Scalarization and saddle points of approximate proper solutions in nearly subconvexlike vector optimization problems

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1. Introduction

It is well known that saddle point assertions play an important role in scalar optimization due to their relations with other fundamental tools and theories such as Kuhn–Tucker optimality conditions, duality, minimax theory, etc. (see [14]).

Motivated by these facts, different authors have also obtained exact and approximate saddle point results for cone convex vector optimization problems with single and set-valued mappings by considering nondominated and Benson proper solutions (see [1, 2, 5–7, 9, 11, 17–19, 23–25, 28, 29, 31, 32] and the references therein). The reader can find some of these scalarization results.

In this paper we focus on linear scalarizations and saddle point theorems for a kind of approximate Benson-proper solutions due to Gao et al. [10] of a generalized convex cone constrained single-valued vector optimization problem. To be precise, we consider nearly subconvexlikeness assumptions on the objective and the cone constraint mappings. Approximate saddle point theorems that characterize suboptimal solutions of convex scalar optimization problems with inequality and equality constraints have been obtained in [8, 26]. In [9] (resp. [31, 11]), these results were stated in convex Pareto (resp. single-valued vector) optimization problems with inequality constraints (resp. equality and cone constraints) for approximate weak solutions in the Kudateladze sense (resp. approximate solutions and approximate weak solutions in

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the Kutateladze sense, and approximate solutions in the Vályi sense, see [16,12,13]) via a scalar (resp. single-valued vector) Lagrangian function. In these papers, the complementary slack condition is not bounded (see [31, Remark 3.1]).

In [5,28] the authors derive approximate saddle point assertions for vector-valued location and approximation problems using a vector Lagrangian mapping which is affine in each variable.

In the last years, some of these approximate saddle point results for single valued vector optimization problems have been extended to vector optimization problems with set-valued mappings. For example, in [24] the authors consider a subconvexlike vector optimization problem with set-valued maps in real locally convex Hausdorff topological vector spaces and derive scalarization results and sufficient approximate saddle point assertions for approximate weak solutions in the Kutateladze sense. As in the previous references, the complementary slack condition is not bounded.

Recently, Gao et al. [10] introduced a notion of approximate proper efficient solution in the Benson sense of a vector optimization problem, which is motivated by the e-efficiency concepts defined by ourselves in [12,13]. By assuming that the problem is subconvexlike, the authors characterize these approximate proper solutions through approximate solutions of scalar optimization problems, but the necessary and sufficient conditions have not the same error.

In this work we have two objectives. First, to improve this characterization in order to obtain the same error in the necessary and sufficient conditions and also to extend it to an "approximate" new type of nearly subconvexlike problems. This new class of vector optimization problems is wider than the usual nearly subconvexlike vector optimization problems and so includes the subconvexlike problems. Second, to complete it with approximate proper saddle point assertions, where the complementary slack condition is bounded. For this last aim we introduce a set-valued Lagrangian and a new concept of approximate proper saddle point.

Our results work with ordering cones non necessarily solid (i.e., their topological interior can be empty). Moreover they reduce to well known scalarization results and proper saddle point theorems for exact solutions, some of which are obtained under weaker assumptions.

The paper is structured as follows. In Section 2, some basic notations are fixed and the vector optimization problem is introduced. Moreover, some notions of approximate efficiency and approximate proper efficiency are recalled. In Section 3, scalarization results for Benson approximate efficient solutions are obtained under nearly subconvexlikeness assumptions. In Section 4, scalar Lagrangian necessary and sufficient optimality conditions for Benson approximate efficient solutions are established. Also in this section approximate proper saddle point theorems for a set-valued Lagrangian based on approximate Benson efficient solutions are obtained. Finally, we state the conclusions.

2. Preliminaries

Let X be an arbitrary nonempty decision set and let Y and Z be two ordered locally convex Hausdorff topological vector spaces with ordering cones D and K, respectively. The topological dual spaces of Y and Z are denoted by Y* and Z*, respectively. We assume the weak* topology σ(Y*,Y) in Y*.

We suppose that D is a pointed (D ∩ (−D) = {0}) closed convex cone and K is a convex cone with nonempty topological interior and proper (|0) ≠ K ≠ |0). As usual, the preference relation in Y is given by D as follows:

\[ y_1, y_2 ∈ Y, \quad y_1 ≤_D y_2 ⇐⇒ y_2 − y_1 ∈ D. \]

A similar relation is defined in Z through the cone K.

Given a set A ⊂ Y, we denote by int A, cl A, A′ and cone A the topological interior, the closure, the complement and the cone generated by A, respectively, and we say that A is solid if int A ≠ ∅. Moreover, we denote the positive polar cone and the strict positive polar cone of D by D+ and D++, respectively, i.e.,

\[ D^+ = \{ μ ∈ Y^*: μ(d) ≥ 0, ∅d ∈ D \}, \]
\[ D^{++} = \{ μ ∈ Y^*: μ(d) > 0, ∅d ∈ D \setminus \{0\} \}. \]

We denote by \( \mathbb{R}^p_+ \) the nonnegative orthant of \( \mathbb{R}^p \) and \( \mathbb{R}_+ = \mathbb{R}^1_+ \).

For each set \( C ⊂ D \setminus \{0\} \), we define the set-valued mapping \( C : \mathbb{R}_+ → 2^D \) as follows:

\[ C(ε) = \begin{cases} εC & \text{if } ε > 0, \\ \text{cone } C \setminus \{0\} & \text{if } ε = 0. \end{cases} \]

Given \( q ∈ D \setminus \{0\} \) and \( μ ∈ D^{++} \), we denote

\[ C_q = q + D, \]
\[ C_μ = B_μ + (D \setminus \{0\}). \]

where \( B_μ = \{ d ∈ D: μ(d) = 1 \}. \)

In order to deal with approximate maximal points of a nonempty set \( Q ⊂ Y \), the following notion will be considered (see [12,13]). We say that \( y_0 ∈ Q \) is a \( (C, ε) \)-maximal point of \( Q \), denoted by \( y_0 ∈ \text{Max}(Q, C, ε) \), if \( Q − y_0) \cap C(ε) = \emptyset. \) If \( C = D \setminus \{0\} \) or \( ε = 0 \) and \( C(0) = D \setminus \{0\} \) then we say that \( y_0 \) is a maximal point of \( Q \) and we denote it by \( y_0 ∈ \text{Max}(Q, D). \)
In this paper, we study the vector optimization problem:

\[
\min f(x) \quad \text{subject to } x \in S, \tag{P}
\]

where \( f: X \to Y \) and the feasible set \( S \subseteq X \) is defined by a cone constraint \( g(x) \in -K \), i.e.,

\[
S = \{ x \in X : g(x) \in -K \},
\]

with \( g: X \to Z \). Let us recall that problem \((P)\) satisfies the Slater constraint qualification if there exists \( x \in X \) such that \( g(x) \in -\text{int} K \). We say that \((P)\) is a Pareto problem if \( Y = \mathbb{R}^p \) and \( D = \mathbb{R}^p. \) In this case we denote \( f = (f_1, f_2, \ldots, f_p) \).

The next approximate version of Benson proper efficiency is due to Gao et al. [10]. It is motivated by an \( \varepsilon \)-efficiency notion due to Gutiérrez et al. (see [12,13]).

**Definition 2.1.** Let \( C \subseteq D \setminus \{0\} \) and \( \varepsilon \geq 0 \). A point \( x_0 \in S \) is a \((C, \varepsilon)\)-proper solution of \((P)\), denoted by \( x_0 \in \text{Be}(f, S, C, \varepsilon) \), if

\[
\text{cl}(\text{cone}(f(S) + C(\varepsilon) - f(x_0))) \cap (-D) = \{0\}. \tag{1}
\]

**Remark 2.2.** (a) It is obvious that

\[
\text{cl}(\text{cone}(f(S) + C(0) - f(x_0))) = \text{cl}(\text{cone}(f(S) + \text{cl}(\text{cone} C) - f(x_0))).
\]

Thus, if \( \text{cl}(\text{cone} C) = D \) then \( \text{Be}(f, S, C, 0) \) is the well known set of Benson proper solutions of problem \((P)\) (see [3]), that we denote as \( \text{Be}(f, S, D) \). In particular, this happens if we consider \( q \in D \setminus \{0\} \) and \( C = C_q \).

(b) Let us observe that in the original \((C, \varepsilon)\)-proper solution concept (see [10, Definition 3.3 and Remark 2(ii)]), Gao et al. considered the following statement instead of \((1)\):

\[
\text{cl}(\text{cone}(f(S) + C(\varepsilon) - f(x_0))) \cap (-C(0)) = \emptyset. \tag{2}
\]

It is obvious that statement \((1)\) reduces to \((2)\) by taking the ordering cone \( D' = C(0) \cup \{0\} \). Reciprocally, statement \((2)\) reduces to \((1)\) by considering a set \( C \) such that generates \( D \), i.e., \( C(0) = D \setminus \{0\} \). So, Definition 2.1 and Definition 3.3 in [10] are equivalent.

In this work, the following new notion of generalized convexity for a vector-valued mapping will be considered. Roughly speaking, it is an “approximate” version of the well known notion of nearly subconvexlikeness due to Yang et al. [32], in the sense that the points near to the boundary of the image set cannot satisfy the condition of nearly subconvexlikeness.

**Definition 2.3.** Consider \( \varepsilon \geq 0 \) and \( C \subseteq D \setminus \{0\} \). The mapping \( f \) is said to be nearly \((C, \varepsilon)\)-subconvexlike on a nonempty set \( M \subseteq X \) if \( \text{cl}(\text{cone}(f(M) + C(\varepsilon))) \) is convex.

**Remark 2.4.** The above notion reduces to the concept of nearly subconvexlikeness (see [25,32]) when \( C \cup \{0\} \) is a cone or \( \varepsilon = 0 \). In both cases we say that \( f \) is nearly \( G \)-subconvexlike on \( M \), where \( G = C(0) \cup \{0\} \). The following example shows that the nearly \((C, \varepsilon)\)-subconvexlikeness is weaker than the usual nearly subconvexlikeness.

Consider \( X = Y = \mathbb{R}^2 \), \( D = \mathbb{R}^2_+ \), \( x_0 = (1, 1) \), \( f(x) = x - x_0 \) for all \( x \in X \) and \( M = \{ x \in \mathbb{R}^2_+: ||x||_\infty \geq 1 \} \). It is clear that

\[
\text{cl}(\text{cone}(f(M) + \mathbb{R}^2_+)) = \mathbb{R}^2_+ - \text{int} \mathbb{R}^2_+
\]

is not a convex set and so \( f \) is not nearly \( \mathbb{R}^2_+ \)-subconvexlike on \( M \). However, for \( \varepsilon = 1 \) and \( C = \{ x \in \mathbb{R}^2_+: ||x||_1 \geq 1 \} \), we have that

\[
\text{cl}(\text{cone}(f(M) + C)) = \{ (x_1, x_2) \in \mathbb{R}^2_+: x_1 + x_2 \geq 0 \},
\]

which is convex. Thus, \( f \) is nearly \((C, 1)\)-subconvexlike on \( M \).

3. Linear scalarizations

Given a scalar function \( h: X \to \mathbb{R} \), a nonempty set \( M \subseteq X \) and \( \varepsilon \geq 0 \), we denote

\[
\varepsilon-\text{argmin}_M h = \{ x \in M : h(x) - \varepsilon \leq h(z), \ \forall z \in M \}
\]

and \( \text{argmin}_M h := 0-\text{argmin}_M h \). Let us observe that the elements of the set \( \varepsilon-\text{argmin}_M h \) are the suboptimal solutions with error \( \varepsilon \) (exact solutions if \( \varepsilon = 0 \)) of the following scalar optimization problem:

\[
\min h(x) \quad \text{subject to } x \in M.
\]

In this section, the \((C, \varepsilon)\)-proper solutions of \((P)\) are characterized through suboptimal solutions of associated scalar optimization problems with nearly \((C, \varepsilon)\)-subconvexlikeness assumptions.
Lemma 3.1. Let \( x_0 \in S, \varepsilon \geq 0 \) and a nonempty set \( C \subset D \setminus \{0\} \). Suppose that \( \text{int } D^+ \neq \emptyset \) and \( f - f(x_0) \) is nearly \((C, \varepsilon)\)-subconvexlike on \( S \). If \( x_0 \in \text{Be}(f, S, C, \varepsilon) \), then there exist \( \mu \in D^+ \) and an open half space \( H \) defined by \( H = \{ y \in Y: \mu(y) > 0 \} \), such that

(a) \( \text{cl cone}(f(S) + C(\varepsilon) - f(x_0)) \cap (-H) = \emptyset \).

(b) \( \text{cl cone}(f, g)(X) + (C(\varepsilon) \times K) - (f(x_0), 0)) \cap (-H \times -\text{int } K) = \emptyset \).

Proof. (a) As \( x_0 \in \text{Be}(f, S, C, \varepsilon) \) it follows that

\[
\text{cl cone}(f(S) + C(\varepsilon) - f(x_0)) \cap (-D) = \emptyset. \tag{3}
\]

Since \( f - f(x_0) \) is nearly \((C, \varepsilon)\)-subconvexlike on \( S \), we have that \( \text{cl cone}(f(S) + C(\varepsilon) - f(x_0)) \) is a closed convex cone. Then, by applying Theorem 3.22 in [15] to statement (3) we deduce that there exists a functional \( \mu \in Y^* \setminus \{0\} \) such that

\[
\mu(y) \geq 0, \quad \forall y \in \text{cl cone}(f(S) + C(\varepsilon) - f(x_0)),
\]

\[
\mu(d) > 0, \quad \forall d \in D \setminus \{0\}.
\]

By the second inequality \( \mu \in D^+ \) and the proof of this part is complete by taking the open half space \( H \) associated to \( \mu \).

(b) Suppose on the contrary that there exist \( (z_1, z_2) \in (-H \times -\text{int } K) \) and nets \( (\alpha_i) \subset \mathbb{R}_+, (x_i) \subset X, (d_i) \subset C(\varepsilon) \) and \( (k_i) \subset K \) such that

\[
\alpha_i(f(x_i) + d_i - f(x_0)) \rightarrow z_1,
\]

\[
\alpha_i(g(x_i) + k_i) \rightarrow z_2.
\]

As \( K \) is proper, \( 0 \notin \text{int } K \) and so we can assume that \( \alpha_i \neq 0 \). Since \( z_2 \notin -\text{int } K \) we can suppose that \( \alpha_i(g(x_i) + k_i) \in -\text{int } K \). As \( \alpha_i \neq 0 \), we deduce that

\[
g(x_i) + k_i \rightarrow -d_i \text{ in int } K \subset -K.
\]

Hence, \( (x_i) \) is a net of feasible points. By (4) it follows that \( z_1 \in \text{cl cone}(f(S) + C(\varepsilon) - f(x_0)) \cap (-H) \), which contradicts part (a) and the proof is complete. \( \Box \)

Let us recall that the support function \( \sigma_Q : Y^* \rightarrow \mathbb{R} \cup \{+\infty\} \) of a nonempty set \( Q \subset Y \) is \( \sigma_Q(y^*) = \sup_{y \in Q} \{y^*(y)\} \), for all \( y^* \in Y^* \).

Theorem 3.2. Let \( x_0 \in S, \varepsilon \geq 0 \) and a nonempty set \( C \subset D \setminus \{0\} \). Suppose that \( \text{int } D^+ \neq \emptyset \) and \( f - f(x_0) \) is nearly \((C, \varepsilon)\)-subconvexlike on \( S \). If \( x_0 \in \text{Be}(f, S, C, \varepsilon) \) then there exists \( \mu \in D^+ \) such that \( x_0 \in -\varepsilon\sigma_{-C}(\mu) - \text{argmin}_S(\mu \circ f) \).

Proof. Let \( x_0 \in \text{Be}(f, S, C, \varepsilon) \). By Lemma 3.1 there exist \( \mu \in D^+ \) and an open half space \( H \) defined by \( H = \{ y \in Y: \mu(y) > 0 \} \) such that

\[
\text{cl cone}(f(S) + C(\varepsilon) - f(x_0)) \cap (-H) = \emptyset.
\]

Hence, it is clear that

\[
\mu(f(x)) + \mu(d) - \mu(f(x_0)) \geq 0, \quad \forall x \in S, \forall d \in C(\varepsilon). \tag{5}
\]

In particular, we have that

\[
\mu(f(x)) \geq \mu(f(x_0)) - \inf_{d \in C(\varepsilon)} \{\mu(d)\} = \mu(f(x_0)) + \sigma_{-C(\varepsilon)}(\mu), \quad \forall x \in S.
\]

Therefore, \( x_0 \in -\sigma_{-C(\varepsilon)}(\mu) - \text{argmin}(\mu \circ f) \).

It is easy to see that

\[
\sigma_{-C(\varepsilon)}(\mu) = \varepsilon\sigma_{-C}(\mu), \quad \forall \varepsilon > 0. \tag{6}
\]

Since \( C(0) \cup \{0\} \) is a cone, statement (6) is also true for \( \varepsilon = 0 \) and the proof is complete. \( \Box \)

Remark 3.3. (a) If \( Y \) is normed then we can suppose in Theorem 3.2 that \( \|\mu\| = 1 \) since one can divide in statement (5) by \( \|\mu\| \).

(b) Suppose that \( C(0) = D \setminus \{0\} \). In [10, Theorem 5.4], the authors obtained the following necessary condition:

\[
\text{Be}(f, S, C, \varepsilon) \subset \bigcup_{\mu \in D^+, \|\mu\|=1} \varepsilon\beta\text{-argmin}_S(\mu \circ f). \tag{7}
\]
where $\beta = \inf\{\|d\|: d \in C\}$. To obtain this result they assume that $Y$ is a normed space, $C$ is a solid coraadiant ($\alpha C \subset C$, for all $\alpha \geq 0$) set and $D$ is locally compact or $D^+$ is solid, and $f$ is $D$-subconvexlike on $S$, i.e., $f(S) + \text{int}D$ is convex.

Theorem 3.2 improves [10, Theorem 5.4] since $Y$ is a Hausdorff locally convex space, $C$ does not need to be coriadiant neither solid (and so $D$ can be not solid) and the function $f - f(x_0)$ is assumed to be nearly $(C, \varepsilon)$-subconvexlike on $S$, which is a weaker generalized convexity condition than the subconvexlkeness of $f$ (see Remark 2.4 and [25]). Moreover, if $Y$ is normed, let us observe that the error $\varepsilon \beta$ in statement (7) is greater than the error $-\varepsilon \sigma_b(\mu)$ in Theorem 3.2, since

$$-\sigma_b(\mu) = \inf\{\mu(d): d \in C\} \leq \inf\{\|d\|: d \in C\} = \beta, \quad \forall \mu \in D^+, \|\mu\| = 1.$$ 

Moreover, the inequality can be strict as it is observed in the following example: let $Y = (\mathbb{R}^2, \|\cdot\|_2)$, $D = \mathbb{R}^2_+$, $C = \{(y_1, y_2) \in \mathbb{R}^2: y_1 + y_2 \geq 1\}$ and $\mu = (1/\sqrt{2}, 2/\sqrt{5})$. It is clear that $\beta = 1/\sqrt{2}$ and $-\sigma_b(\mu) = 1/\sqrt{5}$.

(c) Theorem 3.2 reduces to the single-valued version of [30, Theorem 4.4(i)] by considering $C = C_\varepsilon$.

**Theorem 3.4.** Let $\varepsilon \geq 0$ and a nonempty set $C \subset D \setminus \{0\}$. It follows that,

$$\bigcup_{\mu \in D^+} -\varepsilon \sigma_b(\mu)-\text{argmin}_S(\mu \circ f) \subset \text{Be}(f, S, C, \varepsilon).$$

**Proof.** Let us consider $\mu \in D^+$ and $x_0 \in -\varepsilon \sigma_b(\mu)-\text{argmin}_S(\mu \circ f)$. Suppose on the contrary that $x_0 \notin \text{Be}(f, S, C, \varepsilon)$. Then, there exist $v \in D \setminus \{0\}$ and nets $(\alpha_i) \subset \mathbb{R}^+, (\alpha_i) \subset S$ and $(d_i) \subset C(\varepsilon)$ such that $\alpha_i(f(x_i) + d_i - f(x_0)) \to -v$. Since $\mu \in D^+$ and $v \in D \setminus \{0\}$, we deduce that

$$\alpha_i\left(\mu(f(x_i)) + \mu(d_i) - \mu(f(x_0))\right) \to -\mu(v) < 0.$$ 

Therefore, we can suppose that there exists $i_0$ such that $\alpha_{i_0}(\mu(f(x_{i_0})) + \mu(d_{i_0}) - \mu(f(x_0))) < 0$. In particular,

$$\mu(f(x_{i_0})) + \mu(d_{i_0}) - \mu(f(x_0)) < 0,$$

and as $d_{i_0} \in C(\varepsilon)$ we deduce that

$$\mu(f(x_{i_0})) - \varepsilon \sigma_b(\mu) - \mu(f(x_0)) \leq \mu(f(x_{i_0})) + \mu(d_{i_0}) - \mu(f(x_0)) < 0,$$

which is a contradiction with (8). \qed

**Remark 3.5.** (a) Suppose that $C(0) = D \setminus \{0\}$. In [10, Theorem 5.5], the authors obtained the following sufficient optimality condition for $(C, \varepsilon)$-proper efficient solutions of $(P)$ by assuming that $Y$ is a Banach space, $C$ is coriadiant, $0 \notin \text{cl}C$ and $D^+$ is solid:

$$\bigcup_{\mu \in D^+} \varepsilon \theta-\text{argmin}_S(\mu \circ f) \subset \text{Be}(f, S, C, \varepsilon),$$

where $\theta = \inf\{\|d\|: d \in C\} \cap \text{inf}\{\|\xi - \mu\|: \xi \in Y^* \setminus D^+\}$. By [4, Lemma 2.7] we see that:

$$\inf_{d \in D, \|d\|=1} \{\mu(d)\} \geq \inf\{\|\xi - \mu\|: \xi \in Y^* \setminus D^+\}, \quad \forall \mu \in D^+.$$ 

From this statement is clear that

$$\mu(d) \geq \|d\| \inf\{\|\xi - \mu\|: \xi \in Y^* \setminus D^+\}, \quad \forall d \in D, \forall \mu \in D^+.$$ 

By applying this inequality to elements of $C$ we deduce that

$$-\sigma_b(\mu) = \inf\{\mu(d): d \in C\} \geq \inf\{\|d\|: d \in C\} \cap \text{inf}\{\|\xi - \mu\|: \xi \in Y^* \setminus D^+\} = \theta,$$

for all $\mu \in D^+$. Therefore,

$$\bigcup_{\mu \in D^+} \varepsilon \theta-\text{argmin}_S(\mu \circ f) \subset \bigcup_{\mu \in D^+} -\varepsilon \sigma_b(\mu)-\text{argmin}_S(\mu \circ f)$$

and so [10, Theorem 5.5] is a particular case of Theorem 3.4. Moreover, let us underline that the assumptions of Theorem 3.4 are more general than the assumptions of [10, Theorem 5.5].

(b) Theorem 3.4 reduces to the single-valued version of [30, Theorem 4.4(i)] by considering $C = C_\varepsilon$.

The next corollary is an immediate consequence of Theorems 3.2 and 3.4.
Corollary 3.6. Let \( \varepsilon \geq 0 \) and a nonempty set \( C \subset D \setminus \{0\} \). Suppose that \( \text{int} \, D^+ \neq \emptyset \) and \( f - f(x) \) is nearly \((C, \varepsilon)\)-subconvexlike on \( S \) for all \( x \in S \). Then,

\[
\text{Be}(f, S, C, \varepsilon) = \bigcup_{\mu \in D^+} -\varepsilon \sigma_{-C}(\mu) \text{-argmin}_S (\mu \circ f).
\]

Remark 3.7. (a) The previous corollary shows that there is no gap between the errors obtained in the necessary condition of Theorem 3.2 and in the sufficient condition in Theorem 3.4. Moreover, if \( Y \) is normed and \( \mu \in D^{+1} \), \( \|\mu\| = 1 \), it is easy to check that (see Remarks 3.3(b) and 3.5(a))

\[
\theta \leq -\sigma_{-C}(\mu) \leq \beta
\]

and so the errors of Theorems 5.4 and 5.5 of [10] are not the same. For example, consider \( Y = (\mathbb{R}^2, \| \cdot \|_2), D = \mathbb{R}^2, C = \{(y_1, y_2) \in \mathbb{R}^2: y_1 + y_2 \geq 1\} \) and \( \mu = (1/\sqrt{5}, 2/\sqrt{5}) \). It is clear that \( \theta = 1/\sqrt{20}, -\sigma_{-C}(\mu) = 1/\sqrt{5} \) and \( \beta = 1/\sqrt{2} \).

(b) If \( C = C_q \) then Corollary 3.6 reduces to [23, Theorem 4], where the order cone \( D \) is assumed to be solid and a convexity assumption stronger than nearly \((C, \varepsilon)\)-subconvexlikeness is considered.

(c) When \( \varepsilon = 0 \) and \( C = D \setminus \{0\} \), Corollary 3.6 reduces to well known characterizations of (exact) Benson proper solutions of a vector optimization problem. In particular, it reduces to [2, Corollary 4.1], [3, Theorem 4.2], [7, Theorem 4.1], [18, Theorem 3.1(ii)], the equivalence (ii) \( \iff \) (iii) of [21, Theorem 3.2] and the single-valued version of [32, Theorem 6.2]. Moreover, [3, Theorem 4.2] and [18, Theorem 3.1(ii)] are referred to a finite dimensional objective space, [3, Theorem 4.2], [7, Theorem 4.1] and [18, Theorem 3.1(ii)] use stronger convexity assumptions and [2, Corollary 4.1], [7, Theorem 4.1] and [18, Theorem 3.1(ii)] require a solid order cone.

Next we obtain scalar Lagrangian optimality conditions for \((C, \varepsilon)\)-proper solutions of problem \((P)\). The necessary condition of Theorem 3.8 will be used in Section 4 to prove saddle point results on \((C, \varepsilon)\)-proper solutions of problem \((P)\).

Theorem 3.8. Let \( x_0 \in S, \varepsilon \geq 0 \) and a nonempty set \( C \subset D \setminus \{0\} \). Suppose that \( \text{int} \, D^+ \neq \emptyset \), \( f - f(x_0) \) is nearly \((C, \varepsilon)\)-subconvexlike on \( S \), \( (f - f(x_0), g) \) is nearly \((C \times K, \varepsilon)\)-subconvexlike on \( X \) and the Slater constraint qualification holds. If \( x_0 \in \text{Be}(f, S, C, \varepsilon) \), then there exist \( \mu \in D^{+1} \) and \( \lambda \in K^+ \) such that the following two conditions are satisfied:

(a) \( x_0 \in -\varepsilon \sigma_{-C}(\mu) \text{-argmin}_X (\mu \circ f + \lambda \circ g) \).

(b) \( \sigma_{-C}(\mu) \leq (\lambda \circ g)(x_0) \leq 0 \).

Moreover, we can suppose that \(-\sigma_{-C}(\mu) = 1\) if one of the following additional conditions is satisfied:

(A1) \( C = B + Q \), where \( B \subset D \setminus \{0\} \) is compact and \( Q \subset D \).

(A2) \( Y \) is normed and \( 0 \notin \text{cl} \, C \).

Proof. By Lemma 3.1 it follows that there exists \( \xi \in D^{+1} \) such that the open half space \( H = \{y \in Y: \xi(y) > 0\} \) satisfies

\[
\text{cl} \, \text{cone}(f, g)(X) + (C(\varepsilon) \times K - (f(x_0), 0)) \cap (-H \times \text{int} \, K) = \emptyset.
\]

Since \( (f - f(x_0), g) \) is nearly \((C \times K, \varepsilon)\)-subconvexlike on \( X \), by Eidelheit’s separation theorem (see for instance [15, Theorem 3.16]), we deduce that there exists a functional \((\mu, \lambda) \in (Y^* \times \mathbb{R}^+) \setminus \{(0, 0)\}\) such that

\[
\begin{align*}
\mu(f(x) + d - f(x_0)) + \lambda(g(x) + k) &\geq 0, \quad \forall x \in X, d \in C(\varepsilon), k \in K, \\
\mu(h) + \lambda(k) &> 0, \quad \forall h \in H, k \in \text{int} \, K.
\end{align*}
\]

By (11) it is clear that \( \mu \in H^+ \) and \( \lambda \in K^+ \). Suppose that \( \mu = 0 \). Then, \( \lambda \neq 0 \) and by (10), we obtain that

\[
\lambda(g(x)) \geq 0, \quad \forall x \in X.
\]

As Slater constraint qualification holds, there exists \( \bar{x} \in X \) such that \( g(\bar{x}) \in \text{int} \, K \). As \( \lambda \neq 0 \) we have that \( \lambda(g(\bar{x})) < 0 \), which is a contradiction with (12). Hence, \( \mu \neq 0 \). Moreover, since \( \mu \in H^+ \) and \( D \setminus \{0\} \subset \text{int} \, H \), then \( \mu(d) > 0 \) for all \( d \in D \setminus \{0\} \), i.e., \( \mu \in D^{+1} \).

By taking \( k = 0 \) in (10), it is clear that

\[
\mu(f(x)) + \inf_{d \in C(\varepsilon)} \{\mu(d)\} - \mu(f(x_0)) + \lambda(g(x)) \geq 0, \quad \forall x \in X.
\]

As \( \lambda(g(x_0)) \leq 0 \), it follows that

\[
\mu(f(x)) + \lambda(g(x)) \geq \mu(f(x_0)) + \lambda(g(x_0)) + \varepsilon \sigma_{-C}(\mu), \quad \forall x \in X
\]

and part (a) is proved.

By taking \( x = x_0 \) in (13), we deduce that \( \varepsilon \sigma_{-C}(\mu) \leq \lambda(g(x_0)) \leq 0 \) and the proof of part (b) is finished.
Finally, the last statement of the theorem follows by taking \( -\mu/\sigma_-C(\mu) \) and \( -\lambda/\sigma_-C(\mu) \) instead of \( \mu \) and \( \lambda \), respectively, and the proof is complete if we check that \( -\sigma_-C(\mu) > 0 \). Indeed, if (A1) is true, then we have
\[
-\sigma_-C(\mu) = \inf\{\mu(d_1 + d_2) : d_1 \in B, d_2 \in Q \} \geq \inf\{\mu(d_1) : d_1 \in B \} > 0,
\]
since \( \mu \in D^+ \), \( Q \subset D \) and \( B \subset D \setminus \{0\} \) is compact.

Suppose that (A2) is true. As \( 0 \notin C \) there exists \( \delta > 0 \) such that \( \|d\| \geq \delta \) for all \( d \in C \). By [15, Lemma 3.21(d)] we see that \( D^+ \subset \text{int } D^+ \) and by (9) we deduce that
\[
-\sigma_-C(\mu) \geq \delta \inf\{|\xi - \mu| : \xi \in Y^* \setminus D^+\} > 0,
\]
since \( \mu \in \text{int } D^+ \) and the proof finishes. \( \square \)

**Theorem 3.9.** Let \( x_0 \in S, \varepsilon \geq 0 \) and a nonempty set \( C \subset D \setminus \{0\} \). If there exist \( \mu \in D^+ \) and \( \lambda \in K^+ \) such that \( x_0 \in -\varepsilon \sigma_-C(\mu) - \text{argmin}_x(\mu f + \lambda g) \) and \( \varepsilon \sigma_-C(\mu) \leq \lambda(g(x_0)) \), then \( x_0 \in \text{Be}(f, S, C, \bar{\varepsilon}) \), where \( \bar{\varepsilon} = \varepsilon \) if \( \lambda(g(x_0)) = 0 \) and \( \bar{\varepsilon} = 2\varepsilon \) if \( \lambda(g(x_0)) \neq 0 \).

**Proof.** By hypothesis, we have that
\[
\mu(f(x)) \geq \mu(f(x)) + \lambda(g(x)) \\
\geq \mu(f(x_0)) + \lambda(g(x_0)) + \varepsilon \sigma_-C(\mu) \\
\geq \mu(f(x_0)) + \varepsilon \sigma_-C(\mu), \quad \forall x \in S,
\]
since \( \varepsilon \sigma_-C(\mu) \leq \lambda(g(x_0)) \). Then, \( x_0 \in -\varepsilon \sigma_-C(\mu) - \text{argmin}_x(\mu f + \lambda g) \) and by Theorem 3.4, \( x_0 \in \text{Be}(f, S, C, \bar{\varepsilon}) \). \( \square \)

By applying Theorems 3.8 and 3.9 to \( C = D \setminus \{0\} \) and \( \varepsilon = 0 \) we obtain the following characterization of Benson proper solutions of a generalized convex problem \((P)\) through solutions of an associated scalar Lagrangian optimization problem, which was stated in [25, Corollary 4.1].

**Theorem 3.10.** Suppose that \( x_0 \in S \), \( \text{int } D^+ \neq \emptyset \), \( f - f(x_0) \) is nearly \( D\)-subconvexlike on \( S \), \( (f - f(x_0), g) \) is nearly \( (D \times K)\)-subconvexlike on \( X \) and the Slater constraint qualification holds. Then \( x_0 \in \text{Be}(f, S, D) \) if and only if there exist \( \mu \in D^+ \) and \( \lambda \in K^+ \) such that \( x_0 \in \text{argmin}_x(\mu f + \lambda g) \) and \( (\lambda g)(x_0) = 0 \).

4. Approximate proper saddle points

First, we introduce a new set-valued Lagrangian associated with problem \((P)\) and a new notion of approximate Benson-proper saddle point related to this set-valued Lagrangian.

**Definition 4.1.** Consider a nonempty set \( B \subset D \setminus \{0\} \). The function \( \Phi_B : X \times K^+ \to 2^Y \) defined by
\[
\Phi_B(x, \lambda) = f(x) + \lambda(g(x))B, \quad \forall x \in X, \forall \lambda \in K^+,
\]
is called \( B\)-Lagrangian associated with problem \((P)\).

**Remark 4.2.** Several authors have studied vector Lagrangian mappings \( \mathcal{L} : X \times \Gamma \to Y \), where \( \Gamma = \{T : Z \to Y : T \text{ is linear and } T(K) \subset D\} \)
and
\[
\mathcal{L}(x, T) = f(x) + T(g(x)), \quad \forall x \in X, \forall T \in \Gamma,
\]
that turn problem \((P)\) into an unconstrained vector optimization problem. The functional \( T \) is usually defined as follows:
\[
T(z) = \lambda(z)q, \quad \forall z \in Z,
\]
where \( \lambda \in K^+ \) and \( q \in D \setminus \{0\} \). So the following vector Lagrangian mapping \( \mathcal{L}_q : X \times K^+ \to Y \) is obtained:
\[
\mathcal{L}_q(x, \lambda) = f(x) + \lambda(g(x))q, \quad \forall x \in X, \forall \lambda \in K^+.
\]
Then, the set-valued \( B\)-Lagrangian of Definition 4.1 reduces to \( \mathcal{L}_q \) by considering the singleton \( B = \{q\} \). On the other hand, by using the set-valued \( B\)-Lagrangian we can obtain stronger saddle point conditions than the usual ones.
Definition 4.3. Let $\varepsilon \geq 0$ and $C \subset D \setminus \{0\}$. We say that $(x_0, \lambda_0) \in X \times K^+$ is a proper $\varepsilon$-saddle point with respect to $C$ for the set-valued $B$-Lagrangian associated with problem $(P)$ if the following two conditions are satisfied:

(a) $\text{cl cone}(\Phi_B(X, \lambda_0) + C(\varepsilon) - \Phi_B(x_0, \lambda_0)) \cap (-D) = \{0\}$,
(b) $\Phi_B(x_0, \lambda_0) \subset \text{Max}(\Phi_B(0, K^+), C, \varepsilon)$.

Remark 4.4. (a) Let us observe that condition (a) of Definition 4.3 generalizes (1) from a vector-valued mapping $f$ to a set-valued mapping $\Phi_B$. So, we denote it by $x_0 \in \text{Be}(\Phi_B(\cdot, \lambda_0), X, C, \varepsilon)$ and by $x_0 \in \text{Be}(\Phi_B(\cdot, \lambda_0), X, D)$ if $C = D \setminus \{0\}$, or $\varepsilon = 0$ and $C(0) = D \setminus \{0\}$. In both cases we say that $x_0$ is a proper saddle point for the $B$-Lagrangian associated with problem $(P)$.

(b) In the literature, the saddle point concepts associated with vector Lagrangian mappings are usually based on (non necessarily proper) efficient solutions of the related Lagrangian vector optimization problem. However, the saddle point concept of Definition 4.3 considers in part (a) a kind of proper minimal point of the set-valued Lagrangian $\Phi_B$. Due to this fact we can obtain necessary optimality conditions for approximate or exact Benson proper solutions of problem $(P)$ via saddle point assertions stronger than the usual ones.

The following theorem shows a sufficient condition for the elements of the set $\text{Be}(\Phi_B(\cdot, \lambda_0), X, C, \varepsilon)$ based on suboptimal solutions of associated scalar optimization problems. Given $\mu \in D^{s+}$ we denote

$$B_{\mu} = \{d \in D: \mu(d) = 1\}.$$

Theorem 4.5. Let $\varepsilon \geq 0$. If there exist $\mu \in D^{s+}$ and $\lambda \in K^+$ such that $x_0 \in \varepsilon$-argmin$_X(\mu \circ f + \lambda \circ g)$, then $x_0 \in \text{Be}(\Phi_B(\cdot, \lambda), X, C, \varepsilon)$, for all nonempty set $C \subset D \setminus \{0\}$ such that $-\sigma_{-C}(\mu) \geq 1$.

Proof. Suppose on the contrary that there exists $C \subset D \setminus \{0\}$ such that $-\sigma_{-C}(\mu) \geq 1$ and $x_0 \notin \text{Be}(\Phi_B(\cdot, \lambda), X, C, \varepsilon)$. Then

$$\text{cl cone}(\Phi_B(\cdot, \lambda) + C(\varepsilon) - \Phi_B(x_0, \lambda)) \cap (-D \setminus \{0\}) \neq \emptyset$$

and there exist $w \in -D \setminus \{0\}$ and nets $(\alpha_i) \subset \mathbb{R}_+$, $(y_i) \subset \Phi_B(\cdot, \lambda)$, $(d_i) \subset C(\varepsilon)$ and $(z_i) \subset \Phi_B(x_0, \lambda)$ such that $\alpha_i(y_i + d_i - z_i) \rightarrow w$.

For each $i$, there exist $x_i \in X$ and $q_i, p_i \in B_{\mu}$ with

$$y_i = f(x_i) + \lambda(g(x_i))q_i,$$

$$z_i = f(x_0) + \lambda(g(x_0))p_i,$$

and as $x_0 \in \varepsilon$-argmin$_X(\mu \circ f + \lambda \circ g)$ we deduce that

$$\mu(\alpha_i(f(x_i) + \lambda(g(x_i))q_i + d_i - f(x_0) - \lambda(g(x_0))p_i))$$

$$\geq \alpha_i(\mu \circ f + \lambda \circ g)(x_i) + \varepsilon - (\mu \circ f + \lambda \circ g)(x_0)) \geq 0,$$

(14)

since $\varepsilon > 0$ then

$$\mu(d_i) \geq \varepsilon \inf_{d \in C} \mu(d) = -\varepsilon \sigma_{-C}(\mu) \geq \varepsilon$$

and if $\varepsilon = 0$ then

$$\mu(d_i) \geq \inf_{d \in C(0)} \mu(d) = 0.$$

Thus, taking the limit in (14) it follows that $\mu(w) \geq 0$. But, on the other hand, $w \in -D \setminus \{0\}$ and, since $\mu \in D^{s+}$, we deduce that $\mu(w) < 0$, obtaining a contradiction.

Next we obtain a necessary condition for $(C, \varepsilon)$-proper solutions of problem $(P)$ via $(C, \varepsilon)$-proper solutions of unconstrained $B$-Lagrangians associated with $(P)$ by assuming $(C, \varepsilon)$-subconvexlikeness hypotheses.

Corollary 4.6. Consider $x_0 \in S$, $\varepsilon \geq 0$ and a nonempty set $C \subset D \setminus \{0\}$. Suppose that $\text{int} \, D^+ \neq \emptyset$, $f - f(x_0)$ is nearly $(C, \varepsilon)$-subconvexlike on $S$, $(f - f(x_0), g)$ is nearly $(C \times K, \varepsilon)$-subconvexlike on $X$ and the Slater constraint qualification holds. If $x_0 \in \text{Be}(f, S, C, \varepsilon)$, then there exist $\mu \in D^{s+}$ and $\lambda \in K^+$ such that $x_0 \in \text{Be}(\Phi_B(\cdot, \lambda), X, C', -\varepsilon \sigma_{-C}(\mu))$, for all $C' \subset D \setminus \{0\}$ such that $-\sigma_{-C}(\mu) \geq 1$. In particular, if assumption (A1) or (A2) is satisfied, then $x_0 \in \text{Be}(\Phi_B(\cdot, \lambda), X, C, \varepsilon)$.
Proof. By Theorem 3.8 there exist $\mu \in D^+$ and $\lambda \in K^+$ such that $x_0 \in -\varepsilon C(\mu)$-argmin$_x(\mu \circ f + \lambda \circ g)$ and $\varepsilon C(\mu) \leq (\lambda \circ g)(x_0) \leq 0$. By Theorem 4.5 we deduce that $x_0 \in \text{Be}(\Phi_B(\cdot, \cdot), X, C', -\varepsilon C(\mu))$, for all nonempty set $C' \subset D \setminus \{0\}$ such that $-\varepsilon C(\mu) \geq 1$.

By Theorem 3.8, if assumption (A1) or (A2) is true, then one can consider $C = C'$ since $-\varepsilon C(\mu) = 1$ and the proof is complete. □

In the following result, we obtain a characterization of condition (b) in Definition 4.3 in terms of the feasibility of the point $x_0$ and an approximate complementary slack condition.

Lemma 4.7. Suppose that $K$ is closed. Consider $\varepsilon \geq 0$, $x_0 \in X$, $\lambda_0 \in K^+$ and $B \subset D \setminus \{0\}$, $C \subset D \setminus \{0\}$ such that $\text{cone } B \cap \text{cone } C \neq \{0\}$ and $C + \text{cone } B = C$. Then $\Phi_B(x_0, \lambda_0) \subset \text{Max}(\Phi_B(x_0, K^+), C, \varepsilon)$ if and only if $g(x_0) \in -K$ and $\lambda_0(g(x_0))B \subset -(D \cap C(\varepsilon)^\circ)$.

Proof. It is clear that $\Phi_B(x_0, \lambda_0) \subset \text{Max}(\Phi_B(x_0, K^+), C, \varepsilon)$ if and only if

$$
\lambda(g(x_0))b - \lambda_0(g(x_0))b_0 \notin C(\varepsilon), \quad \forall \lambda \in K^+, \forall b, b_0 \in B.
$$

(15)

In particular, by taking $b = b_0$ it follows that

$$(\lambda - \lambda_0)(g(x_0))b_0 \notin C(\varepsilon), \quad \forall \lambda \in K^+, \forall b_0 \in B
$$

(16)

and by considering $\lambda = 0$ we deduce that

$$
\lambda_0(g(x_0))B \cap (-C(\varepsilon)) = \emptyset.
$$

Let us suppose that $g(x_0) \notin -K$. By applying a standard separation argument (see for instance [15, Theorem 3.18]) we deduce that there exists $\hat{\lambda} \in K^+$ such that $\hat{\lambda}(g(x_0)) > 0$. As cone $B \cap$ cone $C \neq \{0\}$ there exist $b \in B$ and $\alpha > 0$ such that $\alpha b \in C(\varepsilon)$. Consider the functional $\lambda' := \lambda_0 + (\alpha/\hat{\lambda}(g(x_0)))\hat{\lambda}$. It is clear that $\lambda' \in K^+$ and

$$(\lambda' - \lambda_0)(g(x_0))b = \alpha b \in C(\varepsilon),$$

contrary to (16). Then $g(x_0) \in -K$ and so

$$
\lambda_0(g(x_0))B \subset -D,
$$

(17)

since $\lambda_0 \in K^+$ and $B \subset D$. By (15) and (17) we have that

$$
\lambda_0(g(x_0))B \subset -(D \cap C(\varepsilon)^\circ).
$$

(18)

Reciprocally, suppose that $g(x_0) \in -K$ and (18) is true. Then

$$
\lambda(g(x_0))b \in -\text{cone } B, \quad \forall \lambda \in K^+, \forall b \in B
$$

(19)

and

$$
-\lambda_0(g(x_0))b_0 \notin C(\varepsilon), \quad \forall b_0 \in B.
$$

(20)

If there exist $\lambda \in K^+$, $b, b_0 \in B$ such that

$$
\lambda(g(x_0))b - \lambda_0(g(x_0))b_0 \in C(\varepsilon)
$$

then by (19) we see that

$$
-\lambda_0(g(x_0))b_0 = (\lambda(g(x_0))b - \lambda_0(g(x_0))b_0) - \lambda(g(x_0))b \in C(\varepsilon) + \text{cone } B = C(\varepsilon),
$$

contrary to (20). Therefore statement (15) is true and we have $\Phi_B(x_0, \lambda_0) \subset \text{Max}(\Phi_B(x_0, K^+), C, \varepsilon)$. □

Remark 4.8. (a) Let us observe that the closedness of the cone $K$ is not required to prove the sufficient condition of Lemma 4.7.

(b) When $\varepsilon = 0$ and cone $C = D$, statement $\lambda_0(g(x_0))B \subset -(D \cap C(\varepsilon)^\circ)$ of Lemma 4.7 reduces to the well known complementary slack condition. Indeed,

$$
D \cap C(0)^\circ = D \cap (D \setminus \{0\})^\circ = \{0\}
$$

and so

$$
\lambda_0(g(x_0))B \subset -(D \cap C(0)^\circ) \iff \lambda_0(g(x_0)) = 0.
$$

since $0 \notin B$. 

(c) By Lemma 4.7 and [17, Proposition 6.1] we deduce that the concept of proper \(\varepsilon\)-saddle point with respect to \(C\) for the \(B\)-Lagrangian associated with problem \((P)\) reduces to the single valued version of the proper saddle point notion due to Li (see [17, Definition 6.2]) by considering \(\varepsilon = 0\) and \(C = D \setminus \{0\}\). 

(d) Let us observe that Lemma 4.7 essentially reduces to [31, Proposition 3.1] by considering a singleton \(B = \{q\}, \ v \in D \setminus \{0\}\) and \(C = \varepsilon + D\). In this case, the complementary slack condition for \(\varepsilon > 0\) ensures that 

\[
\lambda_0(g(x_0))q \in -D \cap (-\varepsilon e - D)^c
\]

and the set \(-D \cap (-\varepsilon e - D)^c\) is not bounded (see [31, Remark 3.1]). In general, all approximate saddle point concepts in the literature associated with vector optimization problems give unbounded complementary slack conditions.

However, the set \(D \cap \epsilon C \cap C\) can be bounded if we consider a suitable set \(C\). Thus, the notion of \(B\)-Lagrangian and the proper \(\epsilon\)-saddle point concept introduced in Definition 4.3 overcome this drawback. For example, if \(Y\) is normed and the norm \(\|\cdot\|\) is \(D\)-monotone on \(D\) (i.e., \(0 \leq d_1 \leq d_2 \Rightarrow \|d_1\| \leq \|d_2\|\)) then \(B = B \cap (D \setminus \{0\})\), where \(B \subset Y\) denotes the unit open ball, and \(C = B^c \cap D\) satisfy cone \(B = \text{cone} C = D\), \(C + \text{cone} B = C\) and \(D \cap \epsilon C = \epsilon B \cap D\) is bounded.

(e) With respect to the assumptions of Lemma 4.7, let us observe that condition \(C + \text{cone} B = C\) does not imply that cone \(B \cap \text{cone} C \neq \emptyset\). Indeed, consider for example \(Y = \mathbb{R}^2, \ D = \mathbb{R}^2_+, \ C = (1, 1) + \mathbb{R}^2_+\) and \(B = \{(1, 0)\}\). It is clear that cone \(B = \{(\alpha, 0): \alpha > 0\}\) and cone \(C = \text{int} \mathbb{R}^2_+ \cup \{0\}\). Thus, \(C + \text{cone} B = C\) but cone \(B \cap \text{cone} C = \emptyset\).

However, if cone \(C\) is closed then

\[
C + \text{cone} B = C \implies \text{cone} B \subset \text{cone} C \implies \text{cone} B \cap \text{cone} C = \text{cone} B \neq \emptyset.
\] 

Let us check the first implication of (21). Fix a point \(q \in C\) and consider \(\varepsilon > 0\) and an arbitrary element \(b \in B\). Then

\[
\alpha b = \lim_{n \to \infty} ((1/n)q + \alpha b) = \lim_{n \to \infty} (1/n)(q + (\alpha n)b) \in \text{cl cone}(C + \text{cone} B) = \text{cone} C
\]

and we have that cone \(B \subset \text{cone} C\).

**Lemma 4.9.** Let \(\mu \in D^+, \ \alpha_0 > 0, \ \varepsilon > 0\) and consider \(\text{cone} C = \text{cone} B = \text{cone} (D \setminus \{0\})\). Then

\[
\alpha_0 B \mu \subset C_{\mu}(\varepsilon) \iff \alpha_0 \leq \varepsilon.
\]

**Proof.** It is easy to check that \(C_{\mu}(0) = D \setminus \{0\}\). Then

\[
\alpha_0 B \mu \subset C_{\mu}(0) \iff \alpha_0 B \mu \subset D \cap (D \setminus \{0\}) \iff \alpha_0 = 0
\]

and relation (22) is true if \(\varepsilon = 0\).

Consider that \(\varepsilon \neq 0\). Without loss of generality we can suppose \(\varepsilon = 1\) since \(C_{\mu}(\varepsilon) = \varepsilon C_{\mu}\). Assume that \(\alpha_0 B \mu \subset C_{\mu}^c\). If \(\alpha_0 > 1\) and \(b \in B \mu\) is arbitrary we have 

\[
\alpha_0 b = b + (\alpha_0 - 1)b \in B \mu + (D \setminus \{0\}) = C_{\mu},
\]

that is contrary to \(\alpha_0 B \mu \subset C_{\mu}^c\). Thus, \(\alpha \leq 1\) and the necessary condition is true. Reciprocally, if \(\alpha_0 \leq 1\) and there exists \(b \in B \mu\) such that \(\alpha_0 b \in C_{\mu}\) then \(\alpha_0 > 1\), since \(\mu(d) > 1\) for all \(d \in C_{\mu}\), and a contradiction is obtained. Thus the sufficient condition holds and the proof is complete. \(\square\)

Next we obtain a necessary condition for \((C, \varepsilon)\)-proper solutions of a nearly \((C, \varepsilon)\)-subconvexlike problem \((P)\) in terms of proper \(\varepsilon\)-saddle points of \(B\)-Lagrangians.

**Theorem 4.10.** Consider \(x_0 \in S, \ \varepsilon \geq 0\) and a nonempty set \(C \subset D \setminus \{0\}\). Suppose that \(\text{int} D^+ \neq \emptyset, \ f - f(x_0)\) is nearly \((C, \varepsilon)\)-subconvexlike on \(S, \ (f - f(x_0), g)\) is nearly \((C \times K, \varepsilon)\)-subconvexlike on \(X, \ \text{the Slater constraint qualification holds and assumption}\) (A1) or (A2) is true. If \(x_0 \in \text{Be}(f, S, C, \varepsilon)\) then there exist \(\mu \in D^+\) and \(\lambda_0 \in K^+\) such that \(-\sigma_{-C}(\mu) = 1\) and \((x_0, \lambda_0)\) is a proper \(\varepsilon\)-saddle point with respect to \(C_{\mu}\) for the \(B_{\mu}\)-Lagrangian associated with problem \((P)\).

**Proof.** By Theorem 3.8 we deduce that there exist \(\mu \in D^+\) and \(\lambda_0 \in K^+\) such that

\[
-\sigma_{-C}(\mu) = 1, \quad x_0 \in \varepsilon\text{-argmin}_X (\mu \circ f + \lambda_0 \circ g),
\]

\[
-\varepsilon \leq (\lambda_0 \circ g)(x_0).
\]

By (24) and Theorem 4.5 it follows that

\[
x_0 \in \text{Be}(\text{Be}_{B_{\mu}}(\cdot, \lambda_0), X, C_{\mu}, \varepsilon)
\]
(see Remark 4.4(a)), since
\[ -\sigma_{-C_\mu}(\mu) = \inf\{\mu(q + d): q \in B_\mu, \ d \in D \setminus [0]\} = 1 + \inf\{\mu(d): d \in D \setminus [0]\} = 1. \]

On the other hand, it is obvious that \(\lambda(g(x_0))B_\mu \subset \{-D\}, \) since \(x_0\) is feasible, \(\lambda \in K^+\) and \(B_\mu \subset D.\) By (22) and (25) we see that \(\lambda g(x_0)B_\mu \subset -(D \cap C_\mu(\varepsilon^2))\) and by applying the sufficient condition of Lemma 4.7 (see part (a) of Remark 4.8) we have that
\[ \Phi_B(x_0, \lambda_0) \subset \max(\Phi_B(x_0, K^+), C_\mu, \varepsilon) \]
and the result follows by (23), (26) and (27). \(\square\)

In the following result we give an exact version of Theorem 4.10.

**Theorem 4.11.** Consider \(x_0 \in S,\) \(q \in D \setminus [0]\) and suppose that \(\int D^+ \neq \emptyset,\) \(K\) is closed, \(f - f(x_0)\) is nearly \(D\)-subconvexlike on \(S,\) \((f - f(x_0), g)\) is nearly \((D \times K)\)-subconvexlike on \(X\) and the Slater constraint qualification holds. If \(x_0 \in \text{Be}(f, S, D)\) then there exist \(\mu \in D^+\) and \(\lambda_0 \in K^+\) such that \(-\sigma_{-C_\mu}(\mu) = 1\) and \((x_0, \lambda_0)\) is a proper saddle point for the \(B_\mu\) and \([q]\)-Lagrangians associated with problem \((P)\).

**Proof.** Consider \(C_q = q + D\) and \(\varepsilon = 0.\) With these data, the assumptions of Theorem 4.10 are satisfied. Indeed, from Remark 2.2(a) it follows that \(f - f(x_0)\) is nearly \((C_q, 0)\)-subconvexlike on \(S,\) since \(f - f(x_0)\) is nearly \(D\)-subconvexlike on \(S\) and also \(\text{Be}(f, S, D) = \text{Be}(f, S, C_q)\) \(0.\) Analogously, \((f - f(x_0), g)\) is nearly \((C_q \times K, 0)\)-subconvexlike on \(X,\) since \((f - f(x_0), g)\) is nearly \((D \times K)\)-subconvexlike on \(X\) and \(K\) is closed. Moreover, it is obvious that assumption (A1) is satisfied. Then, by Theorem 4.10 we know that there exist \(\mu \in D^+\) and \(\lambda_0 \in K^+\) such that \(-\sigma_{-C_\mu}(\mu) = 1\) and \((x_0, \lambda_0)\) is a proper 0-saddle point with respect to \(C_\mu\) for the \(B_\mu\)-Lagrangian associated with problem \((P).\) As \(C_\mu(0) = D \setminus [0]\) we deduce that \((x_0, \lambda_0)\) is a proper saddle point for the \(B_\mu\)-Lagrangian associated with problem \((P).\) Moreover, it is clear that
\[ -\sigma_{-C_\mu}(\mu) = 1 \iff \mu(q) = 1. \]
So \(q \in B_\mu\) and from the definition of proper \(\varepsilon\)-saddle point and Remark 2.2(a) we see that
\[ \text{cl}\ \text{cone}(\Phi_B(x, \lambda_0) + D - \Phi_B(x_0, \lambda_0)) \cap (-D) \subset \text{cl}\ \text{cone}(\Phi_B(x, \lambda_0) + C_\mu(0) - \Phi_B(x_0, \lambda_0)) \cap (-D) = [0], \]
and
\[ \Phi_B(x_0, \lambda_0) \in \Phi_B(x_0, \lambda_0) \subset \max(\Phi_B(x_0, K^+), D). \]
Since \(\Phi_B(x_0, \lambda_0) \in \Phi_B(x, K^+) \subset \Phi_B(x_0, K^+)\) it follows that \(\Phi_B(x_0, \lambda_0) = \max(\Phi_B(x_0, K^+), D).\) Thus, \((x_0, \lambda_0)\) is a proper saddle point for the \([q]\)-Lagrangian associated with problem \((P)\) and the proof is complete. \(\square\)

**Remark 4.12.** The (exact) saddle point result of Theorem 4.11 is stronger than other similar saddle point results in the literature based on vector valued Lagrangian functions \(L : X \times \Gamma \to Y\) and Benson proper efficient solutions of problem \((P)\), since it considers Benson proper efficient solutions of Lagrangian mappings instead of efficient solutions (see Remarks 4.2 and 4.4(b)), and compare Theorem 4.11 with [1, Corollary 4.2], [18, Theorem 4.4], [19, Theorem 3.2] and [29, Theorem 4.1]).

In the following result we obtain a sufficient condition for \((C, \varepsilon)\)-proper solutions of problem \((P)\) based on \((C, \varepsilon)\)-proper solutions of unconstrained \(B\)-Lagrangian mappings.

**Theorem 4.13.** Consider \(\varepsilon \geq 0, \lambda \in K^+, B \subset D \setminus [0]\) such that \(C = B + P, \) where \(P \subset D\) satisfies \(P(0) \subset P\) and \(P + D = D.\) If \(x_0 \in S,\) \(-\varepsilon \leq \lambda(g(x_0))\) and \(x_0 \in \text{Be}(\Phi_B(\cdot, \lambda), X, C, \varepsilon)\) then \(x_0 \in \text{Be}(f, S, C, \varepsilon - \lambda(g(x_0))).\)

**Proof.** Let us suppose that \(\tilde{\varepsilon} := \varepsilon - \lambda(g(x_0)) > 0.\) As \(-\varepsilon \leq \lambda(g(x_0)) \leq 0,\) if \(\varepsilon = 0\) then \(\lambda(g(x_0)) = 0\) and so \(\tilde{\varepsilon} = 0,\) which is a contradiction. Thus \(\varepsilon > 0.\)

Since \(C(\tilde{\varepsilon}) \subset \varepsilon B - \lambda(g(x_0))B + P\) and \(0 \in \lambda(g(x))B + D\) for all \(x \in S\) we have that
\[ f(x) - f(x_0) + C(\tilde{\varepsilon}) \subset f(x) - f(x_0) + \varepsilon B - \lambda(g(x_0))B + P \]
\[ \subset f(x) + \lambda(g(x))B + D - f(x_0) + \varepsilon B - \lambda(g(x_0))B + P \]
\[ = \Phi_B(x, \lambda) - \Phi_B(x_0, \lambda) + \varepsilon B + P + D \]
\[ = \Phi_B(x, \lambda) - \Phi_B(x_0, \lambda) + C(\varepsilon), \quad \forall x \in S, \]
since $P + D = P$, $\varepsilon > 0$ and $\varepsilon B + P = \varepsilon (B + P) = \varepsilon C$. Therefore,

$$f(S) - f(x_0) + C(\bar{\varepsilon}) \subseteq \Phi_B(X, \lambda) - \Phi_B(x_0, \lambda) + C(\varepsilon).$$

(28)

If $\bar{\varepsilon} = 0$ then $\varepsilon = \lambda(g(x_0)) = 0$ and it follows that

$$f(x) - f(x_0) + C(0)$$

$$= f(x) - f(x_0) + \text{cone} B \setminus \{0\} + P$$

$$\subseteq f(x) + \lambda(g(x))B + D - f(x_0) - \lambda(g(x_0))B + \text{cone} B \setminus \{0\} + P$$

$$= \Phi_B(x, \lambda) - \Phi_B(x_0, \lambda) + \text{cone} B \setminus \{0\} + P + D$$

$$= \Phi_B(x, \lambda) - \Phi_B(x_0, \lambda) + C(0), \quad \forall x \in S.$$}

Then (28) is also true for $\bar{\varepsilon} = 0$ and as $x_0 \in \text{Be}(\Phi_B(\cdot, \lambda), X, C, \varepsilon)$ we deduce that

$$\text{cl cone}(f(S) - f(x_0) + C(\bar{\varepsilon})) \cap (-D)$$

$$\subseteq \text{cl cone}(\Phi_B(X, \lambda) - \Phi_B(x_0, \lambda) + C(\varepsilon)) \cap (-D)$$

$$= \{0\},$$

i.e., $x_0 \in \text{Be}(f, S, C, \varepsilon - \lambda(g(x_0)))$, which finishes the proof. □

**Remark 4.14.** With respect to the assumptions of Theorem 4.13, let us observe that if $0 \in P$, then conditions $P \subseteq D$ and $P + D = P$ imply $P = D$. However, $P = D \setminus \{0\}$ or $P = \text{int} D$ (when $D$ is solid) satisfy the assumptions and $P \neq D$ in both cases.

In the next corollary we give a sufficient condition for $(C, \bar{\varepsilon})$-proper solutions of problem $(P)$ through proper $\varepsilon$-saddle points of $B$-Lagrangians.

**Corollary 4.15.** Consider $\varepsilon \geq 0$, $\mu \in D^+$ and suppose that $K$ is closed.

(a) If $(x_0, \lambda_0) \in X \times K^+$ is a proper $\varepsilon$-saddle point with respect to $C_\mu$ for the $B_\mu$-Lagrangian associated with problem $(P)$ then $x_0 \in \text{Be}(f, S, C_\mu, \varepsilon - \lambda_0(g(x_0)))$.

(b) If $(x_0, \lambda_0) \in X \times K^+$ is a proper saddle point for the $B_\mu$-Lagrangian associated with problem $(P)$ then $x_0 \in \text{Be}(f, S, D)$.

**Proof.** Let $(x_0, \lambda_0) \in X \times K^+$ be a proper $\varepsilon$-saddle point with respect to $C_\mu$ for the $B_\mu$-Lagrangian associated with problem $(P)$. By Lemma 4.7 we deduce that $x_0 \in S$ and $\lambda_0(g(x_0))B_\mu \subset - (D \cap C_\mu(\varepsilon))^\circ$. By (22) we see that

$$-\varepsilon \leq \lambda(g(x_0)) \leq 0.$$  

(29)

Then part (a) follows by applying Theorem 4.13 to $\lambda = \lambda_0$, $B = B_\mu$ and $P = D \setminus \{0\}$.

Suppose that $(x_0, \lambda_0) \in X \times K^+$ is a proper saddle point for the $B_\mu$-Lagrangian associated with problem $(P)$. It is clear that $(x_0, \lambda_0) \in X \times K^+$ is a proper 0-saddle point with respect to $C_\mu$ for the $B_\mu$-Lagrangian associated with problem $(P)$, since $C_\mu(0) = D \setminus \{0\}$ (see Remark 4.4(a)). Then, by part (a) and (29) we deduce that $x_0 \in \text{Be}(f, S, C_\mu, 0) = \text{Be}(f, S, D)$. □

Next we characterize the set of Benson proper solutions of problem $(P)$ through saddle points for $B_\mu$-Lagrangians. The result is a direct consequence of Theorem 4.11 and Corollary 4.15.

**Corollary 4.16.** Let $x_0 \in X$ and suppose that $\text{int} D^+ \neq \emptyset$, $K$ is closed, $f - f(x_0)$ is nearly $D$-subconvexlike on $S$, $(f - f(x_0), g)$ is nearly $(D \times K)$-subconvexlike on $X$ and the Slater constraint qualification holds. Then $x_0 \in \text{Be}(f, S, D)$ if and only if there exist $\mu \in D^+$ and $\lambda_0 \in K^+$ such that $(x_0, \lambda_0)$ is a proper saddle point for the $B_\mu$-Lagrangian associated with problem $(P)$.

**Remark 4.17.** In [17, Theorems 6.1 and 6.2], Li characterized the set of Benson proper solutions of a subconvexlike vector optimization problem with set-valued mappings via proper saddle points by assuming that the ordering cone is solid. Then Corollary 4.16 improves the vector valued version of these theorems, since its assumptions are weaker.
5. Conclusions

In this work we have introduced an approximate version of the well known nearly subconvexlikeness, as well as a new set-valued Lagrangian in vector optimization and a new notion of approximate proper saddle point in this framework.

In our opinion, the paper contains two relevant contributions. First, we have obtained a characterization of approximate Benson-proper solutions of a generalized convex constrained vector optimization problem through approximate solutions of associated scalar optimization problems with the same error in the necessary and sufficient condition. This result improves meaningfully the characterization obtained by Gao et al. (see [10]) not only because there is no gap between the errors in the necessary and sufficient condition of our characterization, but also because the hypotheses required to this end are weaker than the hypotheses used by Gao et al.

Second, our concept of approximate proper saddle point is defined by means of a new set-valued Lagrangian, which generalizes the vector Lagrangian functions that appear in the literature. The most relevant result obtained by using this new notion of saddle point is that the complementary slack condition is bounded while, in general, all approximate saddle point assertions in the literature associated with vector optimization problems give not bounded complementary slack conditions.

Finally, let us observe that our results give “exact statements” by considering $\varepsilon = 0$ and, in this case, they reduce to new and more general optimality conditions for Benson proper solutions of nearly subconvexlike vector optimization problems by linear scalarization and proper saddle point theorems.

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