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Note

On the diameter of a domination dot-critical graph[☆]

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ABSTRACT

A graph G is domination dot-critical, or just dot-critical, if contracting any edge decreases the domination number. It is totally dot-critical if identifying any two vertices decreases the domination number. In this paper, we study an open question concerning of the diameter of a domination dot-critical graph G.

1. Introduction

A vertex of a graph G *dominates* itself and its neighbors. A set of vertices S in a graph G is a *dominating set* if each vertex of $V(G) \setminus S$ is dominated by some vertices of S. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. We denote the *open neighborhood* of a vertex v by $N(v)$ and its closed neighborhood by $N[v]$. We indicate that for two vertices u and v, v is adjacent to u by writing $v \perp u$. A vertex v of a graph G is a *critical vertex* if $\gamma(G - v) < \gamma(G)$. A graph G is *vertex-critical* if any vertex of G is a critical vertex; see [3].

For a pair of vertices v, u of a graph G, we denote as $G.(vu)$ the graph obtained by identifying v and u, and (vu) denotes the identified vertex. So, $G.(vu)$ may be viewed as the graph obtained from G by deleting the vertices v and u and appending a new vertex, labeled by (vu), that is adjacent to all the vertices of $G - v - u$ that were originally adjacent to either of v or u. In the case where v is adjacent to u, $G.(vu)$ is the graph obtained by contracting vu.

Burton and Sumner [1] introduced a new critical condition for the domination number. A graph G is called *domination dot-critical*, or simply *dot-critical*, if contracting any edge decreases the domination number, i.e., $\gamma(G.(vu)) < \gamma(G)$, for any two adjacent vertices v and u. It is *totally dot-critical* if identifying any two vertices decreases the domination number. If G is dot-critical and $\gamma(G) = k$, then G is called k-dot-critical.

Let G' be the set of critical vertices of G. In [1], it is proved that a connected 3-dot-critical graph G with $G' = \emptyset$ has a diameter of at most 3, and a connected totally 3-dot-critical graph G with $G' = \emptyset$ has a diameter of at most 2. The upper bound of the diameter of a k-dot-critical graph G with $G' = \emptyset$, for $k \geq 4$, is left as an open question.

Question 1 (*[1]*). *What are the best bounds for the diameter of a k-dot-critical graph and a totally k-dot-critical graph G with $G' = \emptyset$, for $k \geq 4$?*

Chengye et al. [2] studied total domination dot-critical graphs with no critical vertices, and proved that a connected 4-dot-critical graph G with $G' = \emptyset$ has a diameter of at most 5.

We study Question 1 and give an upper bound for the diameter of a k-dot-critical graph G with $G' = \emptyset$, for $k \geq 5$. We make use of the following:

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Theorem A *([1]). If G is a dot-critical graph and $N[v] \subseteq N[u]$, then $v \in G'$.*

Theorem B *([1]). Let $a, b \in V(G)$ for a graph G. Then $\gamma(G.(ab)) < \gamma(G)$ if and only if either there exists a minimum dominating set S of G such that $a, b \in S$ or at least one of a or b is critical in G.*

2. Main results

Let $k \geq 5$ be an integer. In this section, we determine the maximum diameter of a k-dot-critical graph G with $G' = \emptyset$. First we have the following lemma.

Lemma 1. *Let S be a minimum dominating set in a k-dot-critical graph G with $G' = \emptyset$, $k \geq 5$. Let x be a diameterical vertex, and for $i = 0, 1, 2, \ldots, d$, let V_i denote the set of all vertices of G at distance i from x. If $V_d \cap S = \emptyset$, then $|S \cap (V_{d-2} \cup V_{d-1})| \geq 2$.*

Proof. Let G be a k-dot-critical graph with $G' = \emptyset$, for $k \geq 5$. Let $x, y \in V(G)$ and $d(x, y) = \mathrm{diam}(G) = d$. For $i = 0, 1, 2, \ldots, d$, let V_i denote the set of all vertices of G at distance i from x. So, $V_0 = \{x\}$ and $y \in V_d$. Let S be a minimum dominating set, and let $V_d \cap S = \emptyset$. Assume to the contrary that $|S \cap (V_{d-2} \cup V_{d-1})| \leq 1$. Then $|V_{d-1} \cap S| = 1$. There is a vertex $v_{d-1} \in V_{d-1} \cap S$ such that v_{d-1} dominates y. It follows that $N[y] \subseteq N[v_{d-1}]$. But then Theorem A implies that $y \in G'$, a contradiction. ■

Theorem 2. *A connected k-dot-critical graph G with $G' = \emptyset$ has a diameter of at most the values given by the following:* $\begin{cases} 7, & k = 5 \\ 3k - 9, & k \geq 6 \end{cases}$.

Proof. Suppose G is a k-dot-critical graph with $G' = \emptyset$, for $k \geq 5$. Let $x, y \in V(G)$ and $d(x, y) = \mathrm{diam}(G) = d$. For $i = 0, 1, 2, \ldots, d$, let V_i denote the set of all vertices of G at distance i from x. So, $V_0 = \{x\}$ and $y \in V_d$. Assume that $d \geq 5$. Let $v_4 \in V_4$ and $v_5 \in V_5$ be two adjacent vertices. According to Theorem B, there exists a minimum dominating set S containing both v_4 and v_5. It follows from Theorem A, that $|S \cap (V_0 \cup V_1 \cup \cdots \cup V_5)| \geq 4$.

For $k = 5$ we may assume that $|S \cap (V_0 \cup V_1 \cup \cdots \cup V_5)| = 4$. This implies that $|S \setminus (V_0 \cup V_1 \cup \cdots \cup V_5)| = 1$, and so $d \leq 8$. Suppose that $d = 8$. By Lemma 1, we conclude that $S \cap V_8 \neq \emptyset$. It follows that v_5 is adjacent to any vertex in V_6. Let $u_6 \in V_6 \cap N(v_5)$. There is a minimum dominating set S_0 in G containing both v_5 and u_6. It is obvious that $S_0 \cap (V_7 \cup V_8) \neq \emptyset$. Furthermore, $S_0 \cap V_4 = \emptyset$. Now $(S_0 \cap (V_0 \cup \cdots \cup V_5)) \cup (S \cap V_8)$ is a dominating set for G of size less than 5. This is a contradiction. So $d \leq 7$.

So, we henceforth let $k > 5$. Without loss of generality we assume that $d > 8$. Then, $V_7 \cup V_8 \cup \cdots \cup V_d$ is dominated by at most $k - 4$ vertices of S. Since any vertex of S can dominate at most $V_i \cup V_{i+1} \cup V_{i+2}$ for some integer i, we conclude by Lemma 1 that $d \leq 6 + 3(k - 4) - 1 = 3k - 7$.

Suppose that $d = 3k - 7$. It follows that $|S \cap V_d| = |S \cap V_{d-3}| = 1$, and $S \cap (V_{d-2} \cup V_{d-1}) = \emptyset$. Let $v_{d-3} \in S \cap V_{d-3}$ and $v_{d-2} \in V_{d-2}$ be adjacent to v_{d-3}. There is a minimum dominating set S_1 containing v_{d-3} and v_{d-2}. According to Theorem A, $|S_1 \cap (V_{d-2} \cup V_{d-1} \cup V_d)| \geq 2$. Now, $(S_1 \cap (V_0 \cup \cdots \cup V_{d-3})) \cup (S \cap V_d)$ is a dominating set for G of size less than k, a contradiction. Hence, $d \leq 3k - 8$.

Suppose now that $d = 3k - 8$.

For $k = 6$ we have $d = 10$. If $S \cap V_8 \neq \emptyset$, then $S \cap (V_6 \cup V_7) = \emptyset$. As a result v_5 is adjacent to any vertex in V_6. Let $v_6 \in V_6 \cap N(v_5)$. There is a minimum dominating set D_1 containing both v_5 and v_6. It follows that $|D_1 \cap (V_6 \cup \cdots \cup V_{10})| \geq 3$. This leads to $(D_1 \cap (V_0 \cup \cdots \cup V_5)) \cup (S \cap (V_6 \cup \cdots \cup V_{10}))$ being a dominating set for G of size at most 5. This is a contradiction. So, $S \cap V_8 = \emptyset$. But then by Lemma 1, $|S \cap V_{10}| = |S \cap V_7| = 1$. As an immediate result, the vertex in $S \cap V_{10}$ is adjacent to any vertex in V_9, and the vertex in $S \cap V_7$ is adjacent to any vertex in V_8. Let $v_7 \in V_7 \cap S$, and let $v_8 \in V_8 \cap N(v_7)$. There is a minimum dominating set D_2 containing both v_7 and v_8. It follows that $|D_2 \cap (V_8 \cup V_9 \cup V_{10})| \geq 2$. Now, $(D_2 \cap (V_0 \cup \cdots \cup V_7)) \cup (S \cap V_{10})$ is a dominating set for G of size at most 5, a contradiction.

For $k = 7$ we have $d = 13$. If $S \cap V_{11} \neq \emptyset$, then $S \cap (V_{12} \cup V_{13}) \neq \emptyset$. It follows that $|S \cap V_8| = 1$ and $|S \cap (V_7 \cup V_9 \cup V_{10})| = 0$. Assume that $v_8 \in S \cap V_8$. Then v_8 is adjacent to any vertex in $V_7 \cup V_9$. Let $v_9 \in V_9 \cap N(v_8)$. There is a minimum dominating set F_1 containing both v_8 and v_9. It follows that $|F_1 \cap (V_9 \cup \cdots \cup V_{13})| \geq 3$. But then $(F_1 \cap (V_0 \cup \cdots \cup V_8)) \cup (S \cap (V_9 \cup \cdots \cup V_{13}))$ is a dominating set for G of size at most 6. This is a contradiction. So, $S \cap V_{11} = \emptyset$. By Lemma 1, we may assume that $|S \cap V_{10}| = 1$, and $|S \cap V_9| = |S \cap V_{12}| = 0$. Let $v_{10} \in V_{10} \cap S$, and let $v_{11} \in V_{11} \cap N(v_{10})$. There is a minimum dominating set F_2 containing both v_{10} and v_{11}. It is obvious that $F_2 \cap (V_{12} \cup V_{13}) \neq \emptyset$. Now, $(F_2 \cap (V_0 \cup \cdots \cup V_{10})) \cup (S \cap V_{13})$ is a dominating set for G of size at most 6, a contradiction.

In the rest of the proof we suppose that $k \geq 8$. We proceed with Claim 1.

Claim 1. $|S \cap (V_{d-5} \cup \cdots \cup V_d)| \leq 3$.

To see this, assume to the contrary that $|S \cap (V_{d-5} \cup \cdots \cup V_d)| \geq 4$. Then, $V_7 \cup V_8 \cup \cdots \cup V_{d-7}$ is dominated by at most $k - 8$ vertices of S. But any $k - 8$ vertices of S dominate the vertices of at most $3k - 24$ sets among $V_7, V_8, \ldots, V_{d-7}$, while $(d - 7) - 7 + 1 = 3k - 21$, a contradiction.

Now, $S \cap (V_{d-5} \cup V_{d-4} \cup V_{d-3}) \neq \emptyset$. We need to consider the following cases:

Case 1: $S \cap V_{d-5} \neq \emptyset$. Let $v_{d-5} \in S \cap V_{d-5}$, and let $v_{d-4} \in V_{d-4}$ be adjacent to v_{d-5}. According to Theorem B, there is a minimum dominating set S' containing v_{d-5}, v_{d-4}. It follows that $\left|S' \cap (V_{d-5} \cup \cdots \cup V_d)\right| \geq 4$. Now, $(S' \cap (V_0 \cup \cdots \cup V_{d-5})) \cup (S \cap (V_{d-4} \cup \cdots \cup V_d))$ is a dominating set for G of size less than k, a contradiction.

Case 2: $S \cap V_{d-5} = \emptyset$. By Claim 1, $|S \cap (V_{d-4} \cup \cdots \cup V_d)| \leq 3$.
If $|S \cap (V_{d-4} \cup \cdots \cup V_d)| = 3$, then $V_7 \cup V_8 \cup \cdots \cup V_{d-6}$ is dominated by at most $k-7$ vertices of S. But any $k-7$ vertices of S dominate the vertices of at most $3k-21$ sets among $V_7, V_8, \ldots, V_{d-6}$, while $d-6-7+1 = 3k-20$, a contradiction. So we suppose that $|S \cap (V_{d-4} \cup \cdots \cup V_d)| = 2$. But then by Lemma 1, $|S \cap V_{d-3}| = |S \cap V_d| = 1$. Let $u_{d-3} \in S \cap V_{d-3}$. As a result, u_{d-3} is adjacent to any vertex in $V_{d-4} \cup V_{d-2}$. Let $u_{d-2} \in V_{d-2} \cap N(u_{d-3})$. There is a minimum dominating set S'_1 containing both u_{d-3} and u_{d-2}. It is obvious that $\left|S'_1 \cap (V_{d-3} \cup \cdots \cup V_d)\right| \geq 3$. Now, $(S'_1 \cap (V_0 \cup \cdots \cup V_{d-5})) \cup (S \cap (V_{d-3} \cup \cdots \cup V_d))$ is a dominating set for G of size less than $\gamma(G)$, a contradiction.

Hence, $d \leq 3k-9$. ■

It follows from Theorem 2, and the results in [1,2], that a connected k-dot-critical graph G with $G' = \emptyset$ has a diameter of at most $2k-3$ for $k \in \{3, 4, 5, 6\}$. In the next theorem we show that the diameter of a 2-connected k-dot-critical graph G with $G' = \emptyset$ is at most $2k-3$, for $k \geq 7$.

Theorem 3. *A 2-connected k-dot-critical graph G with $G' = \emptyset$ has a diameter of at most $2k-3$, $k \geq 7$.*

Proof. Let G be a 2-connected k-dot-critical graph with $G' = \emptyset$ for $k \geq 7$. Let $x, y \in V(G)$ and $d(x, y) = \text{diam}(G) = d$. For $i = 0, 1, 2, \ldots, d$, let V_i denote the set of all vertices of G at distance i from x. So, $V_0 = \{x\}$ and $y \in V_d$. Since G is 2-connected, $|V_i| \geq 2$, for $i = 1, 2, \ldots, d-1$. The following fact follows from Theorem A.

Fact 2. If $|S \cap V_i| = 1$ for some integer $7 \leq i \leq d-1$, then $|S \cap (V_{i-1} \cup V_i \cup V_{i+1})| \geq 2$.

Let $v_4 \in V_4$ and $v_5 \in V_5$ be two adjacent vertices. According to Theorem B, there exists a dominating set S containing both v_4 and v_5. For $V_0 \cup V_1 \cup \cdots \cup V_5$ to be dominated by S, it follows that $|S \cap (V_0 \cup V_1 \cup \cdots \cup V_5)| \geq 4$. So, by Fact 2, for any integer $6 \leq i \leq d-3$, $|S \cap (V_i \cup V_{i+1} \cup V_{i+2} \cup V_{i+3})| \geq 2$. If $d = 6 + 4j + r$ where $0 \leq r \leq 3$, then $k \geq 4 + 2j + \lceil \frac{r}{2} \rceil$. Now, it follows that for $r \in \{1, 3\}$, $d \leq 2k-3$, and for $r \in \{0, 2\}$, $d \leq 2k-2$. This completes the proof for $r \in \{1, 3\}$.

Suppose that $d = 2k-2$ for $r \in \{0, 2\}$. We distinguish the following cases, according to the value of r.

Case 1: $r = 0$. It follows that $|S \cap V_{d-5}| = 1$ and $|S \cap (V_{d-5} \cup V_{d-4} \cup \cdots \cup V_d)| = 3$. Let $v_{d-5} \in S \cap V_{d-5}$ and $v_{d-4} \in V_{d-4}$ be adjacent to v_{d-5}. By Theorem B, there is a minimum dominating set S_1 such that $v_{d-4}, v_{d-5} \in S_1$. It follows that $|S_1 \cap (V_{d-5} \cup V_{d-4} \cup \cdots \cup V_d)| = 4$. Now, $(S_1 \cap (V_0 \cup V_1 \cup \cdots \cup V_{d-5})) \cup (S \cap (V_{d-4} \cup \cdots \cup V_d))$ is a dominating set for G of size less than k, a contradiction. Hence, $d \leq 2k-3$.

Case 2: $r = 2$. It follows from Lemma 1 that $|S \cap V_d| = |S \cap V_{d-4}| = |S \cap V_{d-3}| = 1$, and $S \cap (V_{d-1} \cup V_{d-2}) = \emptyset$. Let $v_{d-3} \in S \cap V_{d-3}$, and let $v_{d-2} \in V_{d-2}$ be adjacent to v_{d-3}. There is a minimum dominating set S_2 such that $v_{d-3}, v_{d-2} \in S_2$. Then, $|S_2 \cap (V_{d-3} \cup V_{d-2} \cup V_{d-1} \cup V_d)| = 3$. Now, $(S_2 \cap (V_0 \cup V_1 \cup \cdots \cup V_{d-3})) \cup (S \cap V_d)$ is a dominating set for G of size less than k, a contradiction. Hence, $d \leq 2k-3$. ■

Since every totally k-dot-critical graph is k-dot-critical, the bounds in Theorems 2 and 3 hold for connected totally k-dot-critical graphs. We believe that any connected k-dot-critical graph G with $G' = \emptyset$ is 2-connected. We close with the following question.

Question. *Is it true that a connected k-dot-critical graph G with $G' = \emptyset$ is 2-connected?*

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