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Polynomial identities and noncommutative versal torsors

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Abstract

To any cleft Hopf Galois object, i.e., any algebra $^{\alpha}H$ obtained from a Hopf algebra H by twisting its multiplication with a two-cocycle α, we attach two "universal algebras" $\mathcal{A}_H^{\alpha}$ and $\mathcal{U}_H^{\alpha}$. The algebra $\mathcal{A}_H^{\alpha}$ is obtained by twisting the multiplication of H with the most general two-cocycle σ formally cohomologous to α. The cocycle σ takes values in the field of rational functions on H. By construction, $\mathcal{A}_H^{\alpha}$ is a cleft H-Galois extension of a "big" commutative algebra $\mathcal{B}_H^{\alpha}$. Any "form" of $^{\alpha}H$ can be obtained from $\mathcal{A}_H^{\alpha}$ by a specialization of $\mathcal{B}_H^{\alpha}$ and vice versa. If the algebra $^{\alpha}H$ is simple, then $\mathcal{A}_H^{\alpha}$ is an Azumaya algebra with center $\mathcal{B}_H^{\alpha}$. The algebra $\mathcal{U}_H^{\alpha}$ is constructed using a general theory of polynomial identities that we set up for arbitrary comodule algebras; it is the universal comodule algebra in which all comodule algebra identities of $^{\alpha}H$ are satisfied. We construct an embedding of $\mathcal{U}_H^{\alpha}$ into $\mathcal{A}_H^{\alpha}$; this embedding maps the center $\mathcal{Z}_H^{\alpha}$ of $\mathcal{U}_H^{\alpha}$ into $\mathcal{B}_H^{\alpha}$ when the algebra $^{\alpha}H$ is simple. In this case, under an additional assumption, $\mathcal{A}_H^{\alpha} \cong \mathcal{B}_H^{\alpha} \otimes_{\mathcal{Z}_H^{\alpha}} \mathcal{U}_H^{\alpha}$, thus turning $\mathcal{A}_H^{\alpha}$ into a central localization of $\mathcal{U}_H^{\alpha}$. We completely work out these constructions in the case of the four-dimensional Sweedler algebra.
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0. Introduction

In this paper we deal with algebras $^{\alpha}H$ obtained from a Hopf algebra H by twisting its multiplication with a two-cocycle α. This class of algebras coincides with the class of so-called *cleft Hopf Galois extensions* of the ground field; strongly G-graded algebras and certain G-algebras, where G is a finite group, belong to this class. As has been stressed many times (see e.g. [28]), Hopf Galois extensions can be viewed as noncommutative analogues of G-torsors or of principal fiber bundles where the role of the structural group is played by a Hopf algebra. Hopf Galois extensions abound in the world of quantum groups and of noncommutative geometry (see, e.g., [11–13,19,20,23]).

To the algebra $^{\alpha}H$ we attach two "universal algebras" $\mathcal{U}_H^{\alpha}$ and $\mathcal{A}_H^{\alpha}$, using two constructions of a very different nature. The algebra $\mathcal{U}_H^{\alpha}$ is a graded quotient of the tensor algebra over the underlying vector space of H. To construct this quotient, we set up a theory of *polynomial identities for comodule algebras*. We define $\mathcal{U}_H^{\alpha}$ as the universal comodule algebra in which all H-comodule algebra identities of $^{\alpha}H$ are satisfied.

The second universal algebra, $\mathcal{A}_H^{\alpha}$, is obtained by twisting the multiplication of H with a two-cocycle σ taking values in the field of rational functions on H. The cocycle σ can be viewed as the most general cocycle formally cohomologous to α. By construction, $\mathcal{A}_H^{\alpha}$ is a cleft H-Galois extension of the commutative algebra $\mathcal{B}_H^{\alpha}$ generated by the values of the cocycle σ and of its convolution inverse. We show that any "form" of $^{\alpha}H$ can be obtained from $\mathcal{A}_H^{\alpha}$ by a specialization of $\mathcal{B}_H^{\alpha}$. Conversely, any central specialization of $\mathcal{A}_H^{\alpha}$ is a form of $^{\alpha}H$. Thus, the set of algebra morphisms $\mathrm{Alg}(\mathcal{B}_H^{\alpha}, K)$ parametrizes the isomorphism classes of K-forms of $^{\alpha}H$ and $\mathcal{A}_H^{\alpha}$ can be viewed as a flat deformation of $^{\alpha}H$ over the commutative algebra $\mathcal{B}_H^{\alpha}$; it is a noncommutative analogue of a *versal deformation space* or a *versal torsor* in the sense of Serre (see [18, Chapter I]). We also prove that, if $^{\alpha}H$ is a simple algebra, then $\mathcal{A}_H^{\alpha}$ is an Azumaya algebra with center $\mathcal{B}_H^{\alpha}$.

We relate the algebras $\mathcal{U}_H^{\alpha}$ and $\mathcal{A}_H^{\alpha}$ by constructing an injective comodule algebra morphism $\mu_{\sigma} : \mathcal{U}_H^{\alpha} \to \mathcal{A}_H^{\alpha}$. The morphism μ_{σ} sends the center $\mathcal{Z}_H^{\alpha}$ of $\mathcal{U}_H^{\alpha}$ to the center $\mathcal{B}_H^{\alpha}$ of $\mathcal{A}_H^{\alpha}$ when the algebra $^{\alpha}H$ is simple. Under an additional assumption we prove that μ_{σ} induces an isomorphism of H-comodule algebras $\mathcal{B}_H^{\alpha} \otimes_{\mathcal{Z}_H^{\alpha}} \mathcal{U}_H^{\alpha} \cong \mathcal{A}_H^{\alpha}$. We thus obtain the "versal deformation space" $\mathcal{A}_H^{\alpha}$ as a central localization of the universal H-comodule algebra $\mathcal{U}_H^{\alpha}$.

Most of the results presented here generalize results obtained in the group case by the first-named author jointly with D. Haile and M. Natapov, see [3]. In this important special case, additional results are obtained in [3] such as a characterization of the pairs (G, α) consisting of a finite group G and a two-cocycle α for which $\mathcal{A}_H^{\alpha}$ is a division algebra, see also [4]. Since we deal with arbitrary Hopf algebras, our results, whose proofs are not straightforward extensions of those of [3], cover situations that can be radically different from the group case. Our general framework also leads to the emergence of new interesting questions on Hopf algebras such as the one posed in Section 5.4.

In [21,22] a project aimed at classifying Hopf Galois extensions was started. The results presented in this paper can be considered as first steps towards the construction of universal Hopf Galois extensions. Although we do not obtain universal extensions (which may well not exist), we manage to construct versal deformation spaces. We believe that such deformation spaces are of interest and that they deserve to be computed for many Hopf Galois extensions.

The paper is organized as follows. In Section 1 we recall some well-known facts on comodule algebras.

In Section 2 we define the concept of an H-identity for an H-comodule algebra and in Section 3 we associate to each H-comodule algebra A the universal algebra $\mathcal{U}_H(A)$.

In Section 4 we restrict ourselves to the case of H-comodule algebras A that are isomorphic as comodules to $B \otimes H$, where B is the subalgebra of coinvariants of A, assumed to be central in A. For each such H-comodule algebra we construct a comodule algebra morphism whose kernel turns out to be exactly the ideal of H-identities for A. This allows us to embed the universal algebra $\mathcal{U}_H(A)$ into a more controllable algebra, namely the tensor product of A by a (commutative) polynomial algebra.

We construct the two-cocycle σ and the commutative algebra $\mathcal{B}_H^\alpha$ in Section 5. In Section 6 we define the "universal twisted algebra" $\mathcal{A}_H^\alpha$ and show that, if $^\alpha H$ is (semi)simple, then $\mathcal{A}_H^\alpha$ is "generically" (semi)simple, i.e., $\mathcal{K}_H^\alpha \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$ is (semi)simple, where $\mathcal{K}_H^\alpha$ is the field of fractions of $\mathcal{B}_H^\alpha$.

In Section 7 we define forms of $^\alpha H$ and show how specializations of $\mathcal{B}_H^\alpha$ give rise to forms of $^\alpha H$ and *vice versa*. We also prove that $\mathcal{A}_H^\alpha$ is an Azumaya algebra if $^\alpha H$ is a simple algebra.

In Section 8 we embed $\mathcal{U}_H^\alpha$ into $\mathcal{A}_H^\alpha$ and we show in Section 9 that, if $^\alpha H$ is simple, then the previous embedding maps the center $\mathcal{Z}_H^\alpha$ of $\mathcal{U}_H^\alpha$ into $\mathcal{B}_H^\alpha$, resulting in the isomorphism

$$\mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{U}_H^\alpha \cong \mathcal{A}_H^\alpha.$$

We illustrate our results in Section 10 on the four-dimensional Sweedler Hopf algebra. In this example the base of our versal deformation space is a quadric hypersurface in a five-dimensional affine space deprived of two hyperplanes.

The paper ends with two appendices related to the constructions of Section 5. In Appendix A we construct a map needed for the construction of the cocycle σ. In Appendix B we relate the algebra $S(t_H)_\Theta$ defined in Section 5.3 to Takeuchi's free commutative Hopf algebra over a coalgebra. It follows that $S(t_H)_\Theta$ is obtained from a polynomial algebra by inverting certain canonical polynomials. These polynomials extend Dedekind's group determinants to the framework of coalgebras.

1. Preliminaries

Throughout the paper, we fix a ground field k over which all our constructions are defined. As usual, unadorned tensor products mean tensor products over k.

All algebras that we consider are associative unital k-algebras. The unit of an algebra A will be denoted 1_A, or 1 if no confusion is possible. All algebra morphisms are supposed to preserve the units. We denote the set of algebra morphisms from A to A' by $\mathrm{Alg}(A, A')$.

All coalgebras are coassociative counital k-coalgebras. We denote the comultiplication of a coalgebra C by Δ and its counit by ε. We use the Heyneman–Sweedler sigma notation (see [1, Section 2.1.1], [29, Section 1.2]):

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$$

for the comultiplication of $x \in C$ and

$$\Delta^{(2)}(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$$

for the iterated comultiplication $\Delta^{(2)} = (\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta$, and so on.

Let H be a Hopf algebra with comultiplication Δ, counit ε, and antipode S. An H-*comodule algebra* is an algebra A equipped with a right H-comodule structure whose (coassociative, counital) coaction

$$\delta : A \to A \otimes H$$

is an algebra morphism (we will not use left comodules). A coaction δ is called *trivial* if $\delta(a) = a \otimes 1$ for all $a \in A$.

Given two H-comodule algebras A and A' with respective coactions δ and δ', an algebra morphism $f : A \to A'$ is an H-*comodule algebra morphism* if

$$\delta' \circ f = (f \otimes \mathrm{id}_H) \circ \delta.$$

We denote by Alg^H the category whose objects are H-comodule algebras and arrows are H-comodule algebra morphisms.

The space of *coinvariants* of an H-comodule algebra A is the subspace A^H of A defined by

$$A^H = \left\{ a \in A \mid \delta(a) = a \otimes 1 \right\}.$$

The subspace A^H is a subalgebra and a subcomodule of A. It is the largest subcomodule of A on which the coaction is trivial.

We shall later use the following lemma.

Lemma 1.1. *Let $f : A_1 \to A_2$ be an injective H-comodule algebra morphism. Then*

$$A_1^H = \left\{ a \in A_1 \mid f(a) \in A_2^H \right\}.$$

Proof. If $a \in A_1^H$, then $\delta(a) = a \otimes 1$. Therefore, $\delta(f(a)) = f(a) \otimes 1$, which shows that $f(a)$ belongs to A_2^H.

Conversely, let $a \in A_1$ be such that $\delta(f(a)) = f(a) \otimes 1$. Let $\{1\} \cup \{h_i\}_{i \in I}$ be a basis of H. Expanding $\delta(a) = a_0 \otimes 1 + \sum_{i \in I} a_i \otimes h_i$ in this basis, we obtain

$$f(a) \otimes 1 = \delta\big(f(a)\big) = (f \otimes \mathrm{id}_H)\big(\delta(a)\big)$$

$$= (f \otimes \mathrm{id}_H)\Big(a_0 \otimes 1 + \sum_{i \in I} a_i \otimes h_i \Big)$$

$$= f(a_0) \otimes 1 + \sum_{i \in I} f(a_i) \otimes h_i.$$

This implies that $f(a_i) = 0$ for all $i \in I$. Since f is injective, $a_i = 0$ for all $i \in I$. Therefore, $\delta(a) = a_0 \otimes 1$. Applying $\mathrm{id}_{A_1} \otimes \varepsilon$ to both sides of the previous equality, we obtain $a = a_0$. It follows that $a \in A_1^H$. $\square$

We end these preliminaries with two well-known classes of comodule algebras.

Example 1.2. The group algebra $H = k[G]$ of a group G is a Hopf algebra with comultiplication, counit, and antipode given for all $g \in G$ by

$$\Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1, \qquad S(g) = g^{-1}.$$

It is well known (see [10, Lemma 4.8]) that an H-comodule algebra A is the same as a G-graded algebra

$$A = \bigoplus_{g \in G} A_g.$$

The coaction $\delta : A \to A \otimes H$ is given by $\delta(a) = a \otimes g$ for $a \in A_g$ and $g \in G$. We have $A^H = A_e$, where e is the neutral element of G.

Example 1.3. Let G be a finite group and $H = k^G$ be the algebra of k-valued functions on a finite group G. This algebra can be equipped with a Hopf algebra structure that is dual to the Hopf algebra $k[G]$ above. An H-comodule algebra A is the same as an algebra equipped with a left action of G on A by group automorphisms. If we denote the action of $g \in G$ on $a \in A$ by $^g a$, then the coaction $\delta : A \to A \otimes H$ is given by

$$\delta(a) = \sum_{g \in G} {}^g a \otimes e_g,$$

where $\{e_g\}_{g \in G}$ is the basis of H consisting of the functions e_g defined by $e_g(h) = 1$ if $h = g$, and 0 otherwise. The subalgebra of coinvariants of A coincides with the subalgebra of G-invariant elements: $A^H = A^G$.

2. H-identities

Let H be a Hopf algebra. We first build up the formal setting in which the identities will live.

2.1. The tensor algebra

Let X_H be a copy of the underlying vector space of H. We denote the identity map from H to X_H by $h \mapsto X_h$ for all $h \in H$. By definition we have

$$X_{\lambda_1 h_1 + \lambda_2 h_2} = \lambda_1 X_{h_1} + \lambda_2 X_{h_2}$$

for all $\lambda_1, \lambda_2 \in k$ and $h_1, h_2 \in H$.

The vector space X_H is naturally an H-comodule whose coaction δ is induced by the comultiplication Δ of H, i.e., for all $h \in H$ we have

$$\delta(X_h) = \sum_{(h)} X_{h_{(1)}} \otimes h_{(2)} \in X_H \otimes H. \tag{2.1}$$

Consider the tensor algebra $T(X_H)$ of the vector space X_H over the ground field k:

$$T(X_H) = \bigoplus_{r \geqslant 0} T^r(X_H),$$

where $T^r(X_H) = k$ if $r = 0$, $T^r(X_H) = X_H$ if $r = 1$, and $T^r(X_H) = X_H^{\otimes r}$ for all $r \geqslant 2$. If $\{h_i\}_{i \in I}$ is a basis of H, then $T(X_H)$ is the free noncommutative algebra over the set of indeterminates $\{X_{h_i}\}_{i \in I}$.

We equip $T(X_H)$ with an H-comodule algebra structure with the coaction induced by (2.1). On $T^r(X_H)$ ($r \geqslant 2$) the coaction is given by

$$\delta(X_{h^{[1]}} \cdots X_{h^{[r]}}) = \sum_{(h^{[1]}),\ldots,(h^{[r]})} X_{h^{[1]}_{(1)}} \cdots X_{h^{[r]}_{(1)}} \otimes h^{[1]}_{(2)} \cdots h^{[r]}_{(2)} \tag{2.2}$$

for all $h^{[1]}, \ldots, h^{[r]} \in H$.

By the universal property of $T(X_H)$, any algebra morphism from $T(X_H)$ to an algebra A is determined by its restriction to its degree-one summand $T^1(X_H) = X_H$. We thus have a natural bijection

$$\mathrm{Alg}\big(T(X_H), A\big) \cong \mathrm{Hom}(X_H, A) = \mathrm{Hom}(H, A).$$

If we restrict this bijection to H-comodule algebra morphisms, we obtain a bijection

$$\mathrm{Alg}^H\big(T(X_H), A\big) \cong \mathrm{Hom}^H(X_H, A) = \mathrm{Hom}^H(H, A),$$

which is functorial in $A \in \mathrm{Alg}^H$. Here Alg^H is the category of H-comodule algebras and Hom^H is the category of H-comodules.

Let us give a typical example of a coinvariant element of $T(X_H)$.

Lemma 2.1. *For any* $h^{[1]}, \ldots, h^{[r]} \in H$,

$$P_{h^{[1]},\ldots,h^{[r]}} = \sum_{(h^{[1]}),\ldots,(h^{[r]})} X_{h^{[1]}_{(1)}} \cdots X_{h^{[r]}_{(1)}} X_{S(h^{[1]}_{(2)} \cdots h^{[r]}_{(2)})}$$

is a coinvariant element of $T(X_H)$.

Proof. By (2.2), the element $\delta(P_{h^{[1]},\ldots,h^{[r]}}) \in T(X_H) \otimes H$ is equal to

$$\sum_{(h^{[1]}),\ldots,(h^{[r]})} , X_{h^{[1]}_{(1)}} \cdots X_{h^{[r]}_{(1)}} X_{S(h^{[1]}_{(4)} \cdots h^{[r]}_{(4)})} \otimes h^{[1]}_{(2)} \cdots h^{[r]}_{(2)} S\big(h^{[1]}_{(3)} \cdots h^{[r]}_{(3)}\big)$$

$$= \sum_{(h^{[1]}),\ldots,(h^{[r]})} X_{h^{[1]}_{(1)}} \cdots X_{h^{[r]}_{(1)}} X_{S(h^{[1]}_{(3)} \cdots h^{[r]}_{(3)})} \otimes \varepsilon\big(h^{[1]}_{(2)}\big) \cdots \varepsilon\big(h^{[r]}_{(2)}\big) 1,$$

which is clearly equal to $P_{h^{[1]},\ldots,h^{[r]}} \otimes 1$. $\square$

2.2. *H-identities for comodule algebras*

Let A be an H-comodule algebra. An element $P \in T(X_H)$ is called an *H-identity for A* if $\mu(P) = 0$ for all H-comodule algebra morphisms $\mu : T(X_H) \to A$.

Our H-identities for H-comodule algebras generalize graded identities for group-graded algebras. Indeed, let $H = k[G]$ be the Hopf algebra of a group G. The set $\{X_g\}_{g \in G}$ forms a basis of X_H. Let A be an H-comodule algebra; by Example 1.2, it is a G-graded algebra: $A = \bigoplus_{g \in G} A_g$. An H-comodule algebra morphism $\mu : T(X_H) \to A$ is then an algebra morphism such that $\mu(X_g) \in A_g$ for all $g \in G$, and an H-identity for A is what is called a *graded identity*, see [3,7].

Our definition of identities for Hopf comodule algebras should also be compared to the definition of identities for Hopf module algebras appearing in [6].

Let $I_H(A)$ be the set of all H-identities for A. By definition,

$$I_H(A) = \bigcap_{\mu \in \mathrm{Alg}^H(T(X_H),A)} \mathrm{Ker}\,\mu. \tag{2.3}$$

Proposition 2.2.

(a) *The set $I_H(A)$ is a two-sided ideal and a right H-coideal of $T(X_H)$.*
(b) *The ideal $I_H(A)$ is preserved by all H-comodule algebra endomorphisms of $T(X_H)$.*

This result allows us to paraphrase the classical theory of rings with polynomial identities (see, e.g., [27]) by saying that $I_H(A)$ is a T-ideal.

Proof. (a) For each $\mu \in \mathrm{Alg}^H(T(X_H), A)$ the kernel $\mathrm{Ker}\,\mu$ is a two-sided ideal of $T(X_H)$. Hence so is the intersection $I_H(A)$.

Let us show that $\mathrm{Ker}\,\mu$ is a right H-coideal of $T(X_H)$ for each comodule algebra morphism $\mu : T(X_H) \to A$. If we fix a basis $\{h_i\}_{i \in I}$ of H, then there is a family of linear endomorphisms $\{\delta_i\}_{i \in I}$ of $T(X_H)$ such that the coaction of H on $T(X_H)$ can be written

$$\delta(P) = \sum_{i \in I} \delta_i(P) \otimes h_i$$

for all $P \in T(X_H)$. If $P \in \mathrm{Ker}\,\mu$, then

$$\sum_{i \in I} \mu(\delta_i(P)) \otimes h_i = (\mu \otimes \mathrm{id}_H)\delta(P) = \delta(\mu(P)) = 0.$$

It follows that $\mu(\delta_i(P)) = 0$ for all $i \in I$, hence $\delta(P)$ belongs to $\mathrm{Ker}\,\mu \otimes H$. We have thus established that $\mathrm{Ker}\,\mu$ is a right H-coideal of $T(X_H)$. Using a similar argument, one proves that the intersection $I_H(A)$ is a right H-coideal.

(b) If $f : T(X_H) \to T(X_H)$ is an H-comodule algebra morphism, then so is $\mu \circ f : T(X_H) \to A$ for all $\mu \in \mathrm{Alg}^H(T(X_H), A)$. Hence, if $P \in I_H(A)$, then $\mu(f(P)) = 0$ for all $\mu \in \mathrm{Alg}^H(T(X_H), A)$, which implies that $f(P)$ belongs to $I_H(A)$. $\quad\square$

The tensor algebra has a natural grading for which $T^r(X_H)$ ($r \geqslant 0$) is the subspace of all elements of $T(X_H)$ of degree r. Eq. (2.2) implies that

$$\delta\big(T^r(X_H)\big) \subset T^r(X_H) \otimes H. \tag{2.4}$$

Proposition 2.3. *If k is infinite, then the submodule $I_H(A)$ of $T(X_H)$ is graded with $I_H^r(A) = I_H(A) \cap T^r(X_H)$ for all $r \geqslant 0$.*

Proof. Expand $P \in I_H(A)$ into a finite sum with homogeneous parts:

$$P = \sum_{r \geqslant 0} P_r,$$

where $P_r \in T^r(X_H)$. It suffices to show that $P_r \in I_H(A)$ for all $r \geqslant 0$.

Given $\lambda \in k$, consider the algebra endomorphism λ_* of $T(X_H)$ defined by $\lambda(X_h) = \lambda X_h$ for all $h \in H$. It is easy to check that λ_* is an H-comodule map. If $\mu : T(X_H) \to A$ is an H-comodule algebra morphism, then so is $\mu \circ \lambda_*$. Consequently,

$$\sum_{r \geqslant 0} \lambda^r \mu(P_r) = (\mu \circ \lambda_*)(P) = 0.$$

The A-valued polynomial $\sum_{r \geqslant 0} \lambda^r \mu(P_r)$ takes zero values for all $\lambda \in k$. By the assumption on k, this implies that its coefficients are all zero, i.e., $\mu(P_r) = 0$ for all $r \geqslant 0$. Since this holds for all $\mu \in \mathrm{Alg}^H(T(X_H), A)$, we obtain $P_r \in I_H(A)$ for all $r \geqslant 0$. $\square$

Proposition 2.4. *Let k be an infinite field.*

(a) *We have $I_H^0(A) = \{0\}$.*
(b) *If there is an injective comodule map $H \to A$, then $I_H^1(A) = \{0\}$.*

Proof. (a) The dimension of $I_H^0(A)$ is at most one. Since $\mu(1) = 1 \neq 0$ for all $\mu \in \mathrm{Alg}^H(T(X_H), A)$, there cannot exist nonzero H-identities in degree zero.

(b) Any element of $T(X_H)$ of degree one is of the form X_h for some $h \in H$. Let $u : H \to A$ be an injective comodule map. It extends to a morphism $\mu \in \mathrm{Alg}^H(T(X_H), A)$ uniquely determined by $\mu(X_h) = u(h)$ for all $h \in H$. If $X_h \in I_H(A)$, then $u(h) = \mu(X_h) = 0$. By the injectivity of u, we have $h = 0$, hence $X_h = 0$. $\square$

When the condition of Proposition 2.4(b) is not satisfied, there may exist H-identities of degree one, as shown by Proposition 2.5 below.

Let H be a Hopf algebra and $A = k$ be the trivial one-dimensional H-comodule algebra. (In this case there are no injective comodule maps $H \to A$ unless $\dim H = 1$.) Let us determine the ideal $I_H(k)$ of H-identities for k. It is well known (see [1, Section 3.3.1]) that the vector space of H-comodule maps $H \to k$ coincides with the vector space of right integrals for H and that this vector space is either zero or one-dimensional.

If H has no nonzero right integrals, then any element of the augmentation ideal $T^+(X_H) = \bigoplus_{r \geqslant 1} T^r(X_H)$ is an H-identity for k. Hence,

$$I_H(k) = T^+(X_H). \tag{2.5}$$

Let us now consider the case where H has a nonzero right integral. This is the case for instance when H is a finite-dimensional Hopf algebra.

Proposition 2.5. *If $N : H \to k$ is a nonzero right integral, then*

$$I_H^1(k) = \left\{ X_h \in T^1(X_H) \mid N(h) = 0 \right\},$$

and $I_H(A)$ is the two-sided ideal of $T(X_H)$ generated by $I_H^1(k)$.

Proof. Let $\mu : T(X_H) \to k$ be an H-comodule algebra morphism. Its restriction to $T^1(X_H) \cong H$ is an H-comodule map, hence a right integral for H, which is necessarily a scalar multiple of N. It follows that an element of $T^1(X_H)$ is an H-identity for k if and only if it is in the kernel of N.

Let $\{h_i\}_{i \in I}$ be a basis of $\mathrm{Ker}(N)$ and $h_0 \in H$ such that $N(h_0) = 1$. Any element of $T(X_H)$ is a linear combination of monomials in the indeterminates X_{h_i} ($i \in I$) together with the indeterminate $X = X_{h_0}$. Clearly, μ vanishes on any monomial containing at least an indeterminate X_{h_i}, where $i \in I$. On the contrary, if $\mu \neq 0$, then μ takes a nonzero value on any monomial that is a pure power of X. Therefore, $I_H(A)$ is the ideal generated by the indeterminates X_{h_i}, where $i \in I$. $\quad\square$

Remarks 2.6. (a) Instead of defining the H-identities in the tensor algebra $T(X_H)$, we could define them in the *reduced tensor algebra* $\bar{T}(X_H)$, which we define as the quotient of $T(X_H)$ by the two-sided ideal generated by $X_{1_H} - 1$, where 1_H is the unit of the algebra H. Certain identities would be simpler in $\bar{T}(X_H)$, but we would lose the natural grading of $T(X_H)$ as the ideal generated by $X_{1_H} - 1$ is not graded.

(b) If we have a family $(A_\alpha)_\alpha$ of H-comodule algebras, we define an H-identity for the family $(A_\alpha)_\alpha$ to be an element $P \in T(X_H)$ that is an H-identity for each H-comodule algebra A_α of the family. We may then consider the intersection $\bigcap_\alpha I_H(A_\alpha)$, which is a two-sided ideal and a right H-coideal of $T(X_H)$.

2.3. Central polynomials

Let A be an H-comodule algebra. We say that $P \in T(X_H)$ is a *central polynomial for A* if $PQ - QP$ is an H-identity for A for all $Q \in T(X_H)$. Since

$$P(Q_1 Q_2) - (Q_1 Q_2 P) = (PQ_1 - Q_1 P)Q_2 + Q_1(PQ_2 - Q_2 P)$$

for all $Q_1, Q_2 \in T(X_H)$, we conclude by induction on the degree that P is a central polynomial for A if and only if

$$P X_h - X_h P \in I_H(A)$$

for all $h \in H$. To check that P is a central polynomial, it is therefore enough to check that $\mu(P)$ commutes with $\mu(X_h)$ in A for all $\mu \in \mathrm{Alg}^H(T(X_H), A)$ and all $h \in H$.

In Section 4.4 we shall establish under certain assumptions that each element $P_{h^{[1]}, \ldots, h^{[r]}} \in T(X_H)$ of Lemma 2.1 is a central polynomial.

3. The universal H-comodule algebra

Let A be an H-comodule algebra and $I_H(A)$ the two-sided ideal of H-identities for A defined by (2.3). Consider the quotient algebra

$$\mathcal{U}_H(A) = T(X_H)/I_H(A). \tag{3.1}$$

Proposition 3.1.

(a) *The algebra $\mathcal{U}_H(A)$ has a unique structure of an H-comodule algebra such that the projection $T(X_H) \to \mathcal{U}_H(A)$ is an H-comodule algebra morphism.*
(b) *Any $\mu \in \mathrm{Alg}^H(T(X_H), A)$ induces an H-comodule algebra morphism*

$$\mu : \mathcal{U}_H(A) \to A.$$

Proof. (a) This is an immediate consequence of Proposition 2.2(a).

(b) This follows from the vanishing of $\mu(I_H(A))$. $\quad\square$

By definition of $\mathcal{U}_H(A)$, all H-identities for A are satisfied (i.e., vanish) in $\mathcal{U}_H(A)$ and any quotient H-comodule algebra of $T(X_H)$ in which all H-identities for A are satisfied is a homomorphic image of $\mathcal{U}_H(A)$. In this sense, $\mathcal{U}_H(A)$ is the universal algebra in which all H-identities for A are satisfied. We call $\mathcal{U}_H(A)$ the *universal H-comodule algebra* of A.

By Proposition 2.3, if k is infinite, then $I_H(A)$ is graded. For each $r \geqslant 0$, let $\mathcal{U}_H^r(A)$ be the image of $T^r(X_H)$ in $\mathcal{U}_H(A)$ under the projection map $T(X_H) \to \mathcal{U}_H(A)$. We have $1 \in \mathcal{U}_H^0(A)$.

Proposition 3.2. *If k is infinite, then the H-comodule algebra $\mathcal{U}_H(A)$ is graded, namely*

$$\mathcal{U}_H(A) = \bigoplus_{r \geqslant 0} \mathcal{U}_H^r(A),$$

$$\mathcal{U}_H^r(A)\mathcal{U}_H^s(A) \subset \mathcal{U}_H^{r+s}(A) \quad and \quad \delta\big(\mathcal{U}_H^r(A)\big) \subset \mathcal{U}_H^r(A) \otimes H$$

for all $r, s \geqslant 0$.

Proof. Clearly, $\mathcal{U}_H(A) = \sum_{r \geqslant 0} \mathcal{U}_H^r(A)$. To prove that this sum is direct, it suffices to check that, if $\{u_r\}_{r \geqslant 0}$ is a family of elements of $\mathcal{U}_H(A)$ such that $u_r \in \mathcal{U}_H^r(A)$ for all $r \geqslant 0$ and $\sum_{r \geqslant 0} u_r = 0$, then $u_r = 0$ for all $r \geqslant 0$. Represent each u_r by an element $w_r \in T^r(X_H)$. We have $\sum_{r \geqslant 0} w_r \in I_H(A)$. By Proposition 2.3, $I_H(A)$ is graded. This implies that $w_r \in I_H(A)$ for each $r \geqslant 0$. Therefore, $u_r = 0$ for each $r \geqslant 0$, as desired.

The inclusion $\mathcal{U}_H^r(A)\mathcal{U}_H^s(A) \subset \mathcal{U}_H^{r+s}(A)$ follows from the fact that $T(X_H)$ is a graded algebra, and the inclusion $\delta(\mathcal{U}_H^r(A)) \subset \mathcal{U}_H^r(A) \otimes H$ from (2.4). $\quad\square$

We now use Proposition 2.5 to give a simple example for which we can compute $\mathcal{U}_H(A)$. In this example, A is the trivial H-comodule algebra k.

Proposition 3.3.

(a) *If there are no nonzero right integrals for H, then $\mathcal{U}_H(k) \cong k$.*
(b) *If there is a nonzero right integral for H, then $\mathcal{U}_H(k)$ is isomorphic as an H-comodule algebra to $k[X]$ with trivial coaction.*

Proof. (a) This is a consequence of (2.5).

(b) The isomorphism $\mathcal{U}_H(k) \cong k[X]$ follows from the proof of Proposition 2.5. It remains to show that the coaction on $k[X]$ is given by $\delta(X) = X \otimes 1$. We resume the notation of the second part of the proof of Proposition 2.5. Expand

$$\Delta(h_0) = h_0 \otimes x_0 + \sum_{i \in I} h_i \otimes x_i,$$

where x_0 and $\{x_i\}_{i \in I}$ are elements of H. Since $N : H \to k$ is a comodule map,

$$N(h_0) \otimes x_0 + \sum_{i \in I} N(h_i) \otimes x_i = (N \otimes \mathrm{id}_H)\big(\Delta(h_0)\big) = N(h_0) 1 \otimes 1.$$

Replacing $N(h_i)$ by its value, we obtain $1 \otimes x_0 = 1 \otimes 1$, hence $x_0 = 1$. Therefore, by definition of the coaction of H on $T(X_H)$,

$$\delta(X) = X \otimes 1 + \sum_{i \in I} X_{h_i} \otimes x_i \in X \otimes 1 + I_H(k) \otimes H.$$

We obtain the desired conclusion by passing to the quotient by $I_H(k)$.   □

We end this section with a few questions on the structure of $\mathcal{U}_H(A)$.

 (i) If A is free as a module over the subalgebra of coinvariants, does the same hold for $\mathcal{U}_H(A)$?
 (ii) If A is free as a module over its center, is $\mathcal{U}_H(A)$ free over its center $\mathcal{Z}_H(A)$, possibly after some localization of the latter?
(iii) Is a suitable central localization of $\mathcal{U}_H(A)$ a flat deformation of A over the center $\mathcal{Z}_H(A)$? How is $Z(A) \otimes_{\mathcal{Z}_H(A)} \mathcal{U}_H(A)$ related to A, where $Z(A)$ is the center of A?

In the sequel we shall provide answers to these questions under some hypotheses.

4. Detecting H-identities

We again fix a Hopf algebra H. The aim of this section is to construct an H-comodule algebra morphism whose domain is $T(X_H)$ and whose kernel is $I_H(A)$ for any H-comodule algebra A satisfying the following assumptions.

4.1. The assumptions

Let A be an H-comodule algebra and $B = A^H$ be the subalgebra of coinvariant elements of A. We make the following assumptions on A:

(i) B is central in A, and

(ii) there is a B-linear H-comodule isomorphism $u : B \otimes H \to A$, where the coaction on $B \otimes H$ is equal to $\mathrm{id}_B \otimes \Delta$.

In Section 5 we shall give examples of H-comodule algebras for which these assumptions hold.

4.2. The symmetric algebra

Let t_H be another copy of the underlying vector space of H. We denote the identity map from H to t_H by $h \mapsto t_h$ $(h \in H)$.

Let $S(t_H)$ be the *symmetric algebra* over the k-vector space t_H. It is a quotient of the tensor algebra $T(t_H)$ and inherits the grading of the latter. For each $r \geqslant 0$, let us denote $S^r(t_H)$ the subspace of elements of degree r. We have

$$S(t_H) = \bigoplus_{r \geqslant 0} S^r(t_H)$$

with $S^0(t_H) = k$, and $S^1(t_H) = t_H$. If $\{h_i\}_{i \in I}$ is a basis of H, then $S(t_H)$ is isomorphic to the polynomial algebra over the indeterminates $\{t_{h_i}\}_{i \in I}$.

The linear isomorphism $t : H \to t_H$ extends to an injective linear map $H \to S(t_H)$, which we still denote t and whose image is $S^1(t_H)$. The algebra $S(t_H)$ satisfies the following well-known universal property: for any commutative algebra R and any linear map $g : H \to R$, there is a unique algebra morphism $f : S(t_H) \to R$ such that $f \circ t = g$.

4.3. The universal evaluation morphism

Given an algebra A, we endow the tensor product $S(t_H) \otimes A$ with its natural algebra structure. The grading of $S(t_H)$ induces a grading on the algebra $S(t_H) \otimes A$: for each $r \geqslant 0$, the subspace of elements of degree r in $S(t_H) \otimes A$ is equal to $S^r(t_H) \otimes A$.

If, in addition, A is an H-comodule algebra, then $S(t_H) \otimes A$ is an H-comodule algebra with the $S(t_H)$-linear coaction

$$\mathrm{id}_{S(t_H)} \otimes \delta : S(t_H) \otimes A \to S(t_H) \otimes A \otimes H,$$

where $\delta : A \to A \otimes H$ is the coaction of A.

Suppose now that A satisfies the assumptions of Section 4.1. Using the H-comodule algebra $T(X_H)$ of Section 2, we consider the algebra morphism

$$\mu_A : T(X_H) \to S(t_H) \otimes A$$

defined for all $h \in H$ by

$$\mu_A(X_h) = \sum_{(h)} t_{h_{(1)}} \otimes u(1_B \otimes h_{(2)}), \tag{4.1}$$

where $u : B \otimes H \to A$ is the isomorphism of Section 4.1.

Lemma 4.1.

(a) *The algebra morphism $\mu_A : T(X_H) \to S(t_H) \otimes A$ sends $T^r(X_H)$ into $S^r(t_H) \otimes A$ for each $r \geqslant 0$.*
(b) *The morphism μ_A is injective on $T^0(X_H) \oplus T^1(X_H) = k \oplus X_H$ and its kernel is contained in $\bigoplus_{r \geqslant 2} T^r(X_H)$.*

Proof. (a) This follows from (4.1).

(b) Since $\mu_A(1) = 1 \otimes u(1_B \otimes 1_H) \neq 0$, the morphism μ_A is injective on $T^0(X_H)$. It is injective on $T^1(X_H) = X_H$ since

$$\left((\mathrm{id}_{S(t_H)} \otimes \varepsilon) \circ \left(\mathrm{id}_B \otimes u^{-1}\right) \circ \mu_A\right)(X_h) = \sum_{(h)} t_{h_{(1)}} \otimes \varepsilon(h_{(2)}) 1_B = t_h \otimes 1_B$$

for all $h \in H$. $\quad\square$

The morphism μ_A possesses the following important properties.

Lemma 4.2.

(a) *The morphism $\mu_A : T(X_H) \to S(t_H) \otimes A$ is an H-comodule algebra morphism.*
(b) *For every $\mu \in \mathrm{Alg}^H(T(X_H), A)$ there is a unique algebra morphism*

$$\lambda : S(t_H) \to B$$

such that $\mu = (\lambda \otimes \mathrm{id}_A) \circ \mu_A$.

In view of the previous lemma, we call μ_A the *universal evaluation morphism* for A.

Proof. (a) It is enough to check that the restriction of μ_A to X_H is an H-comodule map. Now, $\mu_A = (\mathrm{id}_{S(t_H)} \otimes u) \circ \mu'_A$, where

$$\mu'_A : X_H \to S(t_H) \otimes B \otimes H$$

is the linear map defined by $\mu'_A(X_h) = \sum_{(h)} t_{h_{(1)}} \otimes 1_B \otimes h_{(2)}$ for all $h \in H$. Using the definition of the coaction on X_H and on $S(t_H) \otimes B \otimes H$, and the coassociativity of the comultiplication of H, one checks easily that μ'_A is a comodule map. Since u also is a comodule map, the conclusion follows.

(b) Since we are dealing with algebra morphisms, it suffices to consider their restriction to X_H. Moreover, $u : B \otimes H \to A$ being B-linear, it is enough to prove that for every $\mu \in \mathrm{Hom}^H(X_H, B \otimes H)$ there is a unique linear map $\lambda : t_H \to B$ such that $\mu = (\lambda \otimes \mathrm{id}_B \otimes \mathrm{id}_H) \circ \mu'_A$.

We first prove the uniqueness of λ. If $\mu = (\lambda \otimes \mathrm{id}_B \otimes \mathrm{id}_H) \circ \mu'_A$, then $(\mathrm{id}_B \otimes \varepsilon) \circ \mu = \lambda \otimes \mathrm{id}_B \otimes \varepsilon$. Hence, for all $h \in H$,

$$(\mathrm{id}_B \otimes \varepsilon)\left(\mu(X_h)\right) = \sum_{(h)} \lambda(t_{h_{(1)}}) 1_B \varepsilon(h_{(2)})$$

$$= \sum_{(h)} \lambda(t_{h_{(1)} \varepsilon(h_{(2)})}) = \lambda(t_h).$$

This shows that λ is uniquely determined.

Let us next prove the existence of λ. For $\mu \in \mathrm{Alg}^H(T(X_H), B \otimes H)$ and $h \in H$, expand $\mu(X_h)$ in $B \otimes H$ in the form

$$\mu(X_h) = \sum_j \mu_j(h) \otimes \nu_j(h),$$

where $\mu_j(h) \in B$ and $\nu_j(h) \in H$. If we set $\lambda(t_h) = \sum_j \mu_j(h) \varepsilon(\nu_j(h)) \in B$, then

$$\big((\lambda \otimes \mathrm{id}_B \otimes \mathrm{id}_H) \circ \mu_A'\big)(X_h) = \sum_{(h)} \lambda(t_{h_{(1)}}) \otimes h_{(2)}$$

$$= \sum_{j,(h)} \mu_j(h_{(1)}) \varepsilon\big(\nu_j(h_{(1)})\big) \otimes h_{(2)}$$

$$= \sum_{j,(\nu_j(h))} \mu_j(h) \otimes \varepsilon\big(\nu_j(h)_{(1)}\big) \nu_j(h)_{(2)}$$

$$= \sum_j \mu_j(h) \otimes \nu_j(h) = \mu(X_h)$$

for all $h \in H$. The third and fourth equalities follow respectively from the facts that μ is a comodule map and ε is the counit. $\square$

Part (b) of the previous proof can be summarized by the following sequence of natural bijections:

$$\mathrm{Alg}^H\big(T(X_H), A\big) \cong \mathrm{Hom}^H(H, A)$$

$$\cong \mathrm{Hom}^H(H, B \otimes H)$$

$$\cong \mathrm{Hom}(H, B)$$

$$\cong \mathrm{Alg}\big(S(t_H), B\big),$$

where the last bijection holds because the algebra B is commutative.

We now state the main result of this section: it implies that the H-identities for A are detected by the morphism μ_A.

Theorem 4.3. *Let A be an H-comodule algebra satisfying the assumptions of Section* 4.1.

(a) *We have* $\mathrm{Ker}(\mu_A : T(X_H) \to S(t_H) \otimes A) \subset I_H(A)$.
(b) *If the ground field k is infinite, then*

$$I_H(A) = \mathrm{Ker}\big(\mu_A : T(X_H) \to S(t_H) \otimes A\big).$$

Proof. (a) Let $P \in T(X_H)$ such that $\mu_A(P) = 0$. By Lemma 4.2(b), any morphism $\mu \in \mathrm{Alg}^H(T(X_H), A)$ is of the form $\mu = (\lambda \otimes \mathrm{id}_A) \circ \mu_A$ for some $\lambda \in \mathrm{Alg}(S(t_H), B)$. Therefore, $\mu(P) = (\lambda \otimes \mathrm{id}_H)(\mu_A(P)) = 0$. Hence, $P \in I_H(A)$.

(b) By part (a) it is enough to check that any $P \in I_H(A)$ belongs to the kernel of μ_A. Since $\mathrm{id}_{S(t_H)} \otimes u$ is a linear isomorphism, it suffices to check that $\mu'_A(P) = 0$ for any $P \in I_H(A)$, where $\mu'_A : X_H \to S(t_H) \otimes B \otimes H$ is the map introduced in the proof of Lemma 4.2. Expanding $\mu'_A(P)$ in a basis $\{h_i\}_{i \in I}$ of H, we have

$$\mu'_A(P) = \sum_{i \in I} \mu_A^{(i)}(P) \otimes h_i \in S(t_H) \otimes B \otimes H,$$

where $\mu_A^{(i)}(P) \in S(t_H) \otimes B$ for all $i \in I$. By Lemma 4.2(b), if $P \in I_H(A)$, then

$$\sum_{i \in I} \lambda\big(\mu_A^{(i)}(P)\big) \otimes h_i = (\lambda \otimes \mathrm{id}_{B \otimes H})\big(\mu'_A(P)\big) = 0$$

for all $\lambda \in \mathrm{Alg}(S(t_H), B)$. Therefore, for each $i \in I$, we have $\lambda(\mu_A^{(i)}(P)) = 0$ for all algebra morphisms $\lambda : S(t_H) \to B$. This means that the B-valued polynomial $\mu_A^{(i)}(P)$ takes only zero values. Since k is an infinite field, this implies that $\mu_A^{(i)}(P) = 0$ for all $i \in I$. Hence, $\mu'_A(P) = 0$. $\quad\square$

We have the following consequence of Lemma 4.1 and of Theorem 4.3 for the universal H-comodule algebra $\mathcal{U}_H(A)$ defined by (3.1).

Corollary 4.4. *Let k be an infinite field and A be an H-comodule algebra satisfying the assumptions of Section* 4.1.

(a) *The universal evaluation morphism μ_A induces an injective H-comodule algebra morphism*

$$\mu_A : \mathcal{U}_H(A) = T(X_H)/I_H(A) \hookrightarrow S(t_H) \otimes A.$$

(b) *We have $\mathcal{U}_H^0(A) \cong k$ and $\mathcal{U}_H^1(A) \cong X_H$.*

The previous corollary has an interesting consequence, which guarantees the existence of nonzero H-identities for A in the finite-dimensional case.

Corollary 4.5. *Let k be an infinite field and A be an H-comodule algebra satisfying the assumptions of Section 4.1. If H and A are finite-dimensional and $H \neq k$, then $I_H(A) \neq 0$.*

Proof. If $I_H(A) = 0$, then Corollary 4.4 implies that $T^r(X_H)$ injects into $S^r(t_H) \otimes A$. Set $n = \dim H$ and $d = \dim A$. We then have

$$n^r = \dim T^r(X_H) \leqslant \dim\big(S^r(t_H) \otimes A\big) = d \binom{r + n - 1}{n - 1},$$

which is impossible for $n > 1$ and large r. $\quad\square$

Remark 4.6. Suppose that the map $u : B \otimes H \to A$ of Section 4.1 preserves the units. Since the universal evaluation morphism μ_α sends X_{1_H} to the central element $t_1 \otimes u(1_B \otimes 1_H) = t_1 \otimes 1_A$, it induces an H-comodule algebra morphism $\bar{\mu}_\alpha : \bar{T}(X_H) \to \bar{S}(t_H) \otimes A$, where $\bar{T}(X_H)$ is the reduced tensor algebra of Remark 2.6(a) and $\bar{S}(t_H) = S(t_H)/(t_{1_H} - 1)$. As can be seen from the

computations in Section 10, using the morphism $\bar{\mu}_\alpha$ would simplify certain formulas, but these formulas would no longer be homogeneous with respect to the gradings of $T(X_H)$ and of $S(t_H)$.

4.4. The coinvariants of $\mathcal{U}_H(A)$

The following proposition shows how we can use μ_A to detect the coinvariant elements of $\mathcal{U}_H(A)$.

Proposition 4.7. *Let k be an infinite field and A be an H-comodule algebra satisfying the assumptions of Section 4.1. Then*

$$\mathcal{U}_H(A)^H = \left\{ P \in \mathcal{U}_H(A) \mid \mu_A(P) \in S(t_H) \otimes B \right\}.$$

Proof. This follows from Lemma 1.1, Corollary 4.4(a), and from the easily checked fact that the subalgebra of coinvariants of $S(t_H) \otimes A$ is $S(t_H) \otimes B$. □

We give an interesting consequence of Proposition 4.7 for the elements $P_{h^{[1]},\dots,h^{[r]}} \in T(X_H)$ introduced in Lemma 2.1.

Corollary 4.8. *Under the hypotheses of Proposition 4.7, $P_{h^{[1]},\dots,h^{[r]}}$ is a central polynomial for A for all $h^{[1]},\dots,h^{[r]} \in H$.*

Proof. By Lemma 2.1, $P_{h^{[1]},\dots,h^{[r]}}$ is coinvariant. Thus, by Proposition 4.7, its image $\mu_A(P_{h^{[1]},\dots,h^{[r]}})$ in $S(t_H) \otimes A$ belongs to $S(t_H) \otimes B$. Since $S(t_H) \otimes B$ is central in $S(t_H) \otimes A$, the element $\mu_A(P_{h^{[1]},\dots,h^{[r]}})$ commutes with $\mu_A(X_h)$ for all $h \in H$. By Lemma 4.2(b), $\mu(P_{h^{[1]},\dots,h^{[r]}})$ then commutes with $\mu(X_h)$ for all $\mu \in \mathrm{Alg}^H(T(X_H), A)$ and all $h \in H$. This shows that $P_{h^{[1]},\dots,h^{[r]}}$ is a central polynomial. □

5. The universal cocycle attached to a twisted algebra

Let H be a Hopf algebra.

5.1. Twisted algebras

Consider the following general construction. Let $\alpha : H \times H \to B$ be a convolution invertible bilinear map with values in a commutative algebra B. Consider the *twisted algebra* $B \otimes {}^\alpha H$, which is the vector space $B \otimes H$ equipped with the product $\cdot_\alpha$ given by

$$(b \otimes x) \cdot_\alpha (c \otimes y) = \sum_{(x),(y)} bc\,\alpha(x_{(1)}, y_{(1)}) \otimes x_{(2)} y_{(2)} \tag{5.1}$$

for all $b, c \in B$ and $x, y \in H$. If α is a two-cocycle of H, which means that it satisfies the equations

$$\sum_{(x),(y)} \alpha(x_{(1)}, y_{(1)})\alpha(x_{(2)} y_{(2)}, z) = \sum_{(y),(z)} \alpha(y_{(1)}, z_{(1)})\alpha(x, y_{(2)} z_{(2)}) \tag{5.2}$$

for all $x, y, z \in H$, then the product $\cdot_\alpha$ is associative. If in addition α is normalized, i.e., if

$$\alpha(x, 1) = \alpha(1, x) = \varepsilon(x)1 \tag{5.3}$$

for all $x \in H$, then $1 \otimes 1$ is a unit for the product $\cdot_\alpha$.

The algebra $A = B \otimes^\alpha H$ is an H-comodule algebra with coaction

$$\delta = \mathrm{id}_B \otimes \Delta : A = B \otimes H \to A \otimes H = B \otimes H \otimes H.$$

Its subalgebra A^H of coinvariants coincides with $B \otimes k1$. It is easy to check from (5.1) that A^H lies in the center of A. The assumptions of Section 4.1 are satisfied for A with $B = A^H$ and $u = \mathrm{id}_{B \otimes H}$.

The twisted algebras of the form $B \otimes^\alpha H$ considered here coincide with the *central cleft H-Galois extensions* of B, see [9,15], [26, Prop. 7.2.3]. (By central we mean that B lies in the center of the extension.)

5.2. The two-cocycle σ

Let $\alpha : H \otimes H \to k$ be a normalized convolution invertible two-cocycle with values in the ground field k and $k \otimes^\alpha H$ be the corresponding twisted algebra. To simplify notation, we henceforth denote $k \otimes^\alpha H$ by $^\alpha H$.

To the two-cocycle α we associate a two-cocycle

$$\sigma : H \times H \to K_H$$

taking values in the field of fractions K_H of $S(t_H)$ and defined by

$$\sigma(x, y) = \sum_{(x),(y)} t_{x_{(1)}} t_{y_{(1)}} \alpha(x_{(2)}, y_{(2)}) t^{-1}_{x_{(3)} y_{(3)}} \tag{5.4}$$

for all $x, y \in H$. In (5.4) we have used the map $t^{-1} : H \to K_H$ uniquely defined by Eq. (A.1) of Appendix A.

By [26, Chapter 7], σ is a two-cocycle on H and it is cohomologous to α in K_H. We call σ the *universal cocycle attached to α*; it can be viewed as the most general two-cocycle on H that is cohomologous to α.

Since α is convolution invertible, so is σ. Its convolution inverse σ^{-1} is given for all $x, y \in H$ by

$$\sigma^{-1}(x, y) = \sum_{(x),(y)} t_{x_{(1)} y_{(1)}} \alpha^{-1}(x_{(2)}, y_{(2)}) t^{-1}_{x_{(3)}} t^{-1}_{y_{(3)}}, \tag{5.5}$$

where α^{-1} is the convolution inverse of α.

The two-cocycle σ is almost normalized in the following sense.

Lemma 5.1. *For all $x \in H$,*

$$\sigma(1, x) = \sigma(x, 1) = \varepsilon(x)t_1 \quad \text{and} \quad \sigma^{-1}(1, x) = \sigma^{-1}(x, 1) = \varepsilon(x)t_1^{-1}.$$

Proof. By (5.3), (5.4), and (A.1),

$$\sigma(1, x) = \sum_{(x)} t_1 t_{x_{(1)}} \alpha(1, x_{(2)}) t_{x_{(3)}}^{-1} = \sum_{(x)} t_1 t_{x_{(1)}} \varepsilon(x_{(2)}) t_{x_{(3)}}^{-1} = \sum_{(x)} t_1 t_{x_{(1)}} t_{x_{(2)}}^{-1} = \varepsilon(x) t_1.$$

The other computations are similar. $\square$

5.3. Two algebras of rational fractions

Let $\alpha : H \otimes H \to k$ and $\sigma : H \otimes H \to K_H$ be as in Section 5.2.

We define $S(t_H)_\Theta$ to be the subalgebra of K_H generated by $t_H = t(H)$ and $t_H^{-1} = t^{-1}(H)$, where $t^{-1} : H \to K_H$ is the map defined in Appendix A.

This algebra depends only on the coalgebra structure of H. A description of $S(t_H)_\Theta$ as an explicit localization of $S(t_H)$ is given in Appendix B.

By definition of the universal two-cocycle σ, both σ and its convolution inverse σ^{-1} take values in $S(t_H)_\Theta$.

We recover the cocycle α from the universal cocycle σ as follows.

Lemma 5.2. *There is a unique algebra morphism* $\epsilon : S(t_H)_\Theta \to k$ *such that*

$$\epsilon(t_x) = \epsilon\big(t_x^{-1}\big) = \varepsilon(x)$$

for all $x \in H$. *Moreover,* $\epsilon(\sigma(x, y)) = \alpha(x, y)$ *and* $\epsilon(\sigma^{-1}(x, y)) = \alpha^{-1}(x, y)$ *for all* $x, y \in H$.

Proof. There is a unique algebra morphism $\epsilon : S(t_H) \to k$ such that $\epsilon(t_x) = \varepsilon(x)$ for all $x \in H$. In order to check that this morphism extends to $S(t_H)_\Theta$, it suffices by (A.1) to check that

$$\sum_{(x)} \epsilon(t_{x_{(1)}}) \epsilon\big(t_{x_{(2)}}^{-1}\big) = \sum_{(x)} \epsilon\big(t_{x_{(1)}}^{-1}\big) \epsilon(t_{x_{(2)}}) = \varepsilon(x) 1. \qquad (5.6)$$

for all $x \in H$. Now,

$$\sum_{(x)} \epsilon(t_{x_{(1)}}) \epsilon\big(t_{x_{(2)}}^{-1}\big) = \sum_{(x)} \varepsilon(x_{(1)}) \varepsilon(x_{(2)}) = \varepsilon\left(\sum_{(x)} x_{(1)} x_{(2)}\right) = \varepsilon(x) 1.$$

The other equality is proved in a similar way.

We also have

$$\epsilon\big(\sigma(x, y)\big) = \epsilon\left(\sum_{(x),(y)} t_{x_{(1)}} t_{y_{(1)}} \alpha(x_{(2)}, y_{(2)}) t_{x_{(3)} h_{(y)}}^{-1}\right)$$

$$= \sum_{(x),(y)} \epsilon(t_{x_{(1)}}) \epsilon(t_{y_{(1)}}) \alpha(x_{(2)}, y_{(2)}) \epsilon\big(t_{x_{(3)} y_{(3)}}^{-1}\big)$$

$$= \sum_{(x),(y)} \varepsilon(x_{(1)}) \varepsilon(y_{(1)}) \alpha(x_{(2)}, y_{(2)}) \varepsilon(x_{(3)} y_{(3)})$$

$$= \sum_{(x),(y)} \varepsilon(x_{(1)})\varepsilon(y_{(1)})\alpha(x_{(2)}, y_{(2)})\varepsilon(x_{(3)})\varepsilon(y_{(3)})$$

$$= \alpha(x, y).$$

The computation of $\epsilon(\sigma^{-1}(x, y))$ is done in a similar way.   □

There is a smaller algebra in which both σ and σ^{-1} take values, namely the subalgebra $\mathcal{B}_H^\alpha$ of K_H generated by the values of σ and of σ^{-1}. The algebra $\mathcal{B}_H^\alpha$ depends on the two-cocycle α and is a subalgebra of $S(t_H)_\Theta$:

$$\mathcal{B}_H^\alpha \subset S(t_H)_\Theta \subset K_H.$$

Proposition 5.3. *The algebras $\mathcal{B}_H^\alpha$ and $S(t_H)_\Theta$ have the following properties.*

(a) *They are augmented.*
(b) *They have no zero-divisors.*
(c) *If H is finite-dimensional, then they are finitely generated.*

Proof. (a) This follows from Lemma 5.2.

(b) This holds since both algebras lie in the field K_H.

(c) If $\{h_i\}_i$ is a basis of H, then the algebra $\mathcal{B}_H^\alpha$ (respectively $S(t_H)_\Theta$) is generated by the elements $\sigma(h_i, h_j)$ and $\sigma^{-1}(h_i, h_j)$ (respectively by the elements t_{h_i} and $t_{h_i}^{-1}$). If H is finite-dimensional, these generators are in finite number.   □

It is natural to ask how $\mathcal{B}_H^\alpha$ depends on the two-cocycle α. In particular, if two arbitrary two-cocycles $\alpha, \beta : H \times H \to k$ are cohomologous, do we have $\mathcal{B}_H^\alpha = \mathcal{B}_H^\beta$ inside $S(t_H)_\Theta$? We do not know the answer to this question in general. Nevertheless, there is a positive answer in the following case.

Proposition 5.4. *Suppose that $\alpha, \beta : H \times H \to k$ are normalized convolution invertible two-cocycles such that there is a convolution invertible linear form $\lambda : H \to k$ verifying the following two conditions for all $x, y \in H$:*

$$\beta(x, y) = \sum_{(x),(y)} \lambda(x_{(1)})\lambda(y_{(1)})\alpha(x_{(2)}, y_{(2)})\lambda^{-1}(x_{(3)}y_{(3)}), \tag{5.7}$$

where λ^{-1} is the convolution inverse of λ, and

$$\sum_{(x)} \lambda(x_{(1)})x_{(2)} = \sum_{(x)} x_{(1)}\lambda(x_{(2)}). \tag{5.8}$$

Then $\mathcal{B}_H^\alpha = \mathcal{B}_H^\beta$.

Condition (5.7) states that α and β are cohomologous two-cocyles. Condition (5.8) states that the linear form λ is *lazy* in the sense of [8, §1]; this condition is automatically satisfied when H is cocommutative.

Proof. Let σ and $\sigma' : H \times H \to S(t_H)_\Theta$ be the universal cocycles attached to α and β, respectively. By definition (see (5.4)),

$$\sigma'(x, y) = \sum_{(x),(y)} t_{x_{(1)}} t_{y_{(1)}} \beta(x_{(2)}, y_{(2)}) t^{-1}_{x_{(3)} y_{(3)}}.$$

Let us express $\sigma'(x, y)$ in terms of the elements $\sigma(x, y)$ given by (5.4). For all $x, y \in H$, we have

$$\begin{aligned}
\sigma'(x, y) &= \sum_{(x),(y)} t_{x_{(1)}} t_{y_{(1)}} \beta(x_{(2)}, y_{(2)}) t^{-1}_{x_{(3)} y_{(3)}} \\
&= \sum_{(x),(y)} t_{x_{(1)}} t_{y_{(1)}} \lambda(x_{(2)}) \lambda(y_{(3)}) \alpha(x_{(3)}, y_{(3)}) \lambda^{-1}(x_{(4)} y_{(4)}) t^{-1}_{x_{(5)} y_{(5)}} \\
&= \sum_{(x),(y)} \lambda(x_{(1)}) \lambda(y_{(1)}) t_{x_{(2)}} t_{y_{(2)}} \alpha(x_{(3)}, y_{(3)}) t^{-1}_{x_{(4)} y_{(4)}} \lambda^{-1}(x_{(5)} y_{(5)}) \\
&= \sum_{(x),(y)} \lambda(x_{(1)}) \lambda(y_{(1)}) \sigma(x_{(2)}, y_{(2)}) \lambda^{-1}(x_{(3)} y_{(3)}).
\end{aligned}$$

The third equality follows from condition (5.8) on λ and from the same condition on λ^{-1} (the latter is an immediate consequence of the former). Denoting σ'^{-1} the convolution inverse of σ', we similarly obtain

$$\sigma'^{-1}(x, y) = \sum_{(x),(y)} \lambda^{-1}(x_{(1)}) \lambda^{-1}(y_{(1)}) \sigma^{-1}(x_{(2)}, y_{(2)}) \lambda(x_{(3)} y_{(3)})$$

for all $x, y \in H$. Thus, the generators $\sigma'(x, y)$ and $\sigma'^{-1}(x, y)$ of $\mathcal{B}_H^\beta$ are linear combinations of the generators of $\mathcal{B}_H^\alpha$. It follows that $\mathcal{B}_H^\beta \subset \mathcal{B}_H^\alpha$. Exchanging the roles of α and β, we obtain the reverse inclusion. $\square$

Remarks 5.5. (i) The transcendence degree of the field of fractions of $\mathcal{B}_H^\alpha$ over k cannot exceed the dimension of H over k.

(ii) In the case where H is finite-dimensional, one can obtain a presentation of $\mathcal{B}_H^\alpha$ by generators and relations using standard monomial order techniques of commutative algebra (see, e.g., [17, Chapter 15]). In Section 10 we shall give a presentation of $\mathcal{B}_H^\alpha$ for Sweedler's Hopf algebra.

5.4. The integrality of $S(t_H)_\Theta$ over $\mathcal{B}_H^\alpha$

We end this section by asking the following question: under which conditions on the Hopf algebra H and on the two-cocycle α is $S(t_H)_\Theta$ integral over $\mathcal{B}_H^\alpha$? If the answer to this question

is positive, then the field of fractions K_H of $S(t_H)_\Theta$ is an algebraic extension of the field of fractions $\mathcal{K}_H^\alpha$ of $\mathcal{B}_H^\alpha$. The integrality of $S(t_H)_\Theta$ over $\mathcal{B}_H^\alpha$ will be needed in Theorem 7.2 below.

We do not know how to answer this question in general. We do not even have an example where $S(t_H)_\Theta$ is not integral over $\mathcal{B}_H^\alpha$. The question has a positive answer in the following case.

Proposition 5.6. *If $H = k[G]$ is the Hopf algebra of a finite group G, then $S(t_H)_\Theta$ is integral over $\mathcal{B}_H^\alpha$.*

Proof. It follows from the definition of the comultiplication in $k[G]$ and the definitions of t^{-1} and of σ that

$$\sigma(g, h) = \alpha(g, h) \frac{t_g t_h}{t_{gh}} \in \mathcal{B}_H^\alpha \tag{5.9}$$

for all $g, h \in G$. Since $\alpha(g, h)$ is nonzero for all $g, h \in G$, the fraction $t_g t_h / t_{gh}$ belongs to $\mathcal{B}_H^\alpha$. By an easy induction, for each grouplike element g and each integer $k \geqslant 1$, we conclude that t_g^k / t_{g^k} belongs to $\mathcal{B}_H^\alpha$. Since by Lagrange's theorem there is N such that $g^N = 1$, it follows that t_g^N / t_1 belongs to $\mathcal{B}_H^\alpha$. Now, $t_1 = \sigma(1, 1) \in \mathcal{B}_H^\alpha$ by Lemma 5.1. Therefore, $t_g^N \in \mathcal{B}_H^\alpha$. Since by Example B.4 of Appendix B, the elements $t_g^{\pm 1}$ ($g \in G$) generate $S(t_H)_\Theta$, the conclusion follows. $\square$

6. The universal twisted algebra

We have observed in Section 5.3 that the universal cocycle σ takes values in $\mathcal{B}_H^\alpha$. Therefore we may apply the construction of Section 5.1 and consider the twisted algebra

$$\mathcal{A}_H^\alpha = \mathcal{B}_H^\alpha \otimes^\sigma H. \tag{6.1}$$

The product $\cdot_\sigma$ of $\mathcal{A}_H^\alpha$ is given for all $b, c \in \mathcal{B}_H^\alpha$ and $x, y \in H$ by

$$(b \otimes x) \cdot_\sigma (c \otimes y) = \sum_{(x),(y)} bc\,\sigma(x_{(1)}, y_{(1)}) \otimes x_{(2)} y_{(2)}. \tag{6.2}$$

Lemma 6.1. *The twisted algebra $\mathcal{A}_H^\alpha$ is an associative unital algebra with unit $t_1^{-1} = 1/t_1$.*

Proof. The associativity follows from the fact that σ is a two-cocycle. The fact that t_1^{-1} is the unit follows from (6.2) and Lemma 5.1. $\square$

We call $\mathcal{A}_H^\alpha$ the *universal twisted algebra* attached to the two-cocycle α. Note that the subalgebra of coinvariants of $\mathcal{A}_H^\alpha$ is equal to $\mathcal{B}_H^\alpha \otimes k1$; this subalgebra is central in $\mathcal{A}_H^\alpha$.

Consider the restriction $\epsilon : \mathcal{B}_H^\alpha \to k$ of the algebra morphism $\epsilon : S(t_H)_\Theta \to k$ of Lemma 5.2 and the maximal ideal

$$\mathfrak{m}_\epsilon = \mathrm{Ker}\big(\epsilon : \mathcal{B}_H^\alpha \to k\big).$$

We can recover the original twisted algebra $^\alpha H$ from $\mathcal{A}_H^\alpha$ as follows.

Proposition 6.2. *There is an isomorphism of H-comodule algebras*

$$\mathcal{A}_H^\alpha / \mathfrak{m}_\epsilon \mathcal{A}_H^\alpha \cong {}^\alpha H.$$

Proof. We have the obvious isomorphisms of H-comodules

$$\mathcal{A}_H^\alpha / \mathfrak{m}_\epsilon \mathcal{A}_H^\alpha \cong \left(\mathcal{B}_H^\alpha / \mathfrak{m}_\epsilon \right) \otimes H \cong H.$$

The fact that the product $\cdot_\sigma$ on $\mathcal{A}_H^\alpha$ given by (6.2) turns into the product $\cdot_\alpha$ on H given by (5.1) follows from the equality $\epsilon \circ \sigma = \alpha$ of Lemma 5.2. $\quad\square$

We record the following structure theorem for $\mathcal{A}_H^\alpha$. It involves the field of fractions $\mathcal{K}_H^\alpha$ of $\mathcal{B}_H^\alpha$.

Theorem 6.3. *Let k be a field of characteristic zero and H a finite-dimensional Hopf algebra. If the algebra ${}^\alpha H$ is semisimple, then so is $\mathcal{K}_H^\alpha \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$.*

By [26, Th. 7.4.2], if H is a finite-dimensional semisimple algebra, then ${}^\alpha H$ is semisimple for any two-cocycle α. Note that there are non-semisimple Hopf algebras with two-cocycles α such that ${}^\alpha H$ is (semi)simple, e.g., Sweedler's algebra detailed in Section 10.

Proof. Let A be a B-algebra, where B is a commutative algebra. Suppose that A is free of finite rank as a B-module with basis $\{a_i\}_{i=1,\ldots,d}$. The trace form of A is the B-bilinear form defined for all $a, a' \in A$ by

$$\langle a, a' \rangle_A = \mathrm{Tr}(R_{aa'}),$$

where $R_{aa'}$ is the right multiplication by aa' and $\mathrm{Tr} : \mathrm{End}_B(A) \to B$ is the trace map. It is well known that, if B is a field, then A is semisimple if and only if the trace form $\langle\,,\,\rangle_A$ is nondegenerate. The latter is equivalent to the nonvanishing of the determinant $\det((\langle a_i, a_j \rangle_A)_{i,j=1,\ldots,d})$.

Let $f : B \to B'$ be a morphism of commutative algebras. We denote by the same letter f the induced algebra morphism $A \to A' = B' \otimes_B A$. The trace forms of A and A' are related by the formula

$$\left\langle f(a), f(a') \right\rangle_{A'} = f\left(\langle a, a' \rangle_A \right)$$

for all $a, a' \in A$.

We apply this to the case where $B = \mathcal{B}_H^\alpha$ and $A = \mathcal{A}_H^\alpha$. If $\{h_i\}_{i=1,\ldots,d}$ is a basis of H, then it is also a basis of $\mathcal{A}_H^\alpha$ considered as a $\mathcal{B}_H^\alpha$-module. Set

$$D = \det\left(\left(\langle h_i, h_j \rangle_{\mathcal{A}_H^\alpha} \right)_{i,j=1,\ldots,d} \right) \in \mathcal{B}_H^\alpha.$$

By the remarks above, the trace form of $\mathcal{K}_H^\alpha \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$ is the image of the trace form of $\mathcal{A}_H^\alpha$ in $\mathcal{K}_H^\alpha$. Therefore, $\mathcal{K}_H^\alpha \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$ is semisimple if $D \neq 0$ in $\mathcal{K}_H^\alpha$, hence in $\mathcal{B}_H^\alpha$. In order to prove the nonvanishing of D, it is enough to map it to a nonzero element. Consider the augmentation map $\epsilon : \mathcal{B}_H^\alpha \to k$ of Section 5.3. By Proposition 6.2,

$$k \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha = \mathcal{A}_H^\alpha / \mathfrak{m}_\epsilon \mathcal{A}_H^\alpha \cong {}^\alpha H.$$

The trace form of $^\alpha H$ is the image under ϵ of the trace form of $\mathcal{A}_H^\alpha$. Now,

$$\epsilon(D) = \det\big(\big(\langle h_i, h_j\rangle_{^\alpha H}\big)_{i,j=1,\ldots,d}\big) \in k$$

is nonzero since $^\alpha H$ is semisimple. Therefore, $D \neq 0$. $\square$

Below we shall show under the same hypotheses as in Theorem 6.3 that $\mathcal{K}_H^\alpha \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$ is a simple algebra if $^\alpha H$ is simple. Let us first establish the following result.

Lemma 6.4. *If the center of the algebra $^\alpha H$ is trivial, that is, consists of the scalar multiples of the unit, then the center of $\mathcal{A}_H^\alpha$ is equal to $\mathcal{B}_H^\alpha \otimes 1$.*

Proof. Since the universal cocycle σ takes values in $S(t_H)_\Theta$ (see Section 5.3), we may apply the construction of Section 5.1 and consider the twisted algebra

$$S(t_H)_\Theta \otimes {}^\sigma H, \tag{6.3}$$

which is clearly isomorphic to the algebra $S(t_H)_\Theta \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$. It follows from Lemma 6.1 that $S(t_H)_\Theta \otimes {}^\sigma H$ is an associative algebra with unit $t_1^{-1} = 1/t_1$.

Eq. (5.4) makes sense in $S(t_H)_\Theta$. It can be interpreted as saying that the two-cocycles σ and α are cohomologous in $S(t_H)_\Theta$. By [9] and [14] (see also [26, Section 7]), the linear map

$$\varphi : S(t_H)_\Theta \otimes {}^\sigma H \to S(t_H)_\Theta \otimes {}^\alpha H$$

given for all $b \in S(t_H)_\Theta$ and $h \in H$ by

$$\varphi(b \otimes h) = \sum_{(h)} b t_{h_{(1)}} \otimes h_{(2)} \tag{6.4}$$

is a $S(t_H)_\Theta$-linear isomorphism of H-comodule algebras. Note that the inverse map

$$\varphi^{-1} : S(t_H)_\Theta \otimes {}^\alpha H \to S(t_H)_\Theta \otimes {}^\sigma H$$

is given for all $b \in S(t_H)_\Theta$ and $h \in H$ by

$$\varphi^{-1}(b \otimes h) = \sum_{(h)} b t_{h_{(1)}}^{-1} \otimes h_{(2)}. \tag{6.5}$$

The center of $S(t_H)_\Theta \otimes {}^\sigma H$ is isomorphic to the center of $S(t_H)_\Theta \otimes {}^\alpha H$ under φ. Now it is easy to check that, under the hypothesis of the lemma, the center of $S(t_H)_\Theta \otimes {}^\alpha H$ is the subalgebra $S(t_H)_\Theta \otimes 1$. The isomorphism φ being $S(t_H)_\Theta$-linear and sending the unit $t_1^{-1} \otimes 1 \in S(t_H)_\Theta \otimes {}^\sigma H$ to the unit $1 \otimes 1 \in S(t_H)_\Theta \otimes {}^\alpha H$, the center of $S(t_H)_\Theta \otimes {}^\sigma H$ is equal to

$$t_1^{-1} S(t_H)_\Theta \otimes 1 = S(t_H)_\Theta \otimes 1.$$

Now, let $\omega \in \mathcal{A}_H^\alpha = \mathcal{B}_H^\alpha \otimes {}^\sigma H$ be a central element. It remains central when considered in $S(t_H)_\Theta \otimes {}^\sigma H$. By the above argument, we have $\omega = b \otimes 1$ for some $b \in S(t_H)_\Theta$. Since $\mathcal{B}_H^\alpha \subset S(t_H)_\Theta$, it follows that $b \in \mathcal{B}_H^\alpha$. $\square$

Corollary 6.5. *Let k be a field of characteristic zero and H a finite-dimensional Hopf algebra. If the algebra $^\alpha H$ is simple, then so is the algebra $\mathcal{K}_H^\alpha \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$.*

Proof. By Theorem 6.3, $\mathcal{K}_H^\alpha \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$ is semisimple. By Lemma 6.4, the center of $\mathcal{A}_H^\alpha$ is $\mathcal{B}_H^\alpha \otimes 1$. Therefore, the center of $K_H \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$ is $K_H \otimes 1$. The conclusion follows immediately. $\quad\square$

Under the hypotheses of Corollary 6.5, we may wonder what the *index* of the central simple algebra $\mathcal{K}_H^\alpha \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$ is and how it depends on the two-cocycle α. Such questions are open even in the case of group algebras; for partial answers, see [3, §3].

7. Forms of $^\alpha H$

As in Section 5, we fix a Hopf algebra H and a normalized convolution invertible two-cocycle $\alpha : H \times H \to k$. We consider the corresponding twisted algebra $^\alpha H$.

Given a normalized convolution invertible two-cocycle $\beta : H \times H \to K$ with values in a (field) extension K of k, we say that the twisted H-comodule algebra $K \otimes {}^\beta H$ is a *K-form* of $^\alpha H$ if there is an extension L of K and an L-linear isomorphism of H-comodule algebras

$$L \otimes_K \left(K \otimes {}^\beta H \right) \cong L \otimes_k {}^\alpha H.$$

We now state two theorems relating forms of $^\alpha H$ to central specializations of the universal twisted algebra $\mathcal{A}_H^\alpha$ introduced in the previous section.

Theorem 7.1. *For any K-form $K \otimes {}^\beta H$ of $^\alpha H$, where $\beta : H \times H \to K$ is a normalized convolution invertible two-cocycle with values in an extension K of k, there are an algebra morphism $\lambda : \mathcal{B}_H^\alpha \to K$ and a K-linear isomorphism of H-comodule algebras*

$$K \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha \cong K \otimes {}^\beta H.$$

The algebra morphism $\lambda : \mathcal{B}_H^\alpha \to K$ equips K with a natural $\mathcal{B}_H^\alpha$-module structure so that it makes sense to consider the algebra $K \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$ as in the previous statement.

Proof. Since there is an L-linear isomorphism of H-comodule algebras

$$L \otimes_K \left(K \otimes {}^\beta H \right) \cong L \otimes_k {}^\alpha H$$

for some extension L of K, by [9,16] there is a convolution invertible linear map $\lambda_0 : H \to L$ such that for all $g, h \in H$,

$$\beta(x, y) = \sum_{(x),(y)} \lambda_0(x_{(1)}) \lambda_0(y_{(1)}) \alpha(x_{(2)}, y_{(2)}) \lambda_0^{-1}(x_{(3)} y_{(3)}), \qquad (7.1)$$

where λ_0^{-1} is the convolution inverse of λ_0. Define an algebra morphism $\lambda : S(t_H) \to L$ by $\lambda(t_h) = \lambda_0(h)$ for all $h \in H$. This morphism extends to an algebra morphism, which we still denote by λ, from $S(t_H)_\Theta$ to L such that $\lambda(t_h^{-1}) = \lambda_0^{-1}(h)$ for all $h \in H$. We obtain an algebra morphism $\lambda : \mathcal{B}_H^\alpha \to L$ by restricting λ to the subalgebra $\mathcal{B}_H^\alpha$.

It follows from (5.4), (5.5), and (7.1) that λ sends the generators $\sigma(x, y)$ and $\sigma^{-1}(x, y)$ of $\mathcal{B}_H^\alpha$ respectively to $\beta(x, y)$ and to $\beta^{-1}(x, y)$, where β^{-1} is the convolution inverse of β. Since $\beta(x, y)$ and $\beta^{-1}(x, y)$ belong to K, we conclude that $\lambda(\mathcal{B}_H^\alpha) \subset K$. Thus, $\lambda \otimes \mathrm{id}_H$ defines a map
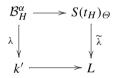
$$\mathcal{A}_H^\alpha = \mathcal{B}_H^\alpha \otimes {}^\sigma H \to K \otimes {}^\beta H.$$

Since $\lambda \circ \sigma = \beta$, this map is an algebra morphism; it is also a comodule map. Therefore, the algebra morphism $\lambda : \mathcal{B}_H^\alpha \to K$ induces an H-comodule algebra isomorphism $K \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha \cong K \otimes {}^\beta H$. $\square$

Under the integrality condition of Section 5.4, Theorem 7.1 has the following converse.

Theorem 7.2. *If the algebra $S(t_H)_\Theta$ is integral over the subalgebra $\mathcal{B}_H^\alpha$, then for any extension K of k and any algebra morphism $\lambda : \mathcal{B}_H^\alpha \to K$, the H-comodule K-algebra $K \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$ is a K-form of ${}^\alpha H$.*

Proof. Let $\mathfrak{m}$ be the kernel of an algebra morphism $\lambda : \mathcal{B}_H^\alpha \to K$; it is a maximal ideal of $\mathcal{B}_H^\alpha$, and the image $k' = \mathcal{B}_H^\alpha/\mathfrak{m}$ of λ is an extension of k contained in K. It follows from the integrality assumption that there is a maximal ideal $\mathfrak{M}$ of $S(t_H)_\Theta$ such that $\mathfrak{m} S(t_H)_\Theta \subset \mathfrak{M}$ (see e.g., [24, Chapter IX, Proposition 9]). Let $\widetilde{\lambda}$ be the projection $S(t_H)_\Theta \to L = S(t_H)_\Theta/\mathfrak{M}$. We have the following commutative square in which the top horizontal map is the inclusion $\mathcal{B}_H^\alpha \hookrightarrow S(t_H)_\Theta$:

$$
\begin{array}{ccc}
\mathcal{B}_H^\alpha & \longrightarrow & S(t_H)_\Theta \\
\lambda \downarrow & & \downarrow \widetilde{\lambda} \\
k' & \longrightarrow & L
\end{array}
$$

The bottom horizontal map $k' \to L$ in the previous square is injective since k' and L are fields.

Applying $\widetilde{\lambda}$ to (5.4) and setting $\beta = \widetilde{\lambda} \circ \sigma : H \times H \to L$, we obtain

$$\beta(g, h) = \widetilde{\lambda}\big(\sigma(g, h)\big)$$
$$= \sum_{(g),(h)} \widetilde{\lambda}(t_{g_{(1)}}) \widetilde{\lambda}(t_{h_{(1)}}) \widetilde{\lambda}\big(\alpha(g_{(2)}, h_{(2)})\big) \widetilde{\lambda}\big(t^{-1}_{g_{(3)}h_{(3)}}\big)$$

for all $g, h \in H$. Since $\widetilde{\lambda}$ is an algebra morphism, $\widetilde{\lambda} \circ t^{-1}$ is the convolution inverse of $\widetilde{\lambda} \circ t$. It follows that β is an invertible two-cocycle, which is cohomologous to $\widetilde{\lambda} \circ \alpha = \alpha$. Therefore,

$$L \otimes {}^\alpha H \cong L \otimes {}^\beta H = L \otimes_K \big(K \otimes_{\mathcal{B}_H^\alpha} (\mathcal{B}_H^\alpha \otimes {}^\sigma H)\big) = L \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha,$$

which shows that $K \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$ is a K-form of ${}^\alpha H$. $\square$

Corollary 7.3. *If $S(t_H)_\Theta$ is integral over $\mathcal{B}_H^\alpha$, then for any extension K of k there is a surjective map*

$$\mathrm{Alg}\big(\mathcal{B}_H^\alpha, K\big) \to \mathrm{Forms}_K\big({}^\alpha H\big)$$

from the set of algebra morphisms $\mathcal{B}_H^\alpha \to K$ to the set of isomorphism classes of K-forms of ${}^\alpha H$.

Proof. For any algebra morphism $\lambda : \mathcal{B}_H^\alpha \to K$, the H-comodule K-algebra

$$\lambda_* \left(\mathcal{B}_H^\alpha \right) = K \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$$

is a K-form of $^\alpha H$ by Theorem 7.2. By Theorem 7.1, the map $\lambda \mapsto \lambda_*(\mathcal{B}_H^\alpha)$ maps $\mathrm{Alg}(\mathcal{B}_H^\alpha, K)$ surjectively onto the set $\mathrm{Forms}_K(^\alpha H)$. $\square$

This corollary allows us to consider the cleft H-Galois extension $\mathcal{B}_H^\alpha \subset \mathcal{A}_H^\alpha$ as a *versal deformation space* for the forms of $^\alpha H$ (this space is not universal since the algebra morphism $\lambda : \mathcal{B}_H^\alpha \to K$ in Theorem 7.1 is not necessarily unique). As a further consequence, if $\mathcal{B}_H^\alpha \cong k[u_1, \dots, u_r]/(P_1, \dots, P_s)$, then the polynomials $P_1, \dots, P_s \in k[u_1, \dots, u_r]$ are invariants of forms of $^\alpha H$.

In case the twisted algebra $^\alpha H$ is simple, we have another important consequence of Theorem 7.2.

Theorem 7.4. *If $S(t_H)_\Theta$ is integral over $\mathcal{B}_H^\alpha$ and if the algebra $^\alpha H$ is simple, then $\mathcal{A}_H^\alpha$ is an Azumaya algebra.*

Proof. By Lemma 6.4, $\mathcal{B}_H^\alpha$ is the center of $\mathcal{A}_H^\alpha$. By [5, Th. 15], in order to prove the theorem, it suffices to check that $\mathcal{A}_H^\alpha / \mathfrak{m} \mathcal{A}_H^\alpha$ is simple for any maximal ideal $\mathfrak{m}$ of $\mathcal{B}_H^\alpha$. Now, by Theorem 7.2,

$$\mathcal{A}_H^\alpha / \mathfrak{m} \mathcal{A}_H^\alpha = \left(\mathcal{B}_H^\alpha / \mathfrak{m} \right) \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$$

is a $\mathcal{B}_H^\alpha / \mathfrak{m}$-form of the simple algebra $^\alpha H$. Therefore, it is simple. $\square$

8. Relating the universal comodule algebra to the universal twisted algebra

From now on we assume that the ground field is infinite. As in Section 5, we fix a Hopf algebra H and a normalized convolution invertible two-cocycle $\alpha : H \times H \to k$, and we consider the corresponding twisted algebra $^\alpha H$. Since $^\alpha H$ satisfies the assumptions of Section 4.1 with $B = k$ and $u = \mathrm{id}_H$, we may apply the constructions and the results of Section 4 to $^\alpha H$.

Let $\mu_\alpha : T(X_H) \to S(t_H) \otimes {}^\alpha H$ be the universal evaluation morphism defined by (4.1). In the present situation,

$$\mu_\alpha(X_h) = \sum_{(h)} t_{h_{(1)}} \otimes h_{(2)} \tag{8.1}$$

for all $h \in H$. For short, we denote $\mathcal{U}_H(^\alpha H)$ by $\mathcal{U}_H^\alpha$. By Corollary 4.4, μ_α induces an injective H-comodule algebra morphism

$$\mathcal{U}_H^\alpha \hookrightarrow S(t_H) \otimes {}^\alpha H,$$

which we still denote μ_α. It follows from Proposition 4.7 that $P \in \mathcal{U}_H^\alpha$ is coinvariant if and only if $\mu_\alpha(P) \in S(t_H) \otimes 1$.

Recall the H-comodule algebra isomorphism

$$\varphi^{-1} : S(t_H)_\Theta \otimes {}^\alpha H \to S(t_H)_\Theta \otimes {}^\sigma H$$

given by (6.5). Define $\mu_\sigma : \mathcal{U}_H^\alpha \to S(t_H)_\Theta \otimes {}^\sigma H$ by

$$\mu_\sigma = \varphi^{-1} \circ \mu_\alpha, \tag{8.2}$$

where we now view μ_α as a morphism with values in $S(t_H)_\Theta \otimes {}^\alpha H$. It follows from the definition and from Corollary 4.4 that μ_σ is an injective H-comodule algebra morphism.

Lemma 8.1.

(a) *We have* $\mu_\sigma(X_h) = 1 \otimes h$ *for all* $h \in H$.
(b) *The morphism* μ_σ *maps* $\mathcal{U}_H^\alpha$ *into* $\mathcal{A}_H^\alpha$.

Proof. (a) It suffices to check that $\varphi(1 \otimes h) = \mu_\alpha(X_h)$ for all $h \in H$. These equalities follow from (6.4) and (8.1).

(b) By part (a) and by (6.2),

$$\mu_\sigma(X_g X_h) = \sum_{(g),(h)} \sigma(g_{(1)}, h_{(1)}) \otimes g_{(2)} h_{(2)}$$

for all $g, h \in H$. Using (6.2) repeatedly, we see that μ_σ sends all monomials in the variables X_h, hence all elements of $\mathcal{U}_H^\alpha$, to $\mathcal{B}_H^\alpha \otimes {}^\sigma H = \mathcal{A}_H^\alpha$. $\quad\square$

Let $Z({}^\alpha H)$ be the center of ${}^\alpha H$. We characterize the center $\mathcal{Z}_H^\alpha$ of $\mathcal{U}_H^\alpha$ as follows.

Proposition 8.2. *An element* $w \in \mathcal{U}_H^\alpha$ *belongs to the center* $\mathcal{Z}_H^\alpha$ *if and only if* $\mu_\alpha(w)$ *belongs to* $S(t_H) \otimes Z({}^\alpha H)$.

Proof. Suppose first that $\mu_\alpha(w) \in S(t_H) \otimes Z({}^\alpha H)$. The elements of the latter are clearly central in $S(t_H) \otimes {}^\alpha H$. Therefore, $\mu_\alpha(w)$ commutes with all elements $\mu_\alpha(X_h)$, where h runs over H. Since μ_α is an injective algebra morphism, w commutes with all generators X_h of $\mathcal{U}_H^\alpha$. Hence, $\mu_\alpha(w) \in \mathcal{Z}_H^\alpha$.

Conversely, any element $w \in \mathcal{Z}_H^\alpha$ commutes with all generators X_h of $\mathcal{U}_H^\alpha$. Therefore, by Lemma 8.1, its image $\mu_\sigma(w)$ commutes with $\mu_\sigma(X_h) = 1 \otimes h$ for all $h \in H$. Since $S(t_H)_\Theta \otimes {}^\sigma H$ is generated as an algebra by $1 \otimes H$ and the central subalgebra $S(t_H)_\Theta \otimes 1$, it follows immediately that $\mu_\sigma(w)$ commutes with all elements of $S(t_H)_\Theta \otimes {}^\sigma H$. Consequently, $\mu_\alpha(w) = \varphi(\mu_\sigma(w))$ commutes with all elements of $S(t_H)_\Theta \otimes {}^\alpha H$, i.e., belongs to the center of the latter. Write

$$\mu_\alpha(w) = \sum_{j \in J} s_j \otimes h_j,$$

where $\{s_j\}_{j \in J}$ is a basis of $S(t_H)_\Theta$ over the ground field and $h_j \in H$ for all $j \in J$. Using brackets to denote commutators, we obtain

$$\sum_{j \in J} s_j \otimes [h_j, h] = \left[\sum_{j \in J} s_j \otimes h_j, h \right] = \left[\mu_\alpha(w), 1 \otimes h \right]$$

for all $h \in H$. This implies that $[h_j, h] = 0$ for all $h \in H$ and $j \in J$. Therefore, $h_j \in Z(^\alpha H)$ for all $j \in J$, from which it follows that $\mu_\alpha(w)$ lies in $S(t_H)_\Theta \otimes Z(^\alpha H)$. Since $\mu_\alpha(w) \in S(t_H) \otimes {}^\alpha H$, it lies in $S(t_H) \otimes Z(^\alpha H)$. $\square$

Corollary 8.3. *The center $\mathcal{Z}_H^\alpha$ of $\mathcal{U}_H^\alpha$ has no zero-divisors if $Z(^\alpha H)$ has none.*

Proof. By Corollary 4.4 and Proposition 8.2, μ_α embeds $\mathcal{Z}_H^\alpha$ into the algebra $S(t_H)_\Theta \otimes Z(^\alpha H)$, which has no zero-divisors if $Z(^\alpha H)$ has none. $\square$

9. The case of nondegenerate cocycles

Let H be a Hopf algebra and $\alpha : H \times H \to k$ a normalized convolution invertible two-cocycle. Generalizing terminology used for group cocycles, we say that α is *nondegenerate* if the center of the twisted algebra $^\alpha H$ is trivial, i.e., if $Z(^\alpha H) = k1$.

Restricting to nondegenerate two-cocycles allows us to relate $\mathcal{Z}_H^\alpha$ to $\mathcal{B}_H^\alpha$ as follows.

Proposition 9.1. *If α is a nondegenerate two-cocycle, then μ_α and μ_σ map $\mathcal{Z}_H^\alpha$ into $\mathcal{B}_H^\alpha \otimes 1$.*

Proof. Since α is nondegenerate, it follows from Proposition 8.2 that μ_α maps $\mathcal{Z}_H^\alpha$ into

$$S(t_H) \otimes Z\big(^\alpha H\big) = S(t_H) \otimes 1.$$

The isomorphism

$$\varphi^{-1} : S(t_H)_\Theta \otimes {}^\alpha H \to S(t_H)_\Theta \otimes {}^\sigma H$$

being $S(t_H)_\Theta$-linear, $\mu_\sigma = \varphi^{-1} \circ \mu_\alpha$ maps $\mathcal{Z}_H^\alpha$ into

$$\varphi^{-1}\big(S(t_H) \otimes 1\big) = S(t_H)\varphi^{-1}(1 \otimes 1) = t_1^{-1} S(t_H) \otimes 1,$$

which is contained in $S(t_H)_\Theta \otimes 1$. Now, by Lemma 8.1(b), μ_σ maps $\mathcal{U}_H^\alpha$ into $\mathcal{A}_H^\alpha = \mathcal{B}_H^\alpha \otimes {}^\sigma H$. Since $\mathcal{B}_H^\alpha \subset S(t_H)_\Theta$, we conclude that

$$\mu_\sigma\big(\mathcal{Z}_H^\alpha\big) \subset \mathcal{B}_H^\alpha \otimes 1.$$

To obtain the same inclusion with μ_α in lieu of μ_σ, we apply the $S(t_H)_\Theta$-linear isomorphism φ to the previous inclusion. We obtain

$$\mu_\alpha\big(\mathcal{Z}_H^\alpha\big) \subset \mathcal{B}_H^\alpha \varphi(1 \otimes 1) = t_1 \mathcal{B}_H^\alpha \otimes 1.$$

We conclude the proof by observing that t_1 is an invertible element of $\mathcal{B}_H^\alpha$ as a consequence of Lemma 5.1 applied to $x = 1$. $\square$

If α is a nondegenerate two-cocycle, as a consequence of Proposition 9.1, we can view the center $\mathcal{Z}_H^\alpha$ of $\mathcal{U}_H^\alpha$ as a subalgebra of $\mathcal{B}_H^\alpha$, and consider the algebra

$$\mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{U}_H^\alpha. \tag{9.1}$$

Remark 9.2. In [2] nondegenerate cocycles are defined in a more restrictive way than above, namely as those cocycles α such that the algebra $^\alpha H$ is k-central simple. Both definitions coincide when H is semisimple. Furthermore, one can show that, when $H = k[G]$ is the Hopf algebra of a group, there exist nondegenerate two-cocycles only if H is semisimple.

Our next objective is to relate $\mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{U}_H^\alpha$ to the universal twisted algebra $\mathcal{A}_H^\alpha$. This will be possible under the following additional condition on the two-cocycle: a nondegenerate two-cocycle $\alpha : H \times H \to k$ is said to be *nice* if $\mathcal{B}_H^\alpha$ is a localization of $\mu_\alpha(\mathcal{Z}_H^\alpha)$ or, equivalently, if their fields of fractions coincide. If α is nice, then $\mathcal{B}_H^\alpha$ is flat as a $\mathcal{Z}_H^\alpha$-module and

$$\mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{B}_H^\alpha \cong \mathcal{B}_H^\alpha. \tag{9.2}$$

Theorem 9.3. *If $\alpha : H \times H \to k$ is a nice nondegenerate two-cocycle, then μ_σ induces an isomorphism of H-comodule algebras*

$$\mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{U}_H^\alpha \cong \mathcal{A}_H^\alpha.$$

Proof. The H-comodule algebra morphism $\mu_\sigma : \mathcal{U}_H^\alpha \to \mathcal{B}_H^\alpha \otimes {}^\sigma H$ induces an H-comodule algebra morphism

$$\mathrm{id} \otimes \mu_\sigma : \mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{U}_H^\alpha \to \mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{B}_H^\alpha \otimes {}^\sigma H.$$

Since μ_σ is injective and $\mathcal{B}_H^\alpha$ is flat as a $\mathcal{Z}_H^\alpha$-module, $\mathrm{id} \otimes \mu_\sigma$ is injective. It then follows from (9.2) that $\mathrm{id} \otimes \mu_\sigma$ maps $\mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{U}_H^\alpha$ injectively into $\mathcal{B}_H^\alpha \otimes {}^\sigma H = \mathcal{A}_H^\alpha$.

As an $\mathcal{B}_H^\alpha$-algebra, $\mathcal{A}_H^\alpha$ is generated by the elements $1 \otimes h$, where h runs over H. By Lemma 8.1(a), these elements are in the image of μ_σ. This proves the surjectivity of $\mathrm{id} \otimes \mu_\sigma$. $\square$

The following is a consequence of Lemma 6.4 and Theorem 9.3.

Corollary 9.4. *If α is a nice nondegenerate two-cocycle, then the center of $\mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{U}_H^\alpha$ is $\mathcal{B}_H^\alpha$.*

We do not know if all nondegenerate two-cocycles are nice. In the group case, we have the following result.

Proposition 9.5. *Any two-cocycle on the Hopf algebra $k[G]$ of a group is nice.*

Proof. For $g, h \in G$, set $Z_g = X_g X_{g^{-1}}$ and $Z_{g,h} = X_g X_h X_{h^{-1}g^{-1}} \in T(X_H)$, where $H = k[G]$. Let $\alpha : H \times H \to k$ be a normalized convolution invertible two-cocycle. Its restriction to $G \times G$ takes nonzero values. An easy computation shows that

$$\mu_\alpha(Z_g) = \alpha(g, g^{-1}) t_g t_{g^{-1}} \otimes 1 \tag{9.3}$$

and

$$\mu_\alpha(Z_{g,h}) = \alpha(g, g^{-1}) \alpha(h, (gh)^{-1}) t_g t_h t_{(gh)^{-1}} \otimes 1. \tag{9.4}$$

It follows from these identities and from Proposition 8.2 that Z_g and $Z_{g,h}$ represent elements of the center $\mathcal{Z}_H^\alpha$. As an easy consequence of (5.9), (9.3), and (9.4), we obtain the following identities for all $g, h \in G$:

$$\sigma(g, h) = \frac{\alpha(g, h)\alpha(gh, (gh)^{-1})}{\alpha(g, g^{-1})\alpha(h, (gh)^{-1})} \cdot \frac{\mu_\alpha(Z_{g,h})}{\mu_\alpha(Z_{gh})}.$$

This shows that the generators $\sigma(g, h)$ and $\sigma^{-1}(g, h) = 1/\sigma(g, h)$ of $\mathcal{B}_H^\alpha$ can be expressed as fractions of elements of $\mu_\alpha(\mathcal{Z}_H^\alpha)$. □

10. The Sweedler algebra

We assume in this section that the characteristic of the ground field k is different from 2.

10.1. Definition

The *Sweedler algebra* H_4 is the algebra generated by two elements x, y subject to the relations

$$x^2 = 1, \qquad y^2 = 0, \qquad xy + yx = 0. \tag{10.1}$$

It is four-dimensional with basis $\{1, x, y, z\}$, where $z = xy$.

It carries a structure of Hopf algebra with comultiplication Δ, counit ε, and antipode S given by

$$\Delta(1) = 1 \otimes 1, \qquad \Delta(x) = x \otimes x,$$
$$\Delta(y) = 1 \otimes y + y \otimes x, \qquad \Delta(z) = x \otimes z + z \otimes 1,$$
$$\varepsilon(1) = \varepsilon(x) = 1, \qquad \varepsilon(y) = \varepsilon(z) = 0,$$
$$S(1) = 1, \qquad S(x) = x, \qquad S(y) = z, \qquad S(z) = -y. \tag{10.2}$$

The tensor algebra $T(X_{H_4})$ is freely generated by the four elements

$$X_1 = E, \qquad X_x = X, \qquad X_y = Y, \qquad X_z = Z,$$

whereas $S(t_{H_4})$ is the polynomial algebra over the elements t_1, t_x, t_y, t_z.

By (A.6) and (A.7), the elements $t_1^{-1}, t_x^{-1}, t_y^{-1}, t_z^{-1}$ of the field of fractions K_{H_4} of $S(t_{H_4})$ are given by

$$t_1^{-1} = \frac{1}{t_1}, \qquad t_x^{-1} = \frac{1}{t_x}, \qquad t_y^{-1} = -\frac{t_y}{t_1 t_x}, \qquad t_z^{-1} = -\frac{t_z}{t_1 t_x}.$$

It follows that the algebra $S(t_{H_4})_\Theta$ is isomorphic to the algebra of Laurent polynomials

$$S(t_{H_4})_\Theta \cong k\big[t_1, t_1^{-1}, t_x, t_x^{-1}, t_y, t_z\big].$$

(This also follows from Proposition B.2.)

10.2. Twisted algebras

Masuoka [25] classified all Hopf Galois extensions for H_4 (we also follow [16]). In particular, he showed that any H_4-Galois extension of k is, up to isomorphism, of the form $^\alpha H_4$, where α is the normalized convolution invertible two-cocycle given by

$$\alpha(x, x) = a, \qquad \alpha(x, y) = 0, \qquad \alpha(x, z) = 0,$$
$$\alpha(y, x) = b, \qquad \alpha(y, y) = c, \qquad \alpha(y, z) = -c,$$
$$\alpha(z, x) = b, \qquad \alpha(z, y) = c, \qquad \alpha(z, z) = -ac \qquad (10.3)$$

for some $a, b, c \in k$ with $a \neq 0$. The four-dimensional algebra $^\alpha H_4$ has the same basis as H_4 and 1 is its unit. In this basis the multiplication $\cdot_\alpha$ of $^\alpha H_4$ is given by

$$x \cdot_\alpha x = a, \qquad x \cdot_\alpha y = z, \qquad x \cdot_\alpha z = ay,$$
$$y \cdot_\alpha x = b - z, \qquad y \cdot_\alpha y = c, \qquad y \cdot_\alpha z = -cx + by,$$
$$z \cdot_\alpha x = bx - ay, \qquad z \cdot_\alpha y = cx, \qquad z \cdot_\alpha z = -ac + bz. \qquad (10.4)$$

To indicate the dependence on the scalars a, b, c, we henceforth denote $^\alpha H_4$ by $A_{a,b,c}$.

It is easy to check that the center of $A_{a,b,c}$ is trivial, i.e., consists of the scalar multiples of the unit. Therefore, the two-cocycle α is nondegenerate in the sense of Section 9 for all values of a, b, c.

It follows from [16, Cor. 2.8] that $A_{a,b,c} \cong H$ when $b^2 - 4ac = 0$ and that $A_{a,b,c}$ is isomorphic to a quaternion algebra when $b^2 - 4ac \neq 0$. Therefore, $A_{a,b,c}$ is a simple algebra when $b^2 - 4ac \neq 0$. If $b^2 - 4ac = 0$, then $A_{a,b,c}$ is not simple since H contains a nonzero nilpotent two-sided ideal, namely the one generated by y.

10.3. The commutative algebra $\mathcal{B}_{H_4}^\alpha$

Let us first compute the values of the universal cocycle σ attached to α, using (5.4).

Lemma 10.1.

(a) *We have*

$$\sigma(x, x) = at_x^2 t_1^{-1},$$
$$\sigma(y, y) = \sigma(z, y) = -\sigma(y, z) = \left(at_y^2 + bt_1 t_y + ct_1^2\right) t_1^{-1},$$
$$\sigma(x, y) = -\sigma(x, z) = (at_x t_y - t_1 t_z) t_1^{-1},$$
$$\sigma(y, x) = \sigma(z, x) = (bt_1 t_x + at_x t_y + t_1 t_z) t_1^{-1},$$
$$\sigma(z, z) = -\left(t_z^2 + bt_x t_z + act_x^2\right) t_1^{-1},$$

(b) *The values of σ^{-1} on the basis elements of H_4 are linear combinations of the values of σ on the same elements, possibly divided by positive powers of t_1 and of $\sigma(x, x) = at_x^2 t_1^{-1}$.*

Proof. To compute the values of σ and of σ^{-1}, we use (5.4), (5.5), and the following values of $\Delta^{(2)} = (\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta\otimes)\Delta$:

$$\Delta^{(2)}(1) = 1 \otimes 1 \otimes 1,$$
$$\Delta^{(2)}(x) = x \otimes x \otimes x,$$
$$\Delta^{(2)}(y) = 1 \otimes 1 \otimes y + 1 \otimes y \otimes x + y \otimes x \otimes x,$$
$$\Delta^{(2)}(z) = x \otimes x \otimes z + x \otimes z \otimes 1 + z \otimes 1 \otimes 1.$$

The rest of the computation is tedious and left to the reader.   □

The algebra $\mathcal{B}^\alpha_{H_4}$ is the subalgebra of $S(t_{H_4})_\Theta$ generated by the above values of σ and σ^{-1}. In order to determine it in terms of the generators of $T(X_{H_4})$ and to obtain a presentation by generators and relations, we compute certain values of the universal evaluation morphism

$$\mu_\alpha : T(X_{H_4}) \to S(t_{H_4}) \otimes{}^\alpha H_4.$$

By (8.1) and (10.2), μ_α is given on the generators E, X, Y, Z by

$$\mu_\alpha(E) = t_1, \qquad \mu_\alpha(X) = t_x x, \tag{10.5}$$
$$\mu_\alpha(Y) = t_y x + t_1 y, \qquad \mu_\alpha(Z) = t_z + t_x z. \tag{10.6}$$

(In the previous formulas we have left tensor product signs and the unit of H out.)

Set $T = XY + YX$, $U = X(XZ + ZX)$, $V = (XZ)^2$. Using (10.4)–(10.6), we obtain the following.

Lemma 10.2. *In the algebra $S(t_{H_4}) \otimes{}^\alpha H_4$ we have the following equalities:*

$$\mu_\alpha(X^2) = at_x^2,$$
$$\mu_\alpha(Y^2) = at_y^2 + bt_1 t_y + ct_1^2,$$
$$\mu_\alpha(T) = t_x(2at_y + bt_1),$$
$$\mu_\alpha(U) = at_x^2(2t_z + bt_x),$$
$$\mu_\alpha(V) = \mu_\alpha((ZX)^2) = at_x^2(t_z^2 + bt_x t_z + act_x^2),$$
$$\mu_\alpha(4X^2V) = \mu_\alpha(U^2 - a^{-1}(b^2 - 4ac)X^6),$$
$$\mu_\alpha(T^2 - 4X^2Y^2) = \mu_\alpha(a^{-1}(b^2 - 4ac)E^2X^2),$$
$$\mu_\alpha(EZ - XY) = t_1 t_z - at_x t_y,$$
$$\mu_\alpha(EU - X^2T) = 2at_x^2(t_1 t_z - at_x t_y).$$

It follows from Proposition 8.2 and Lemma 10.2 that the elements E, X^2, Y^2, T, U, V of $\mathcal{U}^\alpha_{H_4}$ lie in the center $\mathcal{Z}^\alpha_{H_4}$. We henceforth identify these elements with their images in $\mathcal{B}^\alpha_{H_4}$.

Theorem 10.3.

(a) *As a k-algebra, $\mathcal{B}_{H_4}^\alpha$ is generated by $E^{\pm 1}$, $(X^2)^{\pm 1}$, Y^2, T, U, and is isomorphic to $Z_{H_4}^\alpha[E^{-1}, (X^2)^{-1}]$.*

(b) *Any relation between the generators $E^{\pm 1}$, $(X^2)^{\pm 1}$, Y^2, T, U of $\mathcal{B}_{H_4}^\alpha$ is a consequence of the relation*

$$T^2 - 4X^2Y^2 = \frac{b^2 - 4ac}{a} E^2 X^2.$$

Part (a) of Theorem 10.3 implies that the two-cocycle α is nice in the sense of Section 9 for all values of a, b, c.

Proof. (a) From Lemmas 10.1(a) and 10.2 we deduce the relations

$$\sigma(1, 1) = \sigma(1, x) = \sigma(x, 1) = \mu_\alpha(E),$$

$$\sigma(x, x) = \mu_\alpha\left(\frac{X^2}{E}\right),$$

$$\sigma(y, y) = \sigma(z, y) = -\sigma(y, z) = \mu_\alpha\left(\frac{Y^2}{E}\right),$$

$$\sigma(x, y) = -\sigma(x, z) = \mu_\alpha\left(\frac{X^2T - EU}{2EX^2}\right),$$

$$\sigma(y, x) = \sigma(z, x) = \mu_\alpha\left(\frac{X^2T + EU}{2EX^2}\right),$$

$$\sigma(z, z) = -\mu_\alpha\left(\frac{V}{EX^2}\right).$$

By Lemma 10.1(b), the values of σ^{-1} are linear combinations of the above values of σ, possibly divided by positive powers of $t_1 = \mu_\alpha(E)$ and of $\sigma(x, x) = \mu_\alpha(X^2/E)$. This shows that $\mathcal{B}_{H_4}^\alpha$ is generated by $E^{\pm 1}$, $(X^2)^{\pm 1}$, Y^2, T, U, V. Now, by Lemma 10.2, E, X^2, Y^2, T, U, V belong to $Z_{H_4}^\alpha$. Therefore, $\mathcal{B}_{H_4}^\alpha$ is obtained from $Z_{H_4}^\alpha$ by inverting E and X^2, hence is isomorphic to $Z_{H_4}^\alpha[E^{-1}, (X^2)^{-1}]$.

A computation using Lemma 10.2 yields

$$V = \frac{U^2}{4X^2} - \frac{b^2 - 4ac}{4a} X^4,$$

which shows that V can be expressed in terms of $(X^2)^{\pm 1}$ and U.

(b) Let B_0 be the subalgebra of $S(t_{H_4})_\Theta$ generated by $\mu_\alpha(E)^{\pm 1}$, $\mu_\alpha(X^2)^{\pm 1}$, $\mu_\alpha(Y^2)$, and $\mu_\alpha(U)$. Since by part (a),

$$\mu_\alpha(E)^{\pm 1} = \sigma^{\pm 1}(1, 1), \qquad \mu_\alpha\left(X^2\right)^{\pm 1} = \mu_\alpha(E)^{\pm 1}\sigma^{\pm 1}(x, x),$$

$$\mu_\alpha\left(Y^2\right) = \mu_\alpha(E)\sigma(y, y), \qquad \mu_\alpha(U) = \mu_\alpha\left(X^2\right)\left(\sigma(y, x) - \sigma(x, y)\right),$$

we conclude that B_0 is a subalgebra of $\mathcal{B}^\alpha_{H_4}$. Considering the degrees in t_1, t_x, t_y, t_z of the polynomials $\mu_\alpha(E), \mu_\alpha(X^2), \mu_\alpha(Y^2)$, and $\mu_\alpha(U/X^2)$, we easily deduce that these four polynomials are algebraically independent. It follows that

$$B_0 = k\big[E^{\pm 1}, (X^2)^{\pm 1}, Y^2, U\big].$$

By part (a), $\mathcal{B}^\alpha_{H_4}$ is generated by T as a B_0-algebra. By Lemma 10.2,

$$T^2 - 4X^2Y^2 = \frac{b^2 - 4ac}{a} E^2 X^2 \in \mathcal{B}^\alpha_{H_4}. \tag{10.7}$$

To complete the proof, it suffices to check that $\mathcal{B}^\alpha_{H_4}$ is a free B_0-module with basis $\{1, T\}$. By (10.7), the elements 1 and T generate $\mathcal{B}^\alpha_{H_4}$ as a B_0-module. Let us show that they are linearly independent over B_0. Suppose that there exists a relation of the form $P + QT = 0$, where P and $Q \in B_0$. If we denote the degree in t_y by ∂_y and if P, Q are nonzero, we have

$$\partial_y(P) = \partial_y(Q) + \partial_y(T). \tag{10.8}$$

We claim that $\partial_y(P)$ and $\partial_y(Q)$ are even integers: indeed, of the four generators of B_0 only $\mu_\alpha(Y^2) = at_y^2 + bt_1 t_y + ct_1^2$ contains t_y and its degree in t_y is 2. Now, $\partial_y(T) = 1$ is odd. This contradicts (10.8). Therefore, $P = Q = 0$, which shows that $\{1, T\}$ is a basis of $\mathcal{B}^\alpha_{H_4}$ over B_0. We have thus proved that $\mathcal{B}^\alpha_{H_4} = B_0[T]/(T^2 - 4X^2Y^2 - [(b^2 - 4ac)/a]E^2X^2)$. $\quad\square$

Corollary 10.4. *We have the following presentation for $\mathcal{B}^\alpha_{H_4}$:*

$$\mathcal{B}^\alpha_{H_4} \cong k\big[E^{\pm 1}, (X^2)^{\pm 1}, Y^2, T, U\big]/(P_{a,b,c}),$$

where $P_{a,b,c}$ is the polynomial

$$P_{a,b,c} = T^2 - 4X^2Y^2 - \frac{b^2 - 4ac}{a} E^2 X^2.$$

It follows that, in order to specify an algebra morphism from $\mathcal{B}^\alpha_{H_4}$ to an extension K of k, it is enough to pick elements $\mathbf{e}, \mathbf{x}, \mathbf{y}, \mathbf{t}, \mathbf{u} \in K$ verifying $\mathbf{e} \neq 0, \mathbf{x} \neq 0$, and

$$\mathbf{t}^2 - 4\mathbf{xy} = \frac{b^2 - 4ac}{a}\mathbf{e}^2\mathbf{x}.$$

The spectrum of $\mathcal{B}^\alpha_{H_4}$, which is the quadric hypersurface given by the vanishing of $P_{a,b,c}$, is the base space of the universal noncommutative deformation space $\mathcal{A}^\alpha_{H_4}$ computed below.

Proposition 10.5. *The algebra $S(t_{H_4})_\Theta$ is integral over the subalgebra $\mathcal{B}^\alpha_{H_4}$.*

Proof. It suffices to check that the generators $t_1^{\pm 1}$, $t_x^{\pm 1}$, t_y, t_z of $S(t_{H_4})_\Theta$ are integral over $\mathcal{B}_{H_4}^\alpha$. First, $t_1^{\pm 1} = \sigma^{\pm 1}(1, 1)$ belongs to $\mathcal{B}_{H_4}^\alpha$. For the other generators, we use the formulas in Lemma 10.1. For instance,

$$\left(t_x^{\pm 1}\right)^2 = a^{\mp 1}\sigma^{\pm 1}(1, 1)\sigma^{\pm 1}(x, x),$$

which shows that $(t_x)^2$ and $(t_x^{-1})^2$ belong to $\mathcal{B}_{H_4}^\alpha$. The generator t_y satisfies a quadratic equation of the form

$$at_y^2 + bt_1 t_y + ct_1^2 - \sigma(1, 1)\,\sigma(y, y) = 0$$

whose coefficients belong to $\mathcal{B}_{H_4}^\alpha$. We finally use the relation

$$t_1 t_z - at_x t_y = \sigma(1, 1)\,\sigma(x, z)$$

to conclude that t_z is integral over $\mathcal{B}_{H_4}^\alpha$. $\square$

Remark 10.6. The elements $W = Y(YZ + ZY)$ and

$$\nabla_\pm = (XZY + YXZ) \pm (XYZ + YZX)$$

also belong to $\mathcal{Z}_{H_4}^\alpha$. Indeed, we have

$$\mu_\alpha(W) = (2t_z + bt_x)\left(at_y^2 + bt_1 t_y + ct_1^2\right),$$
$$\mu_\alpha(\nabla_+) = t_x(2t_z + bt_x)(2at_y + bt_1),$$
$$\mu_\alpha(\nabla_-) = -\left(b^2 - 4ac\right)t_1 t_x^2.$$

Moreover, the relations

$$X^2\nabla_+ = TU \quad \text{and} \quad E\nabla_- = 4X^2 Y^2 - T^2$$

hold in $\mathcal{Z}_{H_4}^\alpha$. (To prove them, it suffices to check that both sides of each relation have equal polynomial images under μ_α.)

10.4. The universal twisted algebra $\mathcal{A}_{H_4}$

Since the two-cocycle α is nondegenerate and nice, we may apply Theorem 9.3. It states that

$$\mathcal{B}_{H_4}^\alpha \otimes_{\mathcal{Z}_{H_4}^\alpha} \mathcal{U}_{H_4}^\alpha \quad \text{and} \quad \mathcal{A}_{H_4}^\alpha = \mathcal{B}_{H_4}^\alpha \otimes {}^\sigma H_4$$

are isomorphic as H_4-comodule algebras. Let us give a presentation of these algebras by generators and relations.

Theorem 10.7. *The algebra $\mathcal{A}_{H_4}^\alpha$ is isomorphic to the $\mathcal{B}_{H_4}^\alpha$-algebra generated by two variables ξ, η, subject to the relations*

$$\xi^2 = X^2, \qquad \eta^2 = Y^2, \qquad \xi\eta + \eta\xi = T.$$

Proof. Let $\mathcal{A} = \mathcal{B}^\alpha_{H_4} \langle \xi, \eta \rangle / (\xi^2 - X^2, \eta^2 - Y^2, \xi\eta + \eta\xi - T)$. We define an algebra morphism $f : \mathcal{A} \to \mathcal{A}^\alpha_{H_4}$ by $f(\xi) = X$ and $f(\eta) = Y$. It is clear that f is well defined.

Let us first establish that f is surjective. The algebra $\mathcal{A}^\alpha_{H_4}$ is generated by E, X, Y and Z. Now, E belongs to $\mathcal{B}^\alpha_{H_4}$ and X and Y are obviously in the image of f. From Lemma 10.2 we deduce that $2X^2(EZ - XY) = EU - X^2T$ holds in $\mathcal{U}^\alpha_{H_4}$. Hence,

$$Z = \frac{1}{E}XY + \frac{EU - X^2T}{2EX^2}$$

in $\mathcal{A}^\alpha_{H_4}$. Since E^{-1} and $(EU - X^2T)/2EX^2$ belong to $\mathcal{B}^\alpha_{H_4}$, the element Z belongs to the image of f.

We now prove that f is injective. We observe that ξ^2, η^2, $\xi\eta + \eta\xi$ belong to $\mathcal{B}^\alpha_{H_4}$. Hence, $\mathcal{A}$ is spanned by 1, ξ, η, $\xi\eta$ as a $\mathcal{B}^\alpha_{H_4}$-module. Consider an arbitrary element $\omega = \gamma_0 + \gamma_1\xi + \gamma_2\eta + \gamma_3\xi\eta$ of $\mathcal{A}$, where $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ belong to $\mathcal{B}^\alpha_{H_4}$. If $f(\omega) = 0$, then

$$\gamma_0 + \gamma_1 X + \gamma_2 Y + \gamma_3 XY = 0$$

in $\mathcal{A}^\alpha_{H_4}$. Let us replace 1, X, Y, XY by their images under μ_α. We obtain

$$\gamma_0 t_1 \otimes 1 + \gamma_1 (t_x \otimes x) + \gamma_2 (t_y \otimes x + t_1 \otimes y) + \gamma_3 (t_z \otimes 1 + t_x \otimes z) = 0.$$

Since $\{1, x, y, z\}$ is a basis of H_4, we obtain the system of equations

$$\begin{cases} \gamma_0 t_1 + \gamma_3 t_z = 0, \\ \gamma_1 t_x + \gamma_2 t_y = 0, \\ \gamma_2 t_1 = 0, \\ \gamma_3 t_x = 0 \end{cases}$$

in $S(t_{H_4})_\Theta$. Since t_1 and t_x are invertible, $\gamma_i = 0$ for $i = 0, 1, 2, 3$. This proves the injectivity of f. $\square$

Remarks 10.8. (a) Since $A_{a,b,c}$ is simple when $b^2 - 4ac \neq 0$, then so is the algebra $\mathcal{K}^\alpha_{H_4} \otimes_{\mathcal{B}^\alpha_{H_4}} \mathcal{A}^\alpha_{H_4}$ by Corollary 6.5.

If $b^2 - 4ac = 0$, then $P_{a,b,c} = T^2 - 4X^2Y^2$. Let $\theta = 2X^2\eta - T\xi \in \mathcal{A}^\alpha_{H_4}$, where we use the description of $\mathcal{A}^\alpha_{H_4}$ given in Theorem 10.7. It is easy to check that

$$\theta^2 = \theta\xi + \xi\theta = \theta\eta + \eta\theta = 0.$$

Thus, θ ($\neq 0$) generates a nilpotent two-sided ideal in $\mathcal{K}^\alpha_{H_4} \otimes_{\mathcal{B}^\alpha_{H_4}} \mathcal{A}^\alpha_{H_4}$, which implies that the latter is not (semi)simple when $b^2 - 4ac = 0$.

(b) It follows from the results of Section 7 and the computations above that the scalar $(b^2 - 4ac)/a$ is an invariant for the forms of $A_{a,b,c}$. More precisely, if the comodule algebra $A_{a',b',c'}$ is a form of $A_{a,b,c}$, then necessarily

$$\frac{b'^2 - 4a'c'}{a'} = \frac{b^2 - 4ac}{a}. \tag{10.9}$$

It can be shown using [16] that, if (10.9) holds, then $A_{a',b',c'}$ is a form of $A_{a,b,c}$.

(c) Remark (b) has an important consequence. Let $A_{a,b,c}$ be as above. Define $F = k_0((b^2 - 4ac)/a)$ to be the field generated by $(b^2 - 4ac)/a$ over the prime field k_0 of k. Then there exists an H-comodule algebra A_F over the field F such that $k \otimes_F A_F \cong A_{a,b,c}$. Furthermore, F is the unique minimal field with this property. It follows that, if B is an H-comodule algebra over a subfield K of the algebraic closure $\bar{k}$ of k such that $\bar{k} \otimes_K B \cong \bar{k} \otimes_k A_{a,b,c}$, then the universal twisted algebra $\mathcal{A}^{\alpha}_{H_4, F}$ corresponding to A_F specializes to B.

Appendix A. The map t^{-1}

Let C be a k-coalgebra. Pick a copy t_C of the underlying vector space of C and denote the identity map from C to t_C by $x \mapsto t_x$ ($x \in C$). Let $S(t_C)$ be the symmetric algebra over the vector space t_C and K_C the field of fractions of $S(t_C)$.

Lemma A.1. *There is a unique linear map $C \to K_C$, $x \mapsto t_x^{-1}$ such that for all $x \in C$,*

$$\sum_{(x)} t_{x_{(1)}} t_{x_{(2)}}^{-1} = \sum_{(x)} t_{x_{(1)}}^{-1} t_{x_{(2)}} = \varepsilon(x) 1. \tag{A.1}$$

Here we have used the Heyneman–Sweedler sigma notation and ε denotes the counit of C. Observe that Eq. (A.1) is equivalent to

$$t * t^{-1} = t^{-1} * t = \varepsilon\eta, \tag{A.2}$$

where $*$ is the convolution product on $\mathrm{Hom}(C, K_C)$ and $\varepsilon\eta$ is the neutral element for the convolution product (by definition, $(\varepsilon\eta)(x) = \varepsilon(x)1$ for all $x \in C$).

Proof. By (A.2) it suffices to establish that t has a right and a left inverse for the convolution product. The right and the left inverses necessarily coincide and are unique.

Let us prove that t has a right inverse for the convolution product. First assume that C is finite-dimensional. Expressing the comultiplication Δ in a basis $\{x_1, \ldots, x_d\}$ of C, we have

$$\Delta(x_i) = \sum_{p,q=0}^{d} c_i^{p,q} x_p \otimes x_q, \tag{A.3}$$

where $c_i^{p,q} \in k$ are the structure constants of Δ. When we reformulate the equations

$$\sum_{(x)} t_{x_{(1)}} t_{x_{(2)}}^{-1} = \varepsilon(x) 1$$

in the basis, we obtain the matrix equation

$$M \begin{pmatrix} t_{x_1}^{-1} \\ \vdots \\ t_{x_d}^{-1} \end{pmatrix} = \begin{pmatrix} \varepsilon(x_1) \\ \vdots \\ \varepsilon(x_d) \end{pmatrix}, \tag{A.4}$$

where M is the $d \times d$-matrix whose entries are given by

$$M_{i,q} = \sum_{p=1}^{d} c_i^{p,q} t_{x_p} \in S(t_C) \tag{A.5}$$

for all $i, q = 1, \ldots, d$. To prove the existence and the uniqueness of the solution of (A.4), it suffices to check that the determinant $\det(M)$ of M is nonzero. Since $\det(M)$ is a polynomial in the variables $t_{x_1}, \ldots, t_{x_d}$, it suffices to prove that $\det(M)$ is nonzero under a suitable specialization of the variables.

Let $\epsilon : S(t_C) \to k$ be the algebra morphism defined by $\epsilon(t_x) = \varepsilon(x)$ for all $x \in C$. The relation $(\varepsilon \otimes \mathrm{id}_C) \circ \Delta = \mathrm{id}_C$ satisfied by the counit becomes

$$\sum_{p,q=0}^{d} c_i^{p,q} \varepsilon(x_p) x_q = x_i$$

for all $i = 1, \ldots, d$. We can reinterpret this by saying that the scalar matrix $\epsilon(M)$ whose entries $\epsilon(M)_{i,q}$ are given by

$$\epsilon(M)_{i,q} = \sum_{p=1}^{d} c_i^{p,q} \varepsilon(x_p)$$

satisfies

$$\epsilon(M) \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}.$$

Therefore, $\epsilon(M)$ is the identity matrix and $\epsilon(\det(M)) = \det(\epsilon(M)) = 1 \neq 0$.

If C is not finite-dimensional, it is a direct limit of finite-dimensional subcoalgebras C_κ. By what we have just proved, for each κ there is a linear map $C_\kappa \to K_C$, $x \mapsto t_{\kappa,x}^{-1}$ such that

$$\sum_{(x)} t_{x_{(1)}} t_{\kappa,x_{(2)}}^{-1} = \varepsilon(x) 1$$

for all $x \in C_\kappa$. Since

$$\mathrm{Hom}(C, K_C) = \mathrm{Hom}\left(\varinjlim_{\kappa} C_\kappa, K_C \right) \cong \varprojlim_{\kappa} \mathrm{Hom}(C_\kappa, K_C),$$

the maps t_κ^{-1} fit together to form a linear map $t^{-1} : C \to K_C$ satisfying the required property. One proves in an analogous way that t has a left inverse for the convolution product. $\quad\square$

We end this appendix with two elementary computations of values of t^{-1}.

(a) If $g \in C$ is a grouplike element, i.e., satisfying $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$, then

$$t_g^{-1} = \frac{1}{t_g}. \tag{A.6}$$

(b) If $x \in C$ is skew-primitive, i.e., if $\varepsilon(x) = 0$ and $\Delta(x) = g \otimes x + x \otimes h$ for some grouplike elements g, h, then

$$t_x^{-1} = -\frac{t_x}{t_g t_h}. \tag{A.7}$$

Appendix B. The algebra $S(t_C)_\Theta$

We resume the notation of Appendix A, namely C is a coalgebra, $S(t_C)$ is the symmetric algebra over a copy t_C of the underlying vector space of C, and K_C the field of fractions of $S(t_C)$. We recall the linear maps $t : C \to S(t_C) \subset K_C$ and $t^{-1} : C \to K_C$ introduced there.

Let $S(t_C)_\Theta$ be the subalgebra of K_C generated by $t(C)$ and $t^{-1}(C)$. The aim of this appendix is to show that $S(t_C)_\Theta$ is isomorphic to the free commutative Hopf algebra on the coalgebra C introduced by Takeuchi in [30, §11]. This will imply that $S(t_C)_\Theta$ is obtained from $S(t_C)$ by inverting certain "canonical" grouplike elements. We complete the appendix by applying the theory to three interesting examples of coalgebras.

We first observe that, by definition of $S(t_C)_\Theta$, the following property holds: for any couple (g, g^{-1}) of linear maps $C \to R$ with values in a commutative ring R satisfying the relations

$$\sum_{(x)} g(x_{(1)}) g^{-1}(x_{(2)}) = \sum_{(x)} g^{-1}(x_{(1)}) g(x_{(2)}) = \varepsilon(x)1 \tag{B.1}$$

for all $x \in C$, there is a unique algebra morphism $f : S(t_C)_\Theta \to R$ such that $f(t_x) = g(x)$ and $f(t_x^{-1}) = g^{-1}(x)$ for all $x \in C$.

For any $x \in C$, set

$$\Delta(t_x) = \sum_{(x)} t_{x_{(1)}} \otimes t_{x_{(2)}} \in S(t_C)_\Theta \otimes S(t_C)_\Theta, \tag{B.2}$$

$$\Delta(t_x^{-1}) = \sum_{(x)} t_{x_{(2)}}^{-1} \otimes t_{x_{(1)}}^{-1} \in S(t_C)_\Theta \otimes S(t_C)_\Theta, \tag{B.3}$$

$$\varepsilon(t_x) = \varepsilon(t_x^{-1}) = \varepsilon(x) \in k, \tag{B.4}$$

$$S(t_x) = t_x^{-1} \quad \text{and} \quad S(t_x^{-1}) = t_x \in S(t_C)_\Theta. \tag{B.5}$$

Using (B.1), it is easy to check that (B.2)–(B.5) define algebra morphisms $\Delta : S(t_C)_\Theta \to S(t_C)_\Theta \otimes S(t_C)_\Theta$, $\varepsilon : S(t_C)_\Theta \to k$, and an algebra antiautomorphism $S : S(t_C)_\Theta \to S(t_C)_\Theta$, turning $S(t_C)_\Theta$ into a Hopf algebra with comultiplication Δ, counit ε, and antipode S.

The Hopf algebra $S(t_C)_\Theta$ satisfies the following universal property.

Lemma B.1. *For any commutative Hopf algebra H and any morphism of coalgebras $g : C \to H$, there is a unique morphism of Hopf algebras $f : S(t_C)_\Theta \to H$ such that $f \circ t = g$.*

Proof. Since the elements t_x and t_x^{-1} ($x \in C$) generate $S(t_C)_\Theta$ as an algebra, f is uniquely defined by its values on them. By definition, $f(t_x) = g(x)$ for all $x \in C$. The Hopf algebra morphism f preserves the antipodes, which implies for all $x \in C$ that

$$f(t_x^{-1}) = f(S(t_x)) = S(g(x)),$$

where the righmost S denotes the antipode of H. To prove the existence of f, it suffices by (B.1) to check that

$$\sum_{(x)} f(t_{x_{(1)}}) f(t_{x_{(2)}}^{-1}) = \sum_{(x)} f(t_{x_{(1)}}^{-1}) f(t_{x_{(2)}}) = \varepsilon(x)1$$

for all $x \in C$. Now, since g is a coalgebra morphism, we have

$$\sum_{(x)} f(t_{x_{(1)}}) f(t_{x_{(2)}}^{-1}) = \sum_{(x)} g(x_{(1)}) S\big(g(x_{(2)})\big)$$

$$= \sum_{(x)} g(x)_{(1)} S\big(g(x)_{(2)}\big)$$

$$= \varepsilon\big(g(x)\big)1 = \varepsilon(x)1.$$

The relation $\sum_{(x)} f(t_{x_{(1)}}^{-1}) f(t_{x_{(2)}}) = \varepsilon(x)1$ is proved similarly. It is now easy to check that f is a morphism of Hopf algebras. $\square$

Takeuchi's free commutative Hopf algebra generated by the coalgebra C (defined in [30, Def. 62]) also satisfies the universal property of Lemma B.1. Therefore, it is isomorphic to $S(t_C)_\Theta$. This allows us to use the results of [30] to show that $S(t_C)_\Theta$ is obtained from $S(t_C)$ by inverting certain elements, which we now describe.

Assume first that the coalgebra C is finite-dimensional. The dual vector space $A = C^* = \mathrm{Hom}(C, k)$ carries a natural structure of an algebra. Since this algebra is finite-dimensional, we can consider the norm map

$$N : A \otimes S(t_C) \to S(t_C)$$

defined for all $\theta \in A \otimes S(t_C)$ by $N(\theta) = \det(L_\theta)$, where L_θ is the left multiplication by θ. On the other hand, consider the linear map

$$\iota : A \otimes S(t_C) = C^* \otimes S(t_C) \to \mathrm{Hom}\big(C, S(t_C)\big)$$

given by $a \otimes P \mapsto (x \mapsto \langle a, x \rangle P)$, where $a \in A$, $x \in C$, $P \in S(t_C)$, and $\langle \, , \rangle$ is the canonical duality between A and C. The map ι is an isomorphism since A is finite-dimensional. Using the inclusion $t \in \mathrm{Hom}(C, S(t_C))$ and following [30, §11], we define the element $\Theta_C \in S(t_C)$ by

$$\Theta_C = N\big(\iota^{-1}(t)\big). \tag{B.6}$$

By [30, Lemma 58], Θ_C is a grouplike element of $S(t_C)$.

We shall see in Example B.5 below that Θ_C is a generalization of Dedekind's group determinant. For this reason, to denote this element we have used the same symbol Θ as Dedekind. Note also that Θ_C is the determinant of the matrix M defined by (A.5).

In the general case, we have the following.

Proposition B.2. *Let C be a coalgebra and $(C_\kappa)_\kappa$ be a family of finite-dimensional coalgebras such that $\sum_\kappa C_\kappa$ contains the coradical C_0 of C. Then $S(t_C)_\Theta$ is the smallest subalgebra of the fraction field K_C containing $S(t_C)$ and all inverses $\Theta_{C_\kappa}^{-1}$:*

$$S(t_C)_\Theta = S(t_C)\left[\left(\frac{1}{\Theta_{C_\kappa}}\right)_\kappa\right].$$

Moreover, if the coradical C_0 is finite-dimensional, then

$$S(t_C)_\Theta = S(t_C)\left[\frac{1}{\Theta_{C_0}}\right].$$

Proof. This is a consequence of [30, Theorem 61 and Corollary 64] together with our identification of $S(t_C)_\Theta$ with Takeuchi's free commutative Hopf algebra on C. $\square$

As a consequence of Proposition B.2, the spectrum of the commutative algebra $S(t_C)_\Theta$ is obtained from the affine space C by cutting out the zero-loci of the polynomials Θ_{C_κ}. We can reinterpret this as follows: for any field K, a linear map $g : C \to K$ extends to an algebra morphism $S(t_C)_\Theta \to K$ if and only if its extension $f : S(t_C) \to K$ as an algebra morphism satisfies $f(\Theta_\kappa) \neq 0$ for all κ.

We end this appendix with three examples.

Example B.3. Let C be the coalgebra dual to the algebra $M_n(k)$ of $n \times n$-matrices with entries in k. As a vector space, C has a basis $\{X_{i,j}\}_{i,j=1,\dots,n}$ with comultiplication Δ and counit ε given by

$$\Delta(X_{i,j}) = \sum_{k=0}^n X_{i,k} \otimes X_{k,j} \quad \text{and} \quad \varepsilon(X_{i,j}) = \delta_{i,j} \quad \text{(Kronecker symbol)}$$

for all $i, j = 1, \dots, n$. Set $t_{i,j} = t_{X_{i,j}}$. Then $\Theta_C \in S(t_C)$ is equal to the determinant of the "generic" matrix $(t_{i,j})_{i,j}$.

Example B.4. Let C be a coalgebra which has a basis G consisting of grouplike elements (for instance, the coalgebra underlying the Hopf algebra of a group G considered in Example 1.2). The symmetric algebra $S(t_C)$ is the polynomial algebra in the indeterminates $(t_g)_{g \in G}$. By (A.6), $t_g^{-1} = 1/t_g$ for all $g \in G$. Therefore, $S(t_C)_\Theta$ is the algebra of Laurent polynomials

$$S(t_C)_\Theta = k\left[t_g, t_g^{-1} \mid g \in G\right].$$

Observe that $\Theta_C = \prod_{g \in G} t_g$ if G is finite.

Example B.5. Let G be a finite group and $C = k^G$ the coalgebra underlying the Hopf algebra of k-valued functions on G considered in Example 1.3. (Over an algebraically closed field, the coalgebra k^G is a product of matrix coalgebras of type B.3.) In this case Θ_C is equal to *Dedekind's group determinant* Θ_G, which is the determinant of the matrix

$$(t_{gh^{-1}})_{g,h \in G}.$$

The determinant Θ_G was factored into a product of irreducible polynomials by Dedekind when G is abelian and by Frobenius for an arbitrary group. The factors can be expressed in terms of the irreducible characters of G, see [31, Chapter 12, Part B].

If G is an abelian group of exponent N, and k is a field containing a primitive Nth root of unity and having a characteristic not dividing N, then there is an isomorphism of Hopf algebras $C = k^G \cong k[G]$ provided by the discrete Fourier transform. Under these hypotheses, the algebra $S(t_C)_\Theta$ is a ring of Laurent polynomials in view of Example B.4.

If $G = \langle g \mid g^2 = e \rangle$ is the cyclic group of order two and $C = k^G$, then

$$\Theta_C = \begin{vmatrix} t_e & t_g \\ t_g & t_e \end{vmatrix} = t_e^2 - t_g^2 = (t_e - t_g)(t_e + t_g).$$

If k is of characteristic two, then $S(t_C)_\Theta = k[t_e, t_g, (t_e + t_g)^{-2}]$. One can check that this is not an algebra of Laurent polynomials.

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