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Weighted H^p Spaces on Homogeneous Halfplanes*

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1. INTRODUCTION

Let V be an open convex cone in $\mathbb{R}^n$ (the n-dimensional real Euclidean space) and let $T_V = \mathbb{R}^n + iV$ be the "tube" domain in $\mathbb{C}^n$ (the n-dimensional complex Euclidean space), consisting of all points $z = x + iy$ such that $x \in \mathbb{R}^n$ and $y \in V$. These domains are also called halfplanes, since they reduce to the upper halfplane for $n = 1$. The cone V will be supposed to satisfy some rather restrictive conditions: it will be homogeneous (see Section 2) and self-adjoint, which means that its closure $\bar{V}$ equals the dual cone $V^* = \{x \in \mathbb{R}^n : x \cdot y \geqslant 0,\ y \in V\}$.

In this paper we consider spaces of functions in T_V which generalize the classical Hardy spaces H^p on the halfplane. Let w be a given function analytic in T_V. Then $H^p_w(T_V)$ $(0 < p < \infty)$ is the space of all functions F analytic in T_V and such that

$$\sup_{y \in V} \int_{\mathbb{R}^n} |F(x+iy)\, w(x+iy)|^p \, dx < \infty. \tag{1}$$

Similarly, for a given function u on the cone V, the weighted spaces $L^p_u(V)$ are defined to consist of all measurable functions f on V such that

$$\int_V |f(t)\, u(t)|^p \, dt < \infty. \tag{2}$$

We shall write $L^p(V)$ when $u = 1$ and likewise for $H^p(T_V)$.

There is a well-known relationship between H^p spaces and the Laplace transform

$$\mathscr{L}f(z) = \int_V e^{iz \cdot t} f(t)\, dt, \tag{3}$$

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where f is a function supported in V and $z \in T_V$. When the integral (3) is absolutely convergent, the function $\mathscr{L}f(z)$ is analytic in T_V. This is, for example, the case when $f \in L^2(V)$, and it turns out that in this case $\mathscr{L}f \in H^2(T_V)$, as it is easily seen by an application of Plancherel's theorem to (3) (since for $y \in V$, the function $|e^{iz \cdot t}| = e^{-y \cdot t}$ is integrable on V). The converse also holds: every function in $H^2(T_V)$ is characterized as a Laplace transform of some function $f \in L^2(V)$. For $n = 1$, this is the classical theorem of Paley and Wiener, which was generalized to H^p, $1 < p \leqslant 2$, by Doetsch (see [1]). The analogous theorem for $n > 1$ was proved by Stein and Weiss [7]. (Only the case $p = 2$ is treated in this book, but it is obvious that $1 < p \leqslant 2$ can be obtained in the same way. As usual, $p' = p/(p-1)$ is the conjugate index.)

THEOREM A. *Let* $1 < p \leqslant 2$. *Let* $f \in L^p(V)$. *Then* $F = \mathscr{L}f$ *belongs to* $H^{p'}(T_V)$ *and*

$$\left(\int_{\mathbb{R}^n} |F(x+iy)|^{p'}\, dx\right)^{1/p'} \leqslant C\left(\int_V |f(t)|^p\, e^{-py \cdot t}\, dt\right)^{1/p}.$$

THEOREM B. *Let* $1 < p \leqslant 2$. *Let* $F \in H^p(T_V)$. *Then there is an* $f \in L^{p'}(V)$ *such that* $F = \mathscr{L}f$ *and*

$$\left(\int_V |f(t)|^{p'}\, e^{-p'y \cdot t}\, dt\right)^{1/p'} \leqslant C\left(\int_{\mathbb{R}^n} |F(x+iy)|^p\, dx\right)^{1/p}.$$

In the one-dimensional case Rooney [5] proved a weighted analogue of Theorems A and B. He considered weights of power type; that is, he had $w(x+iy) = (x+iy)^\alpha$ and $u(t) = t^\beta$ for some $\alpha, \beta \in \mathbb{R}$. Benedetto and Heinig [1] considered some more general weights.

In the present paper we prove a weighted analogue of Theorems A and B. We shall deal with weights which are n-dimensional "power functions," so that we shall get an n-dimensional version of Rooney's result. To see what the appropriate weights are we must investigate the structure of the cone, in particular its group of automorphisms. This is done in Sections 2 and 5, and the weights are defined and the theorems are stated in Sections 3 and 4. The rest of the paper is concerned with the proofs of these theorems. Let us note that the proofs of the nonweighted results are derived from (a generalization to $1 < p \leqslant 2$ of) Plancherel's theorem. Also, Rooney and Benedetto and Heinig proved the weighted results by first deriving a corresponding weighted Plancherel theorem. We are going to proceed in a different way and reduce the weighted case to the nonweighted. This is done by making use of the convolution property of the Laplace transform and of some properties of a Riemann–Liouville integral operator (of fractional order) which is defined in Section 7. The weighted norm inequality for this operator (Proposition 1) will provide a link between the nonweighted and the weighted case. Since the Riemann–Liouville operator is positive, this inequality is much simpler than a corresponding weighted inequality for the Fourier transform (Plancherel's theorem) would be.

2. Homogeneous Cones

In this section we study some properties of the cone V by considering the group $G(V)$ of its linear automorphisms, i.e., the group of all nonsingular linear transformations A in $\mathbb{R}^n$ such that $AV = V$.

The cone is said to be *homogeneous* if $G(V)$ acts transitively on V; that is if for every $x, y \in V$ there is an $A \in G(V)$ such that $Ax = y$.

A function $f: V \to \mathbb{R}_+$ is said to be *homogeneous* of degree $\alpha \in \mathbb{R}$ if $f(Ax) = |A|^\alpha f(x)$, for all $A \in G(V)$ ($|A|$ is the determinant of the matrix of the transformation).

For an example of a homogeneous function, consider first the partial order in $\mathbb{R}^n$ defined by the cone V, $x < y$ iff $y - x \in V$, and let $\langle a, b \rangle$ be the "interval" with respect to this order. Then

$$s(x) = \int_{\langle 0, x \rangle} dt \tag{4}$$

is homogeneous of degree 1. The function

$$h(x) = \int_V e^{-x \cdot t}\, dt \tag{5}$$

is homogeneous of degree -1.

The following simple lemma will be very useful in all our considerations.

Lemma 1. *If $f: V \to \mathbb{R}_+$ is homogeneous of degree α, then there is a constant $C > 0$ such that*

$$f(x) = Cs^\alpha(x).$$

This means that there are not many homogeneous functions on the cone. For example, $h(x) = Cs^{-1}(x)$. (We are not interested in the exact values of the constants, so we let C stand for values which are not necessarily the same at every appearance.)

A function homogeneous of degree 1 is called a norm for the cone [3]. So (4) is an example of a norm. We shall also write $s_V(x)$ if necessary. The function $s^\alpha(x)$ will play the role of the function x^α in one variable.

There is another important function defined on the cone (the analogue of $1/x$ in one variable). It was defined by Koecher [3] in the following way,

$$x^* = \text{grad} \log s(x). \tag{6}$$

This function is one to one and onto V; moreover it is an involution on V, $x^{**} = x$, and it has a unique fixed point, which we shall denote by a [6]. Additional properties of this function are cited in Section 5. Now we are going to define the rank of the cone (see [2]).

Let G_a be the stability group of a, that is the subgroup of $G(V)$ which leaves a invariant. It was proved by Rothaus [6] that the elements of G_a are rotations (orthogonal transformations), which means that G_a can be identified with $O(m)$, for some $m < n-1$. Then the number $k = n - m$ is called the *rank* of the cone. For example, the rank of the positive octant $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 > 0, ..., x_n > 0\}$ is equal to n, and the rank of the light cone $V^n_+ = \{x \in \mathbb{R}^n : x_1 > 0,\ x_1^2 - x_2^2 - \cdots - x_n^2 > 0\}$ is equal to 2. These are the two extremal cases, since obviously $2 \leqslant k \leqslant n$. Let us put

$$\tau = \frac{k}{n}. \tag{7}$$

Now we are going to define the weights and formulate the theorems, and we return to the properties of $G(V)$ in Section 5.

3. The Weight Functions

We shall choose the function u in (2) to be of the form s^α, $\alpha \in \mathbb{R}$, and write $L^p_\alpha(V)$ for $L^p_{s^\alpha}(V)$.

To define the weights in the H^p spaces consider the Cauchy–Szegö kernel of the cone

$$S(z) = \int_V e^{iz \cdot t}\, dt$$

(see, for example, [7]). This is the Laplace transform of the characteristic function of the cone, and it is an analytic function for $z \in T_V$. This is obvious since

$$|S(z)| \leqslant \int_V e^{-y \cdot t}\, dt = S(iy) \tag{8}$$

and the last integral is absolutely convergent [8]. Note that it follows from (8) by Lemma 1 that

$$S(iy) = Cs^{-1}(y). \tag{9}$$

(See Section 6 for further connections between the functions s and S.)

4. The Main Theorems

In the following theorems V is a homogeneous self-adjoint cone in $\mathbb{R}^n$ and $\tau = \tau(V)$ is the number defined in (7), the ratio of the rank(V) and the dimension of the space.

THEOREM 1. *Let* $1 < p \leqslant 2$. *Let* $2\tau > 1 + 1/p$ *and* $1 - \tau < \alpha < \tau - 1/p$. *If* $f \in L^p_\alpha(V)$, *then* $F = \mathscr{L}f \in H^{p'}_\alpha(T_V)$ *and*

$$\left(\int_{\mathbb{R}^n} |F(x+iy)\, S^\alpha(x+iy)|^{p'}\, dx\right)^{1/p'} \leqslant C\left(\int_V |f(t)|^p\, s^{\alpha p}(t)\, e^{-py\cdot t}\, dt\right)^{1/p}. \tag{10}$$

THEOREM 2. *Let* $1 < p \leqslant 2$. *Let* $1 - \tau < \alpha$ *and* $1/p' < \tau$. *If* $F \in H^p_{-\alpha}(T_V)$, *then there is an* $f \in L^{p'}_{-\alpha}(V)$ *such that* $F = \mathscr{L}f$ *and*

$$\left(\int_V |f(t)|^{p'}\, s^{-\alpha p'}(t)\, e^{-p'y\cdot t}\, dt\right)^{1/p'} \leqslant C\left(\int_{\mathbb{R}^n} |F(x+iy)\, S^{-\alpha}(x+iy)|^p\, dx\right)^{1/p}. \tag{11}$$

Remark. For the cone $\mathbb{R}^n_+$, when $\tau = 1$, the conditions connecting τ and p are automatically satisfied, and the bounds for α are the same as those in the one-dimensional case. In the general case, the values of p for which the theorems are valid depend on the rank of the cone.

The main steps in the proof of both theorems are provided in Section 6 (which shows that the weights in L^p and H^p spaces are related by the Laplace transform) and in Section 7 in which it is shown that the Riemann–Liouville integral, whose kernel is the function s^α, is bounded as an operator on weighted L^p spaces.

5. The Group of Automorphisms

We shall need additional properties of the group $G(V)$. Its elements are connected with the involution (6) by the formula

$$(Ax)^* = A'^{-1}x^* \tag{12}$$

for every $A \in G(V)$ and $x \in V$; since V is self-adjoint then A' (whose matrix is the transpose of A) belongs to $G(V)$ together with A.

We cite for future reference some further properties of $*$ (see [6])

$$s(x)s(x^*) = C \tag{13}$$

$$du^* = s^{-2}(u)\,du \tag{14}$$

$$x < y \Leftrightarrow y^* < x^*. \tag{15}$$

In Section 2 the stability group G_a of the fixed point of the involution was defined. Consider the quotient group $G(V)/G_a$. Every element (coset) of this quotient group is uniquely determined by a point $x \in V$ (such that a is mapped into x by this coset). This group is again transitive (that is the cone is homogeneous with respect to this group). We shall denote it again by $G(V)$ and consider this group only. Now for every $x, y \in V$ there is a unique $A \in G(V)$ such that $Ax = y$ (the group is simply transitive). Let $A(x)$ for $x \in V$ be the unique automorphism which sends a into x; that is $A(x)$ is defined by

$$A(x)a = x. \tag{16}$$

The following properties of $A(x)$ follow from its uniqueness,

$$|A(x)| = Cs(x) \tag{17}$$

$$A(x^*) = A'^{-1}(x). \tag{18}$$

In [6] it is also shown that every automorphism A can be represented in a unique way as a product

$$A = W \cdot B \tag{19}$$

where $W \in G_a$ and B is a positive definite transformation. This is a consequence of the general polar decomposition theorem for nonsingular linear transformations, and it is proved in the same way. It is sufficient to take for B the unique positive square root of $A \cdot A'$.

6. A Formula Relating the Weights

Lemma 2. *Let* $\alpha > -\tau$. *Then for* $z \in T_V$

$$\int_V e^{iz \cdot t} s^\alpha(t)\, dt = C S^{\alpha+1}(z). \tag{20}$$

Proof. When $\alpha > -\tau$ then the integral

$$\int_V |e^{iz \cdot t} s^\alpha(t)|\, dt = \int_V e^{-y \cdot t} s^\alpha(t)\, dt$$

is convergent. We shall postpone the proof of this fact until Section 9. Assuming this the proof of (20) is easy: Equation (20) is an equality between two functions analytic in T_V, so it is sufficient to prove this equality for $z = iy$ (i.e., $x = 0$),

$$\int_V e^{-y \cdot t} s^\alpha(t)\, dt = C S^{\alpha+1}(iy). \tag{21}$$

We claim that the function

$$\varphi(y) = \int_V e^{-y \cdot t} s^\alpha(t)\, dt$$

is homogeneous on the cone. This is easily proved (by a change of variables in the integral): $\varphi(Ay) = |A|^{-\alpha-1} \varphi(y)$, and then by Lemma 1

$$\int_V e^{-y \cdot t} s^\alpha(t)\, dt = C s^{-\alpha-1}(y). \tag{22}$$

On the other hand, we have already seen in (9) that

$$S^{\alpha+1}(iy) = C s^{-\alpha-1}(y)$$

which together with (22) proves (21). Thus the lemma is established.

7. The Riemann–Liouville Integral

Let $\beta < -\tau$. The Riemann–Liouville integral operator of order β is defined by

$$R_\beta f(x) = \int_{\langle 0, x \rangle} s^\beta(x-t) f(t)\, dt. \tag{23}$$

The adjoint operator is

$$R'_\beta f(x) = \int_{\langle t, \infty \rangle} s^\beta(x-t) f(x)\, dx, \tag{24}$$

where $\langle t, \infty \rangle = \{x \in V : t < x\}$. Note that (23) is an operator with a homogeneous kernel, which means that if we put $k(x, t) = s^\beta(x-t)$ then

$$k(Ax, At) = |A|^\beta k(x, t). \tag{25}$$

This fact is exploited in the next lemma. On the other hand, (23) is a convolution of two functions supported in V, i.e.,

$$R_\beta f(x) = (s^\beta * f)(x). \tag{26}$$

This is used in Section 8.

LEMMA 3. *Let* $\beta > -\tau$, $\alpha > -\tau$. *Then*

(a) $R_\beta s^\alpha(x) = C s^{\alpha+\beta+1}(x)$

(b) $R'_\beta s^{-\alpha-\beta-2}(t) = C s^{-\alpha-1}(t)$.

Proof. We postpone again until Section 9 the proof that for $\beta > -\tau$, $\alpha > -\tau$

$$R_\beta s^\alpha(x) = \int_{\langle 0, x \rangle} s^\beta(x-t)\, s^\alpha(t)\, dt \tag{27}$$

is convergent. Assuming this the proof of the lemma is easy (and similar to the proof of (22)). By changing the variables in the integral, we see that the function $R_\beta s^\alpha(x)$ is homogeneous of degree $\alpha + \beta + 1$ and the assertion (a) follows by Lemma 1.

Assertion (b) follows in the very same way, if we show that

$$R'_\beta s^\gamma(t) = \int_{\langle t, \infty \rangle} s^\beta(x-t)\, s^\gamma(x)\, dx < \infty \tag{28}$$

for every $t \in V$. Now we show, by making use of the properties of the group $G(V)$, that the integrals (27) and (28), with $\gamma = -\alpha - \beta - 2$, are equiconvergent. Then (28) follows when we prove (27) in Section 9.

Because of the homogeneity of all functions involved, it is sufficient to consider $t = a$ in (28). If in

$$R'_\beta s^\gamma(a) = \int_{\langle a, \infty \rangle} s^\beta(x-a)\, s^\gamma(x)\, dx$$

we change the variable $x=u^*$, then since the Jacobian is $s^{-2}(u)$, by (14), and $\langle a, \infty\rangle$ is mapped on $\langle 0, a^*\rangle = \langle 0, a\rangle$, by (15), we have

$$R'_\beta s^\gamma(a) = \int_{\langle 0,a\rangle} s^\beta(u^*-a)\, s^\gamma(u^*)\, s^{-2}(u)\, du. \tag{29}$$

Now, using the definition of the automorphism $A(x)$, (16) and (18), we write $s^\beta(u^*-a) = s^\beta(A(u^*)\,a - A(u^*)A(u)a) = |A(u^*)|^\beta\, s^\beta(a-u)$ which, by (18) and (17) equals $s^{-\beta}(u)\, s^\beta(a-u)$. Put this into (29) and use (13) to obtain

$$R'_\beta s^\gamma(a) = \int_{\langle 0,a\rangle} s^\beta(a-u)\, s^{-\beta-\gamma-2}(u)\, du = R_\beta s^\alpha(a).$$

This proves the assertion.

PROPOSITION 1. *Let $1 \leqslant p < \infty$. Let $\beta > -\tau$ and $\mu < \tau - 1/p$. Then*

$$\int_V \left|\frac{R_\beta f(x)}{s^{\beta+1}(x)}\right|^p s^{\mu p}(x)\, e^{-py\cdot x}\, dx$$

$$\leqslant C \int_V |f(x)|^p\, s^{\mu p}(x)\, e^{-py\cdot x}\, dx \tag{30}$$

for $y \in V$.

Proof. If we write

$$|R_\beta f(x)| \leqslant \int_{\langle 0,x\rangle} s^\beta(x-t)\,|f(t)|\, dt$$

$$= \int_{\langle 0,x\rangle} s^{\beta/p}(x-t)\,|f(t)|\, s^\lambda(t)\, s^{\beta/p'}(x-t)\, s^{-\lambda}(t)\, dt$$

and apply Hölder's inequality, then

$$|R_\beta f(x)| \leqslant \left(\int_{\langle 0,x\rangle} s^\beta(x-t)\,|f(t)|^p\, s^{\lambda p}(t)\, dt\right)^{1/p}$$

$$\times \left(\int_{\langle 0,x\rangle} s^\beta(x-t)\, s^{-\lambda p'}(t)\, dt\right)^{1/p'}.$$

We can apply Lemma 3(a) to the last integral if we choose λ so that

$$-\lambda p' > -\tau. \tag{31}$$

Then we shall have

$$|R_\beta f(x)|^p \leqslant C(s^{\beta-\lambda p'+1}(x))^{p/p'} \times \int_{\langle 0,x\rangle} s^\beta(x-t)\,|f(t)|^p\, s^{\lambda p}(t)\,dt.$$

If we substitute this into the left-hand side of (30), which we denote by I for short (since the exponents on s add to $-p(\beta+1)+\mu p+\beta p/p'-\lambda p+p/p'=\mu p-\lambda p-\beta-1$), we get

$$I\leqslant C\int_V s^{\mu p-\lambda p-\beta-1}(x)\,e^{-py\cdot x} \times \int_{\langle 0,x\rangle} s^\beta(x-t)\,|f(t)|^p\, s^{\lambda p}(t)\,dt\,dx$$

and by Fubini's theorem

$$I\leqslant C\int_V |f(t)|^p\, s^{\lambda p}(t)\int_{\langle t,\infty\rangle} e^{-py\cdot x}\, s^\beta(x-t)\, s^{\mu p-\lambda p-\beta-1}(x)\,dx\,dt.$$

Note that $e^{-py\cdot x}$ is monotone decreasing on V (that is if $t<x$, then $e^{-py\cdot x}<e^{-py\cdot t}$). Using this (and the definition of R'_β (25)) we obtain

$$I\leqslant C\int_V |f(t)|^p\, s^{\lambda p}(t)\, e^{-py\cdot t} R'_\beta s^{\mu p-\lambda p-\beta-1}(t)\,dt.$$

Thus assuming that

$$-\mu p+\lambda p-1>-\tau \tag{32}$$

we can apply Lemma 3(b) and obtain

$$I\leqslant C\int_V |f(t)|^p\, s^{\lambda p}(t)\, e^{-py\cdot t} s^{\mu p-\lambda p}(t)\,dt$$

$$=C\int_V |f(t)|^p\, s^{\mu p}(t)\,dt$$

which proves (30).

Thus all that is left to prove is that we can choose λ to satisfy both (31) and (32), i.e.,

$$\lambda<\tau/p' \qquad \text{and} \qquad \lambda>\mu+1/p-\tau/p$$

and this is possible, since by assumption the inequality $\mu + 1/p - \tau/p < \tau/p'$, which is equivalent to $\mu < \tau - 1/p$, is valid. This proves the proposition.

8. Proofs of the Theorems

The proof of both Theorems 1 and 2 is obtained by a reduction to the nonweighted case (to Theorem A and B, respectively).

As mentioned above (see (26)), the Riemann–Liouville operator is defined as a convolution, and it is well known that the Laplace transform transforms convolutions into products. In this case

$$\mathscr{L}(R_\beta f)(z) = \mathscr{L}(f * s^\beta)(z) = \mathscr{L}f(z)\, \mathscr{L}s^\beta(z)$$

and the equality obtained in Section 6 (Lemma 2) gives

$$\mathscr{L}(R_\beta f)(z) = \mathscr{L}f(z)\, S^{\beta+1}(z). \tag{33}$$

Proof of Theorem 1. We shall apply Proposition 1 to $f \in L^p_\alpha(V)$ and to the operator $R_{\alpha-1}$. Since $1 - \tau < \alpha < \tau - 1/p$, we can put $\beta = \alpha - 1$ and $\mu = \alpha$ in Proposition 1 and obtain

$$\int_V \left| \frac{R_{\alpha-1} f(x)}{s^\alpha(x)} \right|^p s^{\alpha p}(x)\, e^{-py\cdot x}\, dx$$
$$\leqslant C \int_V |f(x)|^p\, s^{\alpha p}(x)\, e^{-py\cdot x}\, dx.$$

Thus if we put $g = R_{\alpha-1} f$, we have obtained that

$$\int_V |g(x)|^p\, e^{-py\cdot x}\, dx$$
$$= C \int_V |f(x)|^p\, s^{\alpha p}(x)\, e^{-py\cdot x}\, dx. \tag{34}$$

And this means that Theorem A can be applied to g to see that its Laplace transform $\mathscr{L}g = G$ satisfies

$$\left(\int_{\mathbb{R}^n} |G(x+iy)|^{p'}\, dx \right)^{1/p'} \leqslant C \left(\int_V |g(x)|^p\, e^{-py\cdot x}\, dx \right)^{1/p}. \tag{35}$$

But by (33)

$$G(z) = \mathscr{L}g(z) = \mathscr{L}(R_{\alpha-1} f)(z) = \mathscr{L}f(z)\, S^\alpha(z) = F(z)\, S^\alpha(z)$$

and if we substitute this into the left-hand side of (35), then this inequality together with (34) yields (10), which proves the theorem.

Proof of Theorem 2. If $F \in H^p_{-\alpha}(T_V)$, then by definition $G = FS^{-\alpha}$ belongs to $H^p(T_V)$, so that Theorem B applied to the function G shows that G is the Laplace transform of some function $g \in L^{p'}(V)$ such that

$$\left(\int_V |g(t)|^{p'} e^{-p'y \cdot t}\, dt\right)^{1/p'} \leqslant C\left(\int_{\mathbb{R}^n} |G(x+iy)|^p\, dx\right)^{1/p}. \qquad (36)$$

But now for $F = GS^\alpha$ we have by (33)

$$F(z) = G(z)\, S^\alpha(z) = \mathscr{L}g(z)\, \mathscr{L}s^{\alpha-1}(z) = \mathscr{L}(g * s^{\alpha-1})(z) = \mathscr{L}(R_{\alpha-1}\, g)(z)$$

and if we put $R_{\alpha-1}\, g = f$ we obtain the representation $F(z) = \mathscr{L}f(z)$ for F, and we must prove only that $f \in L^{p'}_{-\alpha}(V)$. This follows again from Proposition 1, in which we put $\beta = \alpha - 1$ and $\mu = 0$,

$$\int_V \left|\frac{R_{\alpha-1}\, g(x)}{s^\alpha(x)}\right|^{p'} e^{-p'y \cdot x}\, dx \leqslant C \int_V |g(x)|^{p'} e^{-p'y \cdot x}\, dx;$$

in other words

$$\int_V |f(x)|^{p'} s^{-\alpha p'}(x)\, e^{-p'y \cdot x}\, dx \leqslant C \int_V |g(x)|^{p'} e^{-p'y \cdot x}\, dx$$

and this together with (36) (in which we put $G(z) = F(z)\, S^{-\alpha}(z)$) proves the theorem.

9. Evaluation of Some Integrals

In this section we indicate a proof of the convergence of integrals (21) and (27) and thus finish the proof of Lemmas 2 and 3. Both these integrals are of the form

$$\varphi(x) = \int_V k(x, t)\, s^\alpha(t)\, dt$$

(with $k(x, t) = e^{-x \cdot t}$ and $k(x, t) = s^\beta(x - t)\theta_{\langle 0, x \rangle}(t)$, respectively). In both cases φ is homogeneous, so it is sufficient to consider it for $x = a$. Thus we must prove

LEMMA 4. *If* $\alpha > -\tau$, *then*

$$\int_V k(a, t)\, s^\alpha(t)\, dt < \infty \tag{37}$$

for $k(a, t) = e^{-a \cdot t}$, *or* $k(a, t) = s^\beta(a - t)\,\theta_{\langle 0, a\rangle}(t)$ *with* $\beta > -\tau$.

Remark. It will be obvious from the proof that a similar lemma holds for some other kernels k.

Proof. (a) We first prove (37) in the simplest case when $\tau = 1$, i.e., $\operatorname{rank}(V) = n$. In this case the cone V is polyhedral, which means that there is a rotation $W \in O(n)$ such that $WV = \mathbb{R}^n_+$. The function $s_V(x)$ is then equal to $s_{\mathbb{R}^n_+}(Wx)$. Since this last function is given by $s_{\mathbb{R}^n_+}(y) = y_1 \cdots y_n$, and both kernels also decompose into products, the integral (37) reduces to an n product of Γ functions, or B functions, respectively, and the lemma is proved in this case.

(b) In the case $\operatorname{rank}(V) = k < n$, the stability subgroup G_a equals $O(m)$ with $m = n - k \geqslant 1$. By renumerating the coordinates we may assume that the space $\mathbb{R}^n$ is decomposed into $\mathbb{R}^{k-1} \times \mathbb{R}^{m+1}$ so that the space $\mathbb{R}^{m+1}$ is invariant under $O(m)$ (then $\mathbb{R}^{k-1}$ is fixed under $O(m)$). By the action of this group every point in $\mathbb{R}^{m+1}$ is brought to the first coordinate line, say, which means that every point $x \in V \subset \mathbb{R}^n$ is brought to $\bar{x} \in V \cap \mathbb{R}^k = V_k$.

By (19) every $A \in G(V)$ is represented as $A = W \cdot B$, where B is positive definite and $W \in G_a$. Now the group G_a acts on positive definite transformations B by the formula $B \to U \cdot B \cdot U' = B_1$ ($U \in G_a$); so by the action of this group every positive definite B is transformed to (a positive definite) B_1 which acts on V_k. It is not difficult to see that these transformations form a transitive group for V_k, and this is sufficient to show that V_k is polyhedral.

Now the integral (37) can be written as

$$\int_V k(a, t)\, s^\alpha(t)\, dt = \int_{V_k} \int_{O(m)} k(a, U\bar{t})\, s^\alpha(U\bar{t})\, dU\, d\bar{t}, \tag{38}$$

where dU is the invariant measure of the group $O(m)$ and $\bar{t} \in V_k$. Since the integrand is invariant under the rotations (which belong to the group of the cone),

$$\begin{aligned} k(a, U\bar{t})\, s^\alpha(U\bar{t}) &= k(Ua, U\bar{t})\, s^\alpha(U\bar{t}) \\ &= |U|^{\kappa + \alpha}\, k(a, \bar{t})\, s^\alpha(\bar{t}) = k(a, \bar{t})\, s^\alpha(\bar{t}) \end{aligned}$$

(here $\kappa = 0$ for the first kernel, and $\kappa = \beta$ for the second), (38) is equal to

$$\int_{V_k} k(a, \bar{t})\, s^{\alpha}(\bar{t})\, d\bar{t}, \tag{39}$$

where $k(a, \bar{t})$ and $s(\bar{t}) = s_V(\bar{t})$ are the restrictions of these functions to the cone V_k. Now V_k has its own norm s_{V_k} and it follows from homogeneity considerations that these two functions are related as

$$s_V(\bar{t}) = (s_{V_k}(\bar{t}))^{n/k}, \qquad \bar{t} \in V_k.$$

By putting this into (39) we obtain for the Riemann–Liouville operator the integral

$$\int_{\langle 0, a \rangle} (s_{V_k}(a - \bar{t}))^{\beta n/k}\, (s_{V_k}(\bar{t}))^{\alpha n/k}\, d\bar{t}$$

and since V_k is polyhedral, (a) shows that this integral converges for $\beta n/k > -1$ and $\alpha n/k > -1$. For $k(a, t) = e^{-a \cdot t}$ the proof is finished in a similar way.

Remarks. (i) The assumption that the cone is homogeneous is essential in all our considerations. On the other hand, the assumption of self-adjointness was made only in order to simplify matters. The group of the non-self-adjoint cone is more complicated, but otherwise the same methods should apply to the non-self-adjoint case also. In this case the Laplace transform transforms functions on V into functions on T_{V^*} (see [7]). For other properties of the cones see [3, 6, 2, 8].

(ii) In [1] and [5] the analogues of inequalities (10) and (11) are given between L^p and H^q spaces, with $p \leqslant q$; we have treated the simpler case $q = p'$ only. To obtain this generalization to different p and q, an (L^p, L^q) inequality for the Riemann–Liouville integral should be used, instead of Proposition 1. This kind of inequality can be proved in a similar way, only instead of Hölder's inequality Young's inequality must be used. In [4] we have considered weighted norm inequalities similar to Proposition 1 for a whole class of operators with homogeneous kernels (25).

(iii) We see that our theorems are more appropriate for cones which have larger rank (i.e., closer to n). The reason is that the function $s(x)$ is defined in the same manner for all cones. It seems that some function dependent on the rank of the cone would give better results, only this function would not be homogeneous (since s is the only one) and this would complicate matters considerably.

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