. If moreover $E_i^{(r)}v=0$ for any $1\leqslant i<m$ and $r\geqslant 1$, then we say that $v$ is a $U(m)$-*high weight vector*.

Throughout $[i..j]$, $(i..j]$, $[i..j)$, $(i..j)$ denote the sets $\{a\in\mathbb{Z}: i\leqslant a\leqslant j\}$, $\{a\in\mathbb{Z}: i<a\leqslant j\}$, $\{a\in\mathbb{Z}: i\leqslant a<j\}$, $\{a\in\mathbb{Z}: i<a<j\}$, respectively. For any condition $\mathscr{P}$, let $\delta_{\mathscr{P}}$ be 1 if $\mathscr{P}$ is true and 0 if it is false. Given a pair of integers $(i,j)$, let $\mathrm{res}_p(i,j)$ denote $(i-j)+p\mathbb{Z}$, which is an element of $\mathbb{Z}/p\mathbb{Z}$. For any set $A\subset\mathbb{Z}$ and two integers $i\leqslant j$, let $A_{i..j}=\{a\in A: i<a<j\}$. If moreover $A\subset(i..j)$ then we put $F_{i,j}^A=F_{a_0,a_1}\cdots F_{a_k,a_{k+1}}$, where $A\cup\{i,j\}=\{a_0<\cdots<a_{k+1}\}$. Thus $F_{i,j}^{\emptyset}=F_{i,j}$. For $i<j$ and $A\subset(i..j)$, the lowering operator $S_{i,j}(A)$ is defined as (see [1, Remark 4.8])

$$S_{i,j}(A) := \sum_{B\subset(i..j)} F_{i,j}^B H_{i,j}(A,B).$$

In this formula, $H_{i,j}(A,B)$ is the element of $U^0(m)$ obtained by evaluating the rational expression

$$\mathscr{H}_{i,j}(A,B) := \sum_{D\subset B\setminus A} (-1)^{|D|} \frac{\prod_{t\in A}(x_t-x_{D_i(t)})}{\prod_{t\in B}(x_t-x_{D_i(t)})},$$

where $D_i(t)=\max\{s\in D\cup\{i\}: s<t\}$, at $x_k:=k-H_k$. Elements $H_{i,j}(A,B)$ are well defined, since $\mathscr{H}_{i,j}(A,B)\in\mathbb{Z}[x_i,\ldots,x_{j-1}]$, which is proved in [1, Lemma 4.6(i)]. We additionally assume that $S_{i,i}(\emptyset)=1$.

Quite easy proofs of all the properties of the operators $S_{i,j}(A)$ we need here, can be found in [1], where the specialization $v \mapsto 1$ should be made.

In this paper, we work with costandard modules $\nabla_n(\lambda)$, where $\lambda \in X^+(n)$, and its non-zero $U(n-1)$-high weight vectors $f_{\mu,\lambda}$, where $\mu \in X^+(n-1)$ and $\lambda_i \geqslant \mu_i \geqslant \lambda_{i+1}$ for $1 \leqslant i < n$. If the last conditions hold we write $\mu \longleftarrow \lambda$. We also denote the element $f_{\bar{\lambda},\lambda}$, where $\bar{\lambda} = (\lambda_1, \dots, \lambda_{n-1})$, by $f_\lambda$. It is a $U(n)$-high weight vector generating the simple submodule $L_n(\lambda)$ of $\nabla_n(\lambda)$. The definitions of all these objects can be found in [5]. Moreover using [5, Lemma 2.6(ii)] and multiplication by a suitable power of the determinant representation of $\mathrm{GL}(n)$, we may assume that $f_\lambda$ and $f_{\mu,\lambda}$, where $\mu \longleftarrow \lambda$ and $a_i := \sum_{s=1}^{i} (\lambda_s - \mu_s)$, are chosen so that $E_1^{(a_1)} \cdots E_{n-1}^{(a_{n-1})} f_{\mu,\lambda} = f_\lambda$.

## 2. Graph of sequences

For the remainder of this paper, we fix an integer $n > 1$ and weights $\lambda \in X^+(n)$, $\mu \in X^+(n-1)$ such that $\mu \longleftarrow \lambda$. For $i = 1, \dots, n-1$, we put $a_i := \sum_{j=1}^{i} (\lambda_j - \mu_j)$. The following formulas can easily be checked by calculations in $U(n, \mathbb{Z})$.

**Lemma 1.** *Let $1 \leqslant i < j \leqslant n$, $1 \leqslant l < n$, $m \geqslant 1$ and $A \subset (i..j)$. We have*

(i) $E_l^{(m)} F_{i,j}^A = F_{i,j}^A E_l^{(m)}$ *if $l \notin A \cup \{i\}$ and $l+1 \notin A \cup \{j\}$,*

(ii) $E_l^{(m)} F_{i,j}^A = F_{i,j}^A E_l^{(m)} - F_{i,l}^{A_{i..l}} F_{l+1,j}^{A_{l+1..j}} E_l^{(m-1)}$ *if $l \in A \cup \{i\}$ and $l+1 \notin A \cup \{j\}$,*

(iii) $E_l^{(m)} F_{i,j}^A = F_{i,j}^A E_l^{(m)} + F_{i,l}^{A_{i..l}} F_{l+1,j}^{A_{l+1..j}} E_l^{(m-1)}$ *if $l \notin A \cup \{i\}$ and $l+1 \in A \cup \{j\}$,*

(iv) $E_l^{(m)} F_{i,j}^A = F_{i,j}^A E_l^{(m)} + F_{i,l}^{A_{i..l}} (H_l - H_{l+1} + 1 - m) F_{l+1,j}^{A_{l+1..j}} E_l^{(m-1)}$ *if $l \in A \cup \{i\}$ and $l+1 \in A \cup \{j\}$.*

We shall use the abbreviation $E(i,j) = E_i \cdots E_j$. Let $1 \leqslant i \leqslant k \leqslant j \leqslant n$ and $A \subset (i..j)$. It follows from Lemma 1 that $E(k, j-1) S_{i,j}(A) = u_k E_k + \cdots + u_{j-1} E_{j-1} + M_{i,j}^k(A)$, where $u_k, \dots, u_{j-1} \in U(n)$ and $M_{i,j}^k(A)$ is a linear combination of elements of the form $F_{i,k}^B H$, where $H \in U^0(n)$. In what follows, we stipulate that any not necessarily commutative product of the form $\prod_{i \in A} x_i$, where $A = \{a_1 < \cdots < a_m\} \subset \mathbb{Z}$, equals $x_{a_1} \cdots x_{a_m}$.

**Lemma 2.** *Given integers $1 \leqslant i_1 < j_1 < \cdots < i_{s-1} < j_{s-1} < i_s < j_s \leqslant n$, sets $A_1 \subset (i_1..j_1)$, $\dots$, $A_s \subset (i_s..j_s)$ and integers $k_1, \dots, k_s$ such that $i_t \leqslant k_t \leqslant j_t$ for $t = 1, \dots, s$ and $j_s = n$ implies $k_s = n$, we put*

$$v = E(k_1, j_1 - 1) S_{i_1,j_1}(A_1) \cdots E(k_s, j_s - 1) S_{i_s,j_s}(A_s) f_{\mu,\lambda}.$$

*Then we have*

(i) $v = X_1 \cdots X_s f_{\mu,\lambda}$, *where each $X_t$ is either $E(k_t, j_t - 1) S_{i_t,j_t}(A_t)$ or $M_{i_t,j_t}^{k_t}(A_t)$,*

(ii) $E_l^{(m)} v = 0$ *if $1 \leqslant l < n-1$ and $m \geqslant 2$,*

(iii) $E_l^{(m)} v = 0$ *if $m \geqslant 1$ and $l \in [1..n-1) \setminus ([i_1..k_1) \cup \cdots \cup [i_s..k_s))$,*

(iv) *If* $i_t < k_t < n$ *then*

$$E_{k_t-1}v = \left(\prod_{r=1}^{t-1} E(k_r, j_r - 1)S_{i_r,j_r}(A_r)\right) E(k_t - 1, j_t - 1)S_{i_t,j_t}(A_t) \times \left(\prod_{r=t+1}^{s} E(k_r, j_r - 1)S_{i_r,j_r}(A_r)\right) f_{\mu,\lambda},$$

(v) *If* $l \in [i_t..k_t - 1)$ *then*

$$E_l v = c\left(\prod_{r=1}^{t-1} E(k_r, j_r - 1)S_{i_r,j_r}(A_r)\right) S_{i_t,l}((A_t)_{i_t..l}) \times E(k_t, j_t - 1)S_{l+1,j_t}((A_t)_{l+1..j_t}) \left(\prod_{r=t+1}^{s} E(k_r, j_r - 1)S_{i_r,j_r}(A_r)\right) f_{\mu,\lambda},$$

*where* $c = 0$ *except the case* $l \in A_t \cup \{i_t\}$, $l + 1 \notin A_t$, *in which* $c = -1$.

**Proof.** (i) Applying Lemma 1, we prove by induction on $t$ (starting from $t = s$) that

$$v = E(k_1, j_1 - 1)S_{i_1,j_1}(A_1) \cdots E(k_{t-1}, j_{t-1} - 1)S_{i_{t-1},j_{t-1}}(A_{t-1}) \times M^{k_t}_{i_t,j_t}(A_t) \cdots M^{k_s}_{i_s,j_s}(A_s) f_{\mu,\lambda}.$$

Using this formula for $t = 1$, we obtain the required result by induction on $s$.

(ii), (iii) follow from part (i) for $X_t = M^{k_t}_{i_t,j_t}(A_t)$ and Lemma 1.

(iv) Applying part (i) (possibly for different parameters), we get

$$\begin{aligned} E_{k_t-1}v &= E_{k_t-1}M^{k_1}_{i_1,j_1}(A_1) \cdots M^{k_{t-1}}_{i_{t-1},j_{t-1}}(A_{t-1}) \\ &\quad \times E(k_t, j_t - 1)S_{i_t,j_t}(A_t) \cdots E(k_s, j_s - 1)S_{i_s,j_s}(A_s) f_{\mu,\lambda} \\ &= M^{k_1}_{i_1,j_1}(A_1) \cdots M^{k_{t-1}}_{i_{t-1},j_{t-1}}(A_{t-1})E(k_t - 1, j_t - 1)S_{i_t,j_t}(A_t) \\ &\quad \times E(k_{t+1}, j_{t+1} - 1)S_{i_{t+1},j_{t+1}}(A_{t+1}) \cdots E(k_s, j_s - 1)S_{i_s,j_s}(A_s) f_{\mu,\lambda}. \end{aligned}$$

Now the required formula follows from part (i).

(v) Since $E_l$ and $E(k_t, j_t - 1)$ commute in this case, we get by [1, 4.11(i), (ii)] and parts (i), (ii) of the current lemma that

$$\begin{aligned} E_l v &= M^{k_1}_{i_1,j_1}(A_1) \cdots M^{k_{t-1}}_{i_{t-1},j_{t-1}}(A_{t-1})E(k_t, j_t - 1)E_l S_{i_t,j_t}(A_t) \\ &\quad \times E(k_{t+1}, j_{t+1} - 1)S_{i_{t+1},j_{t+1}}(A_{t+1}) \cdots E(k_s, j_s - 1)S_{i_s,j_s}(A_s) f_{\mu,\lambda} \\ &= cM^{k_1}_{i_1,j_1}(A_1) \cdots M^{k_{t-1}}_{i_{t-1},j_{t-1}}(A_{t-1})S_{i_t,l}((A_t)_{i_t..l})E(k_t, j_t - 1)S_{l+1,j_t}((A_t)_{l+1..j_t}) \\ &\quad \times E(k_{t+1}, j_{t+1} - 1)S_{i_{t+1},j_{t+1}}(A_{t+1}) \cdots E(k_s, j_s - 1)S_{i_s,j_s}(A_s) f_{\mu,\lambda}. \end{aligned}$$

Now the required formula follows similarly to (iv). □

For $1 \leqslant i < j \leqslant n$ and $A \subset (i..j)$, we define the polynomial $\mathscr{K}_{i,j}(A)$ of $\mathbb{Z}[x_i, \dots, x_{j-1}, y_{i+1}, \dots, y_j]$ as in [1, 4.12] by the formula

$$\mathscr{K}_{i,j}(A) := \sum_{B \subset (i..j)} \left( \mathscr{H}_{i,j}(A, B) \prod_{t \in B \cup \{i\}} (y_{t+1} - x_t) \right).$$

We define $H^{\mu}_{i,j}(A, B)$ by evaluating $\mathscr{H}_{i,j}(A, B)$ at $x_q := \mathrm{res}_p(q, \mu_q)$ and define $K^{\mu,\lambda,k}_{i,j}(A)$ by evaluating $\mathscr{K}_{i,j}(A)$ at

$$\begin{aligned} x_q &:= \mathrm{res}_p(q, \mu_q) \quad \text{for } 1 \leqslant q < n, \\ y_q &:= \mathrm{res}_p(q, \lambda_q + 1) \quad \text{for } 1 < q \leqslant k, \\ y_q &:= \mathrm{res}_p(q, \mu_q + 1) \quad \text{for } k < q < n, \end{aligned} \tag{1}$$

where $1 + \delta_{j=n}(n-1) \leqslant k \leqslant n$. For $1 \leqslant i \leqslant t < n$ and $1 + \delta_{t+1=n}(n-1) \leqslant k \leqslant n$, let $B^{\mu,\lambda,k}(i, t)$ denote the element of $\mathbb{Z}/p\mathbb{Z}$ obtained from $y_{t+1} - x_i$ by substitution (1). We also abbreviate $K^{\mu,\lambda}_{i,j}(A) := K^{\mu,\lambda,n}_{i,j}(A)$ and $B^{\mu,\lambda}(i, t) := B^{\mu,\lambda,n}(i, t)$.

**Remark 1.** Clearly $B^{\mu,\lambda,k}(i, t) = t - i + \mu_i - \mu_{t+1}$ for $k \leqslant t$ and $B^{\mu,\lambda,k}(i, t) = t - i + \mu_i - \lambda_{t+1}$ for $k > t$. In particular, $B^{\mu,\lambda,k}(i, t) = B^{\mu,\lambda,i}(i, t)$ for $k \leqslant i$ and $B^{\mu,\lambda,k}(i, t) = B^{\mu,\lambda,t+1}(i, t)$ for $k > t$.

The next result is actually proved in [5, Proposition 4.5]. Recall that we have defined $a_t = \sum_{j=1}^{t} (\lambda_j - \mu_j)$.

**Proposition 3.** *Given integers* $1 \leqslant d_1 < d'_1 \leqslant d_2 < d'_2 \leqslant \cdots \leqslant d_r < d'_r \leqslant n$, *we have*

$$\left( \prod_{t=1}^{n-1} E_t^{(a_t + \delta_{t \in G})} \right) F_{d_1, d'_1} \cdots F_{d_r, d'_r} f_{\mu,\lambda} = \prod_{q=1}^{r} (\mu_{d_q} - \lambda_{d_q + 1}) f_\lambda,$$

*where* $G = [d_1..d'_1) \cup \cdots \cup [d_r..d'_r)$.

**Lemma 4.** *Under the hypothesis of Lemma* 2, *we have*

$$\left( \prod_{t=1}^{n-1} E_t^{(a_t + \delta_{t \in G})} \right) v = K^{\mu,\lambda,k_1}_{i_1, j_1}(A_1) \cdots K^{\mu,\lambda,k_s}_{i_s, j_s}(A_s) f_\lambda,$$

*where* $G = [i_1..k_1) \cup \cdots \cup [i_s..k_s)$.

**Proof.** By Lemma 1, we have $E(k_t, j_t - 1) F^B_{i_t, j_t} \equiv F^{B_{i_t..k_t}}_{i_t, k_t} \prod_{q \in B \cup \{i_t\}, q \geqslant k_t} (H_q - H_{q+1})$ modulo the left ideal of $U(n)$ generated by $E_{k_t}, \dots, E_{j_t - 1}$. Thus taking into account

[1, Remark 4.8], we get

$$v=\prod_{t=1}^{s}\sum_{B_t\subset(i_t..j_t)}\left(H^{\mu}_{i_t,j_t}(A_t,B_t)F^{(B_t)_{i_t..k_t}}_{i_t,k_t}\prod_{\substack{q\in B_t\cup\{i_t\}\\ q\geqslant k_t}}(\mu_q-\mu_{q+1})\right)f_{\mu,\lambda}. \tag{2}$$

By Proposition 3, we have

$$\left(\prod_{t=1}^{n-1}E_t^{(a_t+\delta_{t\in G})}\right)F^{(B_1)_{i_1..k_1}}_{i_1,k_1}\cdots F^{(B_s)_{i_s..k_s}}_{i_s,k_s}f_{\mu,\lambda}=\prod_{t=1}^{s}\prod_{\substack{q\in B_t\cup\{i_t\}\\ q<k_t}}(\mu_q-\lambda_{q+1})f_{\lambda}.$$

Substituting this into (2) completes the proof. □

Let $V_n$ be the set of all sequences $x=((i_1,k_1,j_1,A_1),\ldots,(i_s,k_s,j_s,A_s))$ such that

$$1\leqslant i_1<j_1<\cdots<i_s<j_s\leqslant n;\quad A_1\subset(i_1..j_1),\ldots,A_s\subset(i_s..j_s),$$

$$i_1\leqslant k_1\leqslant j_1,\ldots,i_s\leqslant k_s\leqslant j_s;\quad j_s=n\ \text{ implies } k_s=n.$$

Moreover, we put $\Phi(x):=E(k_1,j_1-1)S_{i_1,j_1}(A_1)\cdots E(k_s,j_s-1)S_{i_s,j_s}(A_s)$ and $K^{\mu,\lambda}(x):=K^{\mu,\lambda,k_1}_{i_1,j_1}(A_1)\cdots K^{\mu,\lambda,k_s}_{i_s,j_s}(A_s)$. In what follows, we assume that the product of two finite sequences $a=(a_1,\ldots,a_s)$ and $b=(b_1,\ldots,b_t)$ equals $ab=(a_1,\ldots,a_s,b_1,\ldots,b_t)$.

Let $x,x'\in V_n$. We write $x\xrightarrow{l}x'$ if there exists a representation $x=a((i,k,j,A))b$ such that one of the following conditions holds:

- $x'=a((i,k-1,j,A))b$, $l=k-1$, $i<k<n$,
- $x'=a((i+1,k,j,A))b$, $l=i$, $i+1\notin A$, $i<k-1$,
- $x'=a((i,l,l,A_{i..l}),(l+1,k,j,A_{l+1..j}))b$, $l\in(i..k-1)$, $l\in A$, $l+1\notin A$.

The above definitions are made exactly to ensure the following property.

**Lemma 5.** *Let $x,x'\in V_n$. If $x\xrightarrow{l}x'$ then $E_l\Phi(x)f_{\mu,\lambda}=\pm\Phi(x')f_{\mu,\lambda}$.*

**Proof.** Follows directly from Lemma 2(iv), (v). □

We say that $x'$ *follows* from $x$ if there are $x_0,\ldots,x_m\in V_n$ and integers $l_0,\ldots,l_{m-1}$ such that $x=x_0$, $x'=x_m$ and $x_t\xrightarrow{l_t}x_{t+1}$ for $0\leqslant t<m$. In particular, every element of $V_n$ follows from itself.

**Theorem 6.** *Let $x\in V_n$. The equality $\Phi(x)f_{\mu,\lambda}=0$ holds if and only if $K^{\mu,\lambda}(x')=0$ for any $x'$ following from $x$.*

**Proof.** It follows from Lemmas 5 and 4 that $\Phi(x)f_{\mu,\lambda}=0$ implies $K^{\mu,\lambda}(x')=0$ for any $x'$ following from $x$.

Let $x = ((i_1, k_1, j_1, A_1), \ldots, (i_s, k_s, j_s, A_s))$. We prove the reverse implication by induction on $\sum_{t=1}^{s} (k_t - i_t)$. The induction starts by noting that this sum is always nonnegative. So we suppose that the reverse implication is true for smaller values of this sum. By Lemma 2(ii),(iii), we get $E_l^{(m)} \Phi(x) f_{\mu,\lambda} = 0$ if $l < n - 1$ and $m > 1$ or if $m \geqslant 1$ and $l \in [1..n-1) \backslash ([i_1..k_1) \cup \cdots \cup [i_s..k_s))$.

However $E_l \Phi(x) f_{\mu,\lambda} = 0$ also for $l \in [1..n-1) \cap ([i_1..k_1) \cup \cdots \cup [i_s..k_s))$ by Lemma 5 and the inductive hypothesis. Thus $\Phi(x) f_{\mu,\lambda}$ is a $U(n-1)$-high weight vector of weight $\nu = \mu - \sum_{t=1}^{s} (\varepsilon_{i_t} - \varepsilon_{k_t})$, where $\varepsilon_i = (0^{i-1}, 1, 0^{n-1-i})$ for $i < n$ and $\varepsilon_n = (0^{n-1})$. It follows from [5, Corollary 3.3] that $\Phi(x) f_{\mu,\lambda} = 0$ if $\nu \longleftarrow \lambda$ does not hold and that $\Phi(x) f_{\mu,\lambda} = c f_{\nu,\lambda}$ for some $c \in K$ if $\nu \longleftarrow \lambda$. We need to consider only the latter case. By the last equation of the introduction and Lemma 4, we have $c f_\lambda = X(c f_{\nu,\lambda}) = X \Phi(x) f_{\mu,\lambda} = K^{\mu,\lambda}(x) f_\lambda = 0$, where $X = \prod_{t=1}^{n-1} E_t^{(a_t + \delta_{t \in G})}$ and $G = [i_1..k_1) \cup \cdots \cup [i_s..k_s)$. Hence $c = 0$ and $\Phi(x) f_{\mu,\lambda} = 0$. □

The next corollary follows from Theorem 6 and the following simple fact: if $x \in V_n$ and $x = x_1 x_2$ then $x'$ follows from $x$ if and only if there are sequences $x_1'$ and $x_2'$ following from $x_1$ and $x_2$, respectively, such that $x' = x_1' x_2'$.

**Corollary 7.** *Let $x \in V_n$ and $x = x_1 x_2$. Then $\Phi(x) f_{\mu,\lambda} = 0$ if and only if $\Phi(x_1) f_{\mu,\lambda} = 0$ or $\Phi(x_2) f_{\mu,\lambda} = 0$.*

## 3. Main result

We say that a map $\theta : A \to \mathbb{Z}$, where $A \subset \mathbb{Z}$, is *weakly increasing* (*weakly decreasing*) if $\theta(a) \geqslant a$ (resp. $\theta(a) \leqslant a$) for any $a \in A$. We need the following facts about the polynomials $\mathscr{K}_{i,j}(A)$.

**Proposition 8.** *Let $1 \leqslant i < j \leqslant n$, $1 + \delta_{j=n}(n-1) \leqslant k \leqslant n$, $A \subset (i..j)$ and there exists a weakly increasing injection $\theta : (i..j) \backslash A \to (i..j)$ such that $B^{\mu,\lambda,k}(t, \theta(t)) = 0$ for any $t \in (i..j) \backslash A$. Then*

$$K_{i,j}^{\mu,\lambda,k}(A) = \prod_{t \in [i..j) \backslash \operatorname{Im} \theta} B^{\mu,\lambda,k}(i, t).$$

**Proof.** The result is obtained from [5, Lemma 4.4] by substitution (1). □

**Lemma 9.** *For $i < j - 1$ and $A \subset (i..j)$, we have*

(i) $\mathscr{K}_{i,j}(A) = \mathscr{K}_{i,j-1}(A)$ *if* $j - 1 \notin A$,
(ii) $\mathscr{K}_{i,j}(A) = \mathscr{K}_{i,j-1}(A \backslash \{j-1\})(y_j - x_k) + \delta_{k \neq i} \mathscr{K}_{i,j-1}(\{k\} \cup A \backslash \{j-1\})$, *where* $k = \max[i..j) \backslash A$, *if* $j - 1 \in A$.

**Proof.** We put $\bar{A} = (i..j) \backslash A$. In this proof, we use [1, Lemma 4.13(i)] for a self-contained form of $\mathscr{K}_{i,j}(A)$ and the following notation of [1]: if $D \subset (i..j)$ and $k > i$ then $D_i(k) = \max\{t \in D \cup \{i\} : t < k\}$.

(i) If $D \subset \bar{A}\backslash\{j-1\}$ then $(D\cup\{j-1\})_i(t) = D_i(t)$ for $t < j$, $(D\cup\{j-1\})_i(j) = j-1$ and $D_i(j) = D_i(j-1)$. Hence we get

$$\begin{aligned}
&\mathscr{K}_{i,j}(A)\\
&= \sum_{D\subset\bar{A}\backslash\{j-1\}} (-1)^{|D|}\left(\frac{\prod_{t\in(i..j]}(y_t - x_{D_i(t)})}{\prod_{t\in\bar{A}}(x_t - x_{D_i(t)})} - \frac{\prod_{t\in(i..j]}(y_t - x_{(D\cup\{j-1\})_i(t)})}{\prod_{t\in\bar{A}}(x_t - x_{(D\cup\{j-1\})_i(t)})}\right)\\
&= \sum_{D\subset\bar{A}\backslash\{j-1\}} (-1)^{|D|}\left(\frac{\prod_{t\in(i..j-1]}(y_t - x_{D_i(t)})}{\prod_{t\in\bar{A}\backslash\{j-1\}}(x_t - x_{D_i(t)})}\,\frac{(y_j - x_{D_i(j-1)}) - (y_j - x_{j-1})}{x_{j-1} - x_{D_i(j-1)}}\right)\\
&= \mathscr{K}_{i,j-1}(A).
\end{aligned}$$

(ii) If $k = i$ then $A = (i..j)$, $\mathscr{K}_{i,j}(A) = \prod_{t\in(i..j]}(y_t - x_i)$, $\mathscr{K}_{i,j-1}(A\backslash\{j-1\}) = \prod_{t\in(i..j-1]}(y_t - x_i)$ (by part (i)) and the required formula follows.

Therefore, we consider the case $k \neq i$. We have

$$\begin{aligned}
\mathscr{K}_{i,j}(A) = (y_j - x_k)\sum_{D\subset\bar{A}}(-1)^{|D|}\frac{\prod_{t\in(i..j-1]}(y_t - x_{D_i(t)})}{\prod_{t\in\bar{A}}(x_t - x_{D_i(t)})}\\
+ \sum_{D\subset\bar{A}}(-1)^{|D|}(x_k - x_{D_i(j)})\frac{\prod_{t\in(i..j-1]}(y_t - x_{D_i(t)})}{\prod_{t\in\bar{A}}(x_t - x_{D_i(t)})}.
\end{aligned}$$

Part (i) shows that the first sum equals $\mathscr{K}_{i,j}(A\backslash\{j-1\})$. Let us look at the second sum. If $k \in D$ then $D_i(j) = k$ and the summands corresponding to such sets $D$ can be omitted. If $k \notin D$ then $D_i(j) = D_i(k)$ and this summand equals

$$(-1)^{|D|}\frac{\prod_{t\in(i..j-1]}(y_t - x_{D_i(t)})}{\prod_{t\in\bar{A}\backslash\{k\}}(x_t - x_{D_i(t)})}.$$

Thus the second sum equals $\mathscr{K}_{i,j-1}(\{k\}\cup A\backslash\{j-1\})$. □

Next, we are going to prove the result similar to [4, Proposition 3.2], where we replace the $U(n)$-high weight vector $v_+$ by the $U(n-1)$-high weight vector $f_{\mu,\lambda}$. The general scheme of proof is borrowed from [4, Proposition 3.2], although some changes are necessary. We shall use Theorem 6 and Lemma 9 to make them. In what follows, we say that a formula $M = [b_1..c_1]\cup\cdots\cup[b_N..c_N]$ is *the decomposition of M into the union of connected components* if $b_i \leqslant c_i$ for $1\leqslant i\leqslant N$ and $c_i < b_{i+1} - 1$ for $1\leqslant i < N$.

**Definition 10.** Let $1\leqslant i < j\leqslant n$, $M \subset (i..j)$ and $M = [b_1..c_1]\cup\cdots\cup[b_N..c_N]$ be the decomposition of $M$ into the union of connected components. We say that $M$ satisfies the condition $\pi^{\mu,\lambda}_{i,j}(v)$ if $1\leqslant v\leqslant N+1$ and for any $k = 1 + \delta_{b_v - 1 = n}(n-1),\ldots,n$ there exists a weakly increasing injection $\theta_k : \{i\}\cup[b_1..c_1]\cup\cdots\cup[b_{v-1}..c_{v-1}] \to [i..b_v - 1)$ such that $B^{\mu,\lambda,k}(x,\theta_k(x)) = 0$ for any admissible $x$, where we assume $b_{N+1} = j+1$.

**Lemma 11.** *Let* $1 \leqslant i < j \leqslant n$ *and* $A \subset (i..j)$ *be such that* $(i..j)\backslash A$ *satisfies* $\pi_{i,j}^{\mu,\lambda}(v)$ *for some* $v$. *Then* $K_{i,j}^{\mu,\lambda,k}(A) = 0$ *for* $1 + \delta_{j=n}(n-1) \leqslant k \leqslant n$.

**Proof.** Let $(i..j)\backslash A = [b_1..c_1] \cup \cdots \cup [b_N..c_N]$ be the decomposition into the union of connected components. Note that if $v = N + 1$, then the required equalities immediately follow from Proposition 8.

Indeed, take any $k = 1 + \delta_{j=n}(n-1), \dots, n$. Since in this case $b_v - 1 = j$, Definition 10 ensures that there exists a weakly increasing injection $\theta_k : \{i\} \cup ((i..j)\backslash A) \to [i..j)$ such that $B^{\mu,\lambda,k}(x, \theta_k(x)) = 0$ for any admissible $x$. Taking the restriction of $\theta_k$ to $(i..j)\backslash A$ for $\theta$ in Proposition 8, we obtain

$$K_{i,j}^{\mu,\lambda,k}(A) = \prod_{t \in [i..j)\backslash \operatorname{Im}\theta} B^{\mu,\lambda,k}(i, t).$$

The last product equals zero, since $B^{\mu,\lambda,k}(i, \theta_k(i)) = 0$ and $\theta_k(i) \in [i..j)\backslash \operatorname{Im}\theta$.

Let us prove the lemma by induction on $j - i$. The case $j - i = 1$ follows from the above remark. Now let $v \leqslant N$, $j - i > 1$ and suppose that the lemma is true for smaller values of this difference. Take any $k = 1 + \delta_{j=n}(n-1), \dots, n$. By Lemma 9, we have

$$K_{i,j}^{\mu,\lambda,k}(A) = K_{i,j-1}^{\mu,\lambda,k}(A\backslash\{j-1\})B + K_{i,j-1}^{\mu,\lambda,k}(\{c_N\} \cup A\backslash\{j-1\})$$

if $c_N < j - 1$ and

$$K_{i,j}^{\mu,\lambda,k}(A) = K_{i,j-1}^{\mu,\lambda,k}(A)$$

if $c_N = j - 1$, where $B$ is the element of $\mathbb{Z}/p\mathbb{Z}$ obtained from $y_j - x_{c_N}$ by substitution (1). Clearly, the sets $(i..j-1)\backslash(A\backslash\{j-1\})$ and $(i..j-1)\backslash(\{c_N\} \cup A\backslash\{j-1\})$ in the former case and the set $(i..j-1)\backslash A$ in the latter case satisfy the condition $\pi_{i,j-1}^{\mu,\lambda}(v)$. $\square$

**Theorem 12.** *Let* $1 \leqslant i < j \leqslant n$ *and* $A \subset (i..j)$. *Then* $S_{i,j}(A) f_{\mu,\lambda} = 0$ *if and only if* $(i..j)\backslash A$ *satisfies* $\pi_{i,j}^{\mu,\lambda}(v)$ *for some* $v$.

**Proof.** Let $\bar{A} = (i..j)\backslash A$ and $\bar{A} = [b_1..c_1] \cup \cdots \cup [b_N..c_N]$ be the decomposition into the union of connected components. We put $x_k = ((i, k, j, A))$ for brevity. It should be kept in mind that $\Phi(x_j) = S_{i,j}(A)$.

We prove the theorem by induction on $|\bar{A}|$. Suppose $\bar{A} = \emptyset$. Then all the sequences following from $x_j$ are $x_k$, where $i + \delta_{j=n}(j-i) \leqslant k \leqslant j$. By Theorem 6, $\Phi(x_j) f_{\mu,\lambda} = 0$ if and only if $K^{\mu,\lambda}(x_k) = 0$ for any $k = i + \delta_{j=n}(j-i), \dots, j$. Applying Proposition 8, we see that $\Phi(x_j) f_{\mu,\lambda} = 0$ if and only if for any $k = i + \delta_{j=n}(j-i), \dots, j$ there is $t_k \in [i..j)$ such that $B^{\mu,\lambda,k}(i, t_k) = 0$. In view of Remark 1, this assertion is equivalent to $\pi_{i,j}^{\mu,\lambda}(1)$.

Now suppose that $\bar{A} \neq \emptyset$ and that the theorem holds for smaller values of $|\bar{A}|$.

"*If part*": By [1, 4.11(ii)] for any $m = 1, \ldots, N$, we have $E_{b_m-1} S_{i,j}(A) f_{\mu,\lambda} = -S_{i,b_m-1}(A_{i..b_m-1})\, S_{b_m,j}(A_{b_m..j}) f_{\mu,\lambda}$. Note that

$$\begin{aligned} &A_{i..b_m-1} = (i..b_m - 1)\backslash([b_1..c_1] \cup \cdots \cup [b_{m-1}..c_{m-1}]), \\ &A_{b_m..j} = (b_m..j)\backslash((b_m..c_m] \cup \cdots \cup [b_N..c_N]). \end{aligned} \tag{3}$$

If $m \leqslant v - 1$ then $(b_m..c_m] \cup \cdots \cup [b_N..c_N]$ satisfies $\pi^{\mu,\lambda}_{b_m,j}(v - m + 1 - \delta_{b_m=c_m})$, whence by the inductive hypothesis $S_{b_m,j}(A_{b_m..j}) f_{\mu,\lambda}=0$. If $m \geqslant v$ then $i < b_m-1$ and $[b_1..c_1]\cup\cdots\cup[b_{m-1}..c_{m-1}]$ satisfies $\pi^{\mu,\lambda}_{i,b_m-1}(v)$, whence by the inductive hypothesis $S_{i,b_m-1}(A_{i..b_m-1}) f_{\mu,\lambda}=0$. Since the elements $S_{i,b_m-1}(A_{i..b_m-1})$ and $S_{b_m,j}(A_{b_m..j})$ commute, we have in both cases

$$E_{b_m-1} S_{i,j}(A) f_{\mu,\lambda} = 0. \tag{4}$$

Let us prove by induction on $s = 0, \ldots, j - i$ that in the case $j < n$ the conditions

$$K^{\mu,\lambda,j}_{i,j}(A) = 0, \ldots, \quad K^{\mu,\lambda,j-s+1}_{i,j}(A) = 0, \quad \Phi(x_{j-s}) f_{\mu,\lambda} = 0 \tag{5}$$

imply $\Phi(x_j) f_{\mu,\lambda} = 0$. It is obviously true for $s = 0$. Suppose that $0 < s \leqslant j - i$, conditions (5) hold and the assertion is true for smaller values of $s$. By the inductive hypothesis it suffices to prove that $\Phi(x_{j-s+1}) f_{\mu,\lambda} = 0$. Let $x_{j-s+1} \xrightarrow{l} x'$. We have either $x' = x_{j-s}$ or $l = b_m - 1 < j - s$. Since in the former case $\Phi(x') f_{\mu,\lambda} = 0$ by (5), we shall consider the latter case. We have

$$\begin{aligned} \Phi(x') f_{\mu,\lambda} &= E_{b_m-1} \Phi(x_{j-s+1}) f_{\mu,\lambda} = E_{b_m-1} E(j - s + 1, j - 1) S_{i,j}(A) f_{\mu,\lambda} \\ &= E(j - s + 1, j - 1) E_{b_m-1} S_{i,j}(A) f_{\mu,\lambda} = 0. \end{aligned}$$

To obtain the last equality, we used (4). Since $K^{\mu,\lambda}(x_{j-s+1}) = K^{\mu,\lambda,j-s+1}_{i,j}(A) = 0$, we get $\Phi(x_{j-s+1}) f_{\mu,\lambda} = 0$ by Theorem 6.

Note that nothing follows from $x_i$ except itself. Therefore, applying the above assertion for $s = j - i$ and Theorem 6, we see that to prove $\Phi(x_j) f_{\mu,\lambda} = 0$ in the case $j < n$, it suffices to prove $K^{\mu,\lambda,k}_{i,j}(A) = 0$ for $i \leqslant k \leqslant j$. The last equalities follow from Lemma 11.

If $j = n$ then $x_j \xrightarrow{l} x'$ holds if and only if $l = b_m - 1$, where $1 \leqslant m \leqslant N$. In that case $\Phi(x') f_{\mu,\lambda} = 0$ by (4). Therefore, applying Theorem 6, we see that to prove $\Phi(x_j) f_{\mu,\lambda} = 0$ in the case $j = n$, it suffices to prove $K^{\mu,\lambda}_{i,j}(A) = 0$. The last equality follows from Lemma 11.

"*Only if part*": Suppose $\bar{A}$ satisfies the condition $\pi^{\mu,\lambda}_{i,j}(v)$ for no $v$. Multiplying the equality $\Phi(x_j) f_{\mu,\lambda} = 0$ by $E_{b_m-1}$, where $1 \leqslant m \leqslant N$, we get $S_{i,b_m-1}(A_{i..b_m-1}) S_{b_m,j}(A_{b_m..j}) f_{\mu,\lambda} = 0$ according to [1, 4.11(ii)]. By Corollary 7, either $S_{i,b_m-1}(A_{i..b_m-1}) f_{\mu,\lambda} = 0$ or $S_{b_m,j}(A_{b_m..j}) f_{\mu,\lambda} = 0$. The former case is impossible since the inductive hypothesis would yield that $(i..b_m - 1)\backslash A_{i..b_m-1}$ satisfies $\pi^{\mu,\lambda}_{i,b_m-1}(v)$ for some $v \leqslant m$ (see (3)). But then $\bar{A}$ would satisfy $\pi^{\mu,\lambda}_{i,j}(v)$, which is wrong. Therefore $S_{b_m,j}(A_{b_m..j}) f_{\mu,\lambda} = 0$ for any $m = 1, \ldots, N$.

We shall use this fact to prove by downward induction on $u = 1, \dots, N+1$ the following property:

$$\text{for any } k = 1 + \delta_{j=n}(n-1), \dots, n, \text{ there is a weakly increasing injection } d_k : [b_u..c_u] \cup \dots \cup [b_N..c_N] \to (i..j) \text{ such that } B^{\mu,\lambda,k}(x, d_k(x)) = 0 \text{ for any admissible } x. \tag{6}$$

This is obviously true for $u = N+1$. Therefore, we suppose that $1 \leqslant u \leqslant N$ and property (6) is proved for greater $u$. Fix an arbitrary $k = 1 + \delta_{j=n}(n-1), \dots, n$. Since $S_{b_u,j}(A_{b_u..j}) f_{\mu,\lambda} = 0$, the inductive hypothesis asserting that the current lemma is true for smaller values of $|\bar{A}|$ implies that $(b_u..j) \backslash A_{b_u..j}$ satisfies $\pi^{\mu,\lambda}_{b_u,j}(v)$ for some $v$. As a consequence, there is a weakly increasing injection $e_k : [b_u..c_u] \cup \dots \cup [b_{u+w-1}..c_{u+w-1}] \to [b_u..b_{u+w} - 1)$ such that $B^{\mu,\lambda,k}(x, d_k(x)) = 0$ for any admissible $x$ (here $w = v - 1 + \delta_{b_u = c_u}$ and $b_{N+1} = j+1$). The inductive hypothesis asserting that property (6) holds for $u + w$ allows us to extend $e_k$ to the required injection $d_k$. Thus property (6) is proved.

Take any $k = i + \delta_{j=n}(j-i), \dots, j$. Applying property (6) for $u = 1$, the fact that $x_k$ follows from $x_j$, and Proposition 8, we get

$$0 = K^{\mu,\lambda}(x_k) = K^{\mu,\lambda,k}_{i,j}(A) = \prod_{t \in [i..j) \backslash \operatorname{Im} d_k} B^{\mu,\lambda,k}(i,t).$$

Therefore, there is $t' \in [i..j) \backslash \operatorname{Im} d_k$ such that $B^{\mu,\lambda,k}(i,t') = 0$. Putting $\theta_k(t) = d_k(t)$ for $t \in [b_1..c_1] \cup \dots \cup [b_N..c_N]$ and $\theta_k(i) = t'$, we get a map required in Definition 10. This fact together with Remark 1 shows that $\bar{A}$ satisfies $\pi^{\mu,\lambda}_{i,j}(N+1)$, contrary to assumption. □

Following [5], we introduce the following sets:

$$\mathfrak{C}^{\mu}(i,j) := \{a : i < a < j,\ C^{\mu}(i,a) = 0\},$$
$$\mathfrak{B}^{\mu,\lambda}(i,j) := \{a : i \leqslant a < j,\ B^{\mu,\lambda}(i,a) = 0\},$$

where $C^{\mu}(i,a)$ is the residue class of $a - i + \mu_i - \mu_a$ modulo $p$ as in [5].

**Theorem 13.** *Let* $1 \leqslant i < n$.

(i) *Let* $A \subset (i..n)$. *Then* $S_{i,n}(A) f_{\mu,\lambda}$ *is a non-zero* $U(n-1)$*-high weight vector if and only if there is a weakly increasing injection* $d : (i..n) \backslash A \to (i..n)$ *such that* $B^{\mu,\lambda}(x, d(x)) = 0$ *for any admissible* $x$ *and* $B^{\mu,\lambda}(i,t) \neq 0$ *for any* $t \in [i..n) \backslash \operatorname{Im} d$.

(ii) *There is some* $A \subset (i..n)$ *such that* $S_{i,n}(A) f_{\mu,\lambda}$ *is a non-zero* $U(n-1)$*-high weight vector if and only if there is a weakly decreasing injection from* $\mathfrak{B}^{\mu,\lambda}(i,n)$ *to* $\mathfrak{C}^{\mu}(i,n)$.

**Proof.** (i) It is clear from [1, 4.11(ii)], Theorem 12 and Proposition 8 that $S_{i,n}(A) f_{\mu,\lambda}$ is a non-zero $U(n-1)$-high weight vector for such $A$. Conversely, if $S_{i,n}(A) f_{\mu,\lambda}$ is a non-zero $U(n-1)$-high weight vector then, arguing as in the "only if part" of Theorem 12, we get that there is a weakly increasing injection $d : (i..n) \backslash A \to (i..n)$ such that

$B^{\mu,\lambda}(x, d(x))=0$ for any admissible $x$. Now by Proposition 8, we have $0 \neq K^{\mu,\lambda}(i, n)(A)= \prod_{t\in[i..n)\setminus \mathrm{Im}\, d} B^{\mu,\lambda}(i, t)$.

(ii) If $\varepsilon$ is such an injection, then it suffices to put $A = (i..n)\setminus \mathrm{Im}\, \varepsilon$, take for $d$ the inverse map of $\varepsilon$ and apply part (i). Conversely, let $S_{i,n}(A) f_{\mu,\lambda}$ be a non-zero $U(n-1)$-high weight vector for some $A \subset (i..n)$ and let $d$ be an injection, whose existence is claimed by part (i). Now the result follows from the following two observations: $\mathfrak{B}^{\mu,\lambda}(i, n) \subset \mathrm{Im}\, d$; $d(x) \in \mathfrak{B}^{\mu,\lambda}(i, n)$ implies $x \in \mathfrak{C}^{\mu}(i, n)$. □

**Remark 2.** If we obtain a non-zero $U(n-1)$-high weight vector in Theorem 13, then it is a scalar multiple of $f_{\nu,\lambda}$, where $\nu = \mu - \varepsilon_i$ and $\varepsilon_i = (0^{i-1}, 1, 0^{n-1-i})$.

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